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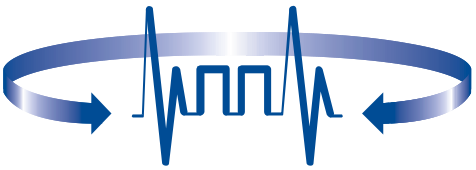

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Данный сборник содержит тезисы докладов международной конференции, посвящённой 80-летию со дня рождения Роланда Львовича Добрушина (1929-1995).

Тематика конференции соответствует основным областям научных интересов Р.Л.Добрушина, и включает Теорию Вероятностей, Теорию Информации, Математическую Физику, Статистическую Механику и Математическую Лингвистику.

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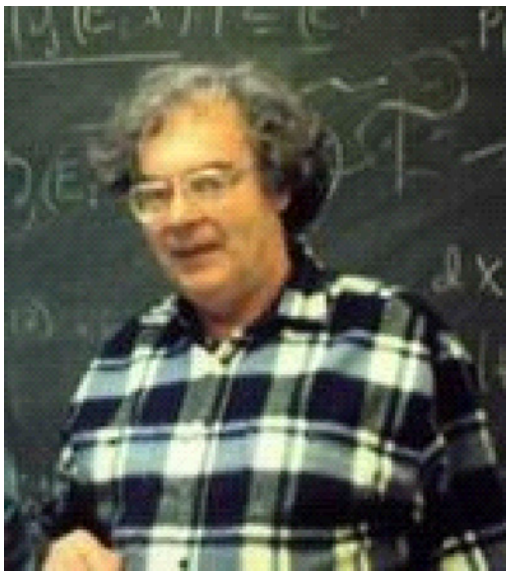

This volume contains extended abstracts of the talks presented at the “Dobrushin-80” international conference, dedicated to the 80-th birthday of Roland L’vovich Dobrushin (1929-1995).

The topics of the conference correspond to the main areas of Roland Dobrushin scientific interests and include Probability Theory, Mathematical Physics, Statistical Mechanics, Information Theory and Mathematical Linguistics.

The conference is organized by the Institute for Information Transmission Problems of the Russian Academy of Sciences (IITP RAS), Independent University of Moscow and French-Russian Poncelet Lab, with financial support of Russian Foundation for Fundamental Research and Russian Academy of Sciences.

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On exponential mixing rate for degenerate 2D diffusion

N. Abourashchi,* A. Yu. Veretennikov†

June 10, 2009

Abstract

A new existence of weak solution of a degenerate stochastic differential equation is established. A new method for a verification of a "local mixing condition" is proposed. An extension of Girsanov–Beneš' result on a martingale property of stochastic exponential is established. As an application, an exponential uniform beta–mixing rate is shown for a degenerate two– dimensional diffusion of Langevin type studied earlier by Campillo et al.

1 Introduction

In a series of papers by F. Campillo et al. [2], [3], [4] the following system of SDEs in R^2 has been investigated for recurrence, invariant measure, approximation, etc.,

$$\begin{aligned} dX_t &= Y_t\,dt, \qquad X_0 = x, \\ dY_t &= b(X_t, Y_t)\,dt + dW_t, \qquad Y_0 = y, \end{aligned} \tag{1}$$

where W is a standard Wiener process, and drift b is a Borel measurable function satisfying a linear growth condition and has a special form,

$$b(x, y) = -u(x, y)y - \beta\, x - \gamma\ \mathrm{sign}(y), \tag{2}$$

where β and γ are some positive constants, and u satisfies Assumption (A2)below. The system describes a mechanical "semi–active" suspension device in a vehicle under external stochastic perturbations treated as a white noise. The term with γ corresponds to *friction,* β is a *spring* coefficient, uY corresponds to *damping* (control related to the velocity of the device), and the function u here stands for tuning of this damping control. Under appropriate assumptions, existence of a (unique) invariant measure has been proved [2]; however, the question of convergence remained open. In this paper we show exponential bound on rate of convergence toward the stationary measure in the distance of total variation for the system (1)–(2) and a bit more general, and a similar exponential bound of beta-mixing, under suitable assumptions on the coefficients. The method of establishing *local mixing* proposed below is applicable to the equation (1), and should be suitable for a wider class of processes, in particular, not necessarily 2D.

Remind the definition of beta-mixing coefficient,

$$\beta_t^{x,y} := \sup_{s\ge 0} E_{x,y} \sup_{B\in\mathcal{B}^2} (P_{x,y}((X_{t+s}, Y_{t+s}) \in B) - P_{x,y}((X_{t+s}, Y_{t+s}) \in B \mid F_s^{X,Y})), \tag{3}$$

where (x, y) is the initial condition for the equation. The coefficient $\beta_t^{x,y}$ dominates the (non-stationary) *alpha-mixing* coefficient introduced (in the stationary form) by Rosenblatt, and the latter is widely used for establishing all kinds of limit theorems. Hence, naturally, $\beta_t^{x,y}$ is also suitable for this goal. The stationary version of the coefficient β_t is widely known as Kolmogorov's coefficient, although for the first time it appeared in the joint work by his students Volkonskii and Rosanov. In his lectures in 1970s, Kolmogorov posed general problems of studying mixing coefficients for general processes. The non-stationary version of beta-coefficient for Markov processes (3) was investigated, in particular, in a series of papers by the second author. Apart from interest for engineers, there are some mathematical issues that make this system special. In terms of recurrence properties, we apply Lyapunov's approach, using the same Lyapunov function as in [2], based on simple quadratic forms. Apparently, the use of

*University of Leeds, UK, e-mail: niloufar @ leeds.ac.uk

†University of Leeds, UK, and Institute of Information Transmission Problems, Moscow, Russia, e-mail: A.Veretennikov @ leeds.ac.uk

such simple functions is limited to a relatively narrow class of processes. However, for the equation (1–2) they are quite sufficient, and possibly could serve certain even wider classes of diffusions. Nevertheless, for more general systems (1) possibly some other Lyapunov functions could be useful. In terms of local mixing properties, a real obstacle is a genuine high degeneracy of the SDE system. It is comparatively not very difficult to verify a local version of the so called general Doeblin–Doob condition (see [5]), however, this kind of condition even in its global form provides only a rather reduced result about mixing (formally, about convergence in total variation) just for one particular Markov process, not for a class of processes. Clearly, for the system (1) there is no global Doeblin–Doob condition available, and the question how to work with its local version is yet open. There is one more, most standard tool frequently used in similar situations, which does provide bounds uniform on some class of processes, "pétite set" condition. However, it apparently fails here completely in an even more severe fashion in compare to a non-degenerate diffusion, where it is not of any real help either. We tackle this problem by establishing some appropriate local version of *Dobrushin's ergodicity condition,* see (16) below. Notice that this kind of condition is also rather useful in the non-degenerated case, where it is provided by Harnack's inequalities, see [15], [14]. After having a good Lyapunov function and verifying a local Dobrushin condition, the remaining part of the proof is based on the method of estimating the upper bounds for mixing rate from [14]. To work with the system (1), we need, at least, weak uniqueness. The latter may be established by using Girsanov's transformation with the help of a method similar to [1]. Under (2) this approach was suggested in [2] with a reference to [1]. Nevertheless, we should notice that apparently the direct reference does not work, neither applied to (1), nor under the restriction (2), since the paper [1] does not consider degenerate SDEs, nor does the presentation of its results in [9]. The extension of this result to our degenerate case being done in the Section 3, the authors realised that the same Lyapunov function method as for the system (1)–(2) may also work for slightly more general systems of SDEs, which satisfy (1) & (**??**). The verification of a local Dobrushin type condition below is also based on Girsanov's transformation, although it is not a direct corollary from the section about weak solutions.

In the Section 2 we formulate our main results along with the assumptions. An extension of the approach from [1] is provided in the Section 3. In particular, in that Section we briefly discuss weak existence of solution of our system (1) with a strong Markov property. The calculus which we suggest, of course, resembles the one in [1], – and even more the one in [9], – and may be considered as a complement to the latter. It also simultaneously provides Novikov's condition (see [13], [12]) for this particular case, although this observation does not lead to any further simplification, being just another view on the problem. In the Section 4 a Lyapunov function is presented for this system, together with some hitting times inequalities. In the Section 5 a "local Dobrushin's condition" is established. To the best of the authors' knowledge, this is the most general local condition which guarantees "computable" mixing and convergence in total variation bounds, with practically the best constants. Whether this condition could be further relaxed, is an open question.

The proof of convergence and mixing rate is given in the last Section 6.

2 Main results

Assumptions for (1) & (2)

(A1) The function b in (1) is Borel measurable, and there exists C such that $|b(x,y)| \le C(1+|x|+|y|)$.

(A2) The function u in (2) is Borel measurable, and there exist constants $0 < u_1 \le u_2 < \infty$ such that $u_1 \le u \le u_2$; β and γ are strictly positive constants.

In the sequel, $\mu_t^{x,y}$ denotes the marginal distribution of (X_t, Y_t), the couple with the initial state (x,y), and μ_∞ stands for its (unique) invariant distribution if the latter exists.

Theorem 1 *Let the system (1) satisfy (A1). Then the following holds true.*

1. *The equation (1) has a (weak) solution unique in distribution, which is a strong Markov process.*
2. *If additionally the drift satisfies (2) as well as (A2), then there exists a unique probability distribution μ_∞ and there exist $C, c > 0$ such that*

$$\|\mu_t - \mu_\infty\|_{TV} \le C\exp(-ct)(1+x^2+y^2), \tag{4}$$

and also

$$\beta_t^{x,y} \le C\exp(-ct)(1+x^2+y^2). \tag{5}$$

Now as we accomplished the results from [2] et al. by rate of convergence, let us describe what more general SDE systems could be tackled in a similar way. Consider the class of drifts f satisfying the following conditions. Firstly, we require (A1), which turns out to be sufficient both for the local Dobrushin's condition and weak existence and uniqueness. To tackle recurrence, we require the following.

Assumptions for (1) without (2)

(A3) The function b in (1) satisfies $b(x,y) = b_0(x,y) - u(x,y)y - v(x,y)x$. where the term b_0 is a Borel bounded function, u satisfies the inequalities from (A2) with some $0 < u_1 \le u_2 < \infty$, and the function v *is bounded* and satisfies $\lim_{(x,y)\to\infty} v(x,y) = \beta > 0$.

As we shall see below, the latter condition can be relaxed so as to allow for some $\beta > 0$ a "small enough" limit $\limsup_{(x,y)\to\infty} |v(x,y) - \beta| << 1$. For the precise formulation as to in which sense the left hand side is $<< 1$ see the calculus in the Lemma 4 and remark 3 below. Now we have a version of Theorem 1 as follows.

Theorem 2 *Let the system (1) satisfy (A1) and (A3). Then again there exists a unique invariant probability distribution μ_∞ and constants $C, c > 0$ such that (4) and (5) hold true.*

3 Weak solution & Girsanov's transformation

First of all let us show that there exists a weak solution of the system (1), and that it possesses a weak uniqueness property. Emphasize that neither (2) nor (A3) is assumed in this section. Basically, there are two methods available: one based on approximations; and another based on Girsanov's transformations. In the general case, if we want to use approximations and weak convergence, then we do have a good *a priori* bound, – e.g., for the second moment, – but the function u may be discontinuous, in particular, in variable x, while the component X has no diffusion term at all. This is an obstacle while using approximations and passing to a limiting measure. So, we will work with Grisanov's transformations. We start with a couple $(X, \tilde W)$ on *some* probability space $(\Omega, \mathcal{F}, \tilde P)$, where $\tilde W$ is a Wiener process, and $X_t = x + \int_0^t \tilde W_s\, ds$. In the other words, the process $(X, \tilde W)$ solves the system (1) in the trivial case $b \equiv 0$. We will use Girsanov's exponential to solve a general case. Let $\tilde\rho_T := \exp\left(\int_0^T \left(b(X_t, y + \tilde W_t)\, d\tilde W_t \ -\frac{1}{2}\int_0^T \left| b(X_t, y + \tilde W_t) \right|^2 \right) dt$. We ought to show that this is a probability density, i.e., that $\tilde E \tilde\rho_T = 1$.

Lemma 1 *Under the assumption (A1), there exists $T > 0$ small enough, such that for every $R > 0$,*

$$\sup_{(x,y)\in B_R} \tilde E_{x,y} \tilde\rho_T^2 < \infty. \tag{6}$$

Moreover, for every $(x,y) \in B_R$ and every $T > 0$ (not only small),

$$\tilde E_{x,y} \tilde\rho_T = 1. \tag{7}$$

Emphasize that the value of the left hand side in (6), of course, may depend on R, however, the value T may be chosen unique for all $R > 0$.

Proof. Notice that the assertion (6) guarantees uniform integrability of $\tilde\rho_T$ with respect to the measure $\tilde P$, for every $(x,y) \in B_R$, which implies (7) for small values of T. However, the latter equality is extended on any T by simple induction based on Markov property (remind that *small* T in (6) does not depend on initial data), see [1] or [9, Corollary 3.5.14]. Hence, it suffices to prove only (6). We estimate, using Cauchy–Bouniakovsky–Schwarz' inequality (known widely as Cauchy–Schwarz' or Cauchy's),

$$\left(\tilde E_{x,y} \tilde\rho_T^2 \right)^2 \le C(T, R, x, y)\, \tilde E \exp\left(C\,(T^2 + T^4) \sup_{0\le t\le 1} |\tilde W_t|^2 \right).$$

it is, indeed, easy to see that with any constant β, the latter expectation is finite if $T > 0$ is chosen small enough. The Lemma 1 is proved.

Theorem 3 *Under the assumption (A1), there exists a weak solution of the system (1) on $[0, \infty)$ which is unique in distribution. Any solution (X, Y) on any probability space with a Wiener process W is a strong Markov process. Also, for any $T > 0$,*

$$E\rho_T = 1, \textit{ where } \quad \rho_T := \exp\left(-\int_0^T (b(X_t, Y_t)\, dW_t \ -\frac{1}{2}\int_0^T |b(X_t, Y_t)|^2 \ dt \right).$$

Lemma 2 *Under the assumption (A1), there exists $T > 0$ small enough, such that for every $R > 0$,*

$$\sup_{(x,y)\in B_R} E^{\rho}_{x,y}\rho_T < \infty. \tag{8}$$

Proof. Notice that since $E^{\rho}_{x,y}\rho_T = E\rho^2_T$, the assertion (8) guarantees uniform integrability of ρ_T with respect to the measure P, for every $(x, y) \in B_R$, which, by the way, again implies theorem 3, at least, for $T > 0$ small enough. The inequality (8) can be rewritten as

$$\sup_{(x,y)\in B_R} E^{\rho}_{x,y}\rho_T = \sup_{(x,y)\in B_R} \tilde{E}_{x,y}(\tilde{\rho}_T)^{-1} < \infty.$$

In this form, it follows from the calculus quite similar to that in the proof of the Lemma 1. The Lemma 2 is proved.

Remark . The result from [1] about Girsanov's transformation relates to the following SDE in R^d with a d-dimensional Wiener process (we use another notation Z_t for the process, to distinguish it from the setting (1)),

$$dZ_t = b(t, Z_t)\, dt + dW_t, \qquad Z_0 = z. \tag{9}$$

In this Remark, drift b is a d-dimensional Borel measurable vector–function, and it satisfies a linear growth condition with some constant $L > 0$,

$$|b(t, z)| \le L\,(1 + |z|), \qquad \forall\, z \in R^d. \tag{10}$$

The following Theorem is a reformulation of some combination of Lemma 0 and Theorem 1 and a discussion around them from [1], and the Lemma 7 from [7]. However, it is easier for us to cite a later presentation from [9, Corollary 3.5.16 & Proposition 5.3.6]. As usual (e.g., as above in the Lemma 1), to solve (9), we consider a probability space $(\Omega, \mathcal{F}, \tilde{P})$ with a (another) Wiener process $\tilde{W}_t$, $t \ge 0$.

Theorem 4 *[Benes 1971] Under (10), for any T,*

$$\tilde{E}\zeta_T = 1, \quad \zeta_T := \exp(-\int_0^T b(s, \tilde{W}_s)\, d\tilde{W}_s - \frac{1}{2}\int_0^T |b(s, \tilde{W}_s)|^2\, ds),$$

the process $W_t := \tilde{W}_t - \int_0^t b(s, \tilde{W}_s)\, ds$, $0 \le t \le T$, is d-dimensional Wiener under the new measure $dP \equiv d\tilde{P}^{\zeta} := \zeta_T\, d\tilde{P}$, and, hence, the equation (9) has a weak solution unique in the sense of distribution.

4 Lyapunov functions and hitting time bounds

Lemma 3 *Let (A1)–(A2) be satisfied. Then for the system (1–2) there exists a constant C such that*

$$\sup_{t\ge 0} E(|X_t|^2 + |Y_t|^2) \le C(1 + x^2 + y^2). \tag{11}$$

Of course, the constant C depends on the initial data (x, y). *Proof* follows from [2], with the Lyapunov function suggested there,$f(x, y) = \beta x^2 + \epsilon xy + y^2$, with $\epsilon > 0$ small enough.

Lemma 4 *Let (A1) and (A3) be satisfied. Then for the system (1–(A3)) there exists a constant C such that(11)holds.*

Proof. We will use the same Lyapunov function as in (11), (with just a notation β changed to β_0), where ϵ is to be chosen. The calculus is similar to the one in the previous Lemma. We apply Itô's formula to $f(X_t, Y_t)$:

$$\begin{aligned} &df(X_t, Y_t) = 2\beta X_t\, dX_t + 2Y_t\, dY_t + (dY_t)^2 + \epsilon X_t\, dY_t + \epsilon Y_t\, dX_t \\ &\le 2Y_t\, dW_t + 2(Y_t + \epsilon X_t)b_0(X_t, Y_t)\, dt - ((u_1 - \epsilon)Y_t^2 + v(X_t, Y_t)\epsilon X_t^2 + (2v(X_t, Y_t) - 2\beta + \epsilon u(X_t, Y_t))X_tY_t)\, dt. \end{aligned}$$

Here the inequality sign, of course, relates to the dt terms, while the term dW_t remains the same. Clearly, to establish the Lyapunov condition, the terms of the first order are not important if $|(X_t, Y_t)| > R$ and if R is chosen large enough. Next, since the difference $2v(X_t, Y_t) - 2\beta$ is *small enough* by modulus for $|(X_t, Y_t)| > R$ due to (7), clearly we can choose $\epsilon > 0$ small enough, so that the expression

$$\ell_t := ((u_1 - \epsilon)Y_t^2 + v(X_t, Y_t)\epsilon X_t^2 + (2v(X_t, Y_t) - 2\beta + \epsilon u(X_t, Y_t))X_tY_t) \tag{12}$$

is no less than some positive definite quadratic form, say, $\frac{u_1}{2}Y_t^2 + c\epsilon X_t^2$ $(c>0)$, if $|(X_t,Y_t)|>R$.
Also, of course, $\frac{u_1}{2}Y_t^2 + c\epsilon X_t^2 \geq C^{-1}f(X_t,Y_t) \qquad (C>0)$.
On the other hand, if $|(X_t,Y_t)|\leq R$, then the whole expression in (12) is bounded. Hence, with the same notation $g(t):=Ef(X_t,Y_t)$ as above, – and with a notice that $\sup_{s\leq t} Ef(X_s,Y_s)<\infty$ for any $t>0$, – taking expectations, we get,

$$g'(t) \leq -C^{-1}E\ell_t 1(|(X_t,Y_t)|>R) - E\ell_t\, 1(|(X_t,Y_t)|\leq R) \leq -E\ell_s - 2\,E\ell_s\, 1(|(X_s,Y_s)|\leq R) \leq -C^{-1}g(s)\,ds + 2C_0,$$

with $C_0 = \sup_{|(x,y)|\leq R}|\ell(x,y)|$, and $\ell(x,y) = ((u_1-\epsilon)y^2 + v(x,y)\epsilon x^2 + (2v(x,y) - 2\beta + \epsilon u(x,y))xy)$.
This shows that

$$\frac{d}{dt}g(t) \leq -Cg(t) + C, \qquad \text{and hence,} \qquad (0\leq)\quad g(t)\leq C(1+\exp(-Ct)). \tag{13}$$

The Lemma 4 is proved.

Lemma 5 *Let (A1)–(A2) be satisfied, and R be large enough. Then for the system (1–2) there exist $C,\alpha>0$ such that*

$$E_{x,y}\exp(\alpha\tau) \leq C(1+f(x,y)), \tag{14}$$

Lemma 6 *Let (A1) and (A3) be satisfied, and R be large enough. Then for the system (1) & (A3) there exist $C,\alpha>0$ such that,(14)holds.*

The proofs of both Lemmas 5 and 6 follow easily from the standing inequality above (13), similarly to the calculus in [15] or [14].
We will need a similar technical inequality for a process in a double–dimension state space. Namely, we consider another independent copy $(\bar X_t, \bar Y_t,\, t\geq 0)$ of the process $(X_t, Y_t,\, t\geq 0)$, possibly with another initial condition. Let $Z_t = (X_t, Y_t)$, $\bar Z_t = (\bar X_T, \bar Y_t)$.

Lemma 7 *Let (A1)–(A2) be satisfied, and R be large enough. Then for the system (1–2) there exist $C,\alpha>0$ and γ, defined as , $\gamma := \inf(t\geq 0:\ |Z_t|\vee|\bar Z_t|\leq R)$, such that*

$$E_{z,z'}\exp(\alpha\gamma) \leq C\,(1+f(z)+f(z')), \tag{15}$$

Lemma 8 *Let (A1) and (A3) be satisfied, and R be large enough. Then for the system (1) & (A3) there exist $C,\alpha>0$ such that (15)hold.*

The proofs of the Lemmas 7 and 8 follow similarly from the Lyapunov inequality above(13) or cf. [15] or [14].

5 Dobrushin's local mixing condition

The next result is the second part of the method used in this paper and our main contribution to the technique of verification of mixing rate here. We consider any solution to the equation (1), without restrictions (2).

Lemma 9 *Let (A1) be satisfied. Then for any $R>0$ there exists $c>0$ such that*

$$\inf_{(x_0,y_0),(x_1,y_1)\in B_R}\int_{B_R}\left(\frac{\mu_{x_0,y_0}(dx\,dy)}{\mu_{x_1,y_1}(dx\,dy)}\wedge 1\right)\mu_{x_1,y_1}(dx\,dy) \geq c>0. \tag{16}$$

Proof. First of all, notice that $\dfrac{\mu_{x_0,y_0}(dx\,dy)}{dx\,dy} > 0$, a.s. Indeed, by virtue of Girsanov's transformation (cf., e.g., Theorem 3 above), under the measure P^ρ we have a representation,

$$\rho_T = \exp\left(-\int_0^T b(x_0+\int_0^t \tilde W_s\,ds, y+\tilde W_t)\,d\tilde W_t\ -\frac{1}{2}\int_0^T\left|b(x_0+\int_0^t\tilde W_s\,ds, y_0+\tilde W_t)\right|^2\,dt\right).$$

Denote $\mu^\rho_{x_0,y_0}(dx\,dy) := E^\rho_{x_0,y_0}1(X_T\in dx, Y_T\in dy)$. We have,

$$\frac{\mu_{x_0,y_0}(dx\,dy)}{dx\,dy} = \frac{\mu^\rho_{x_0,y_0}(dx\,dy)}{dx\,dy}E_{x_0,y_0}(\rho^{-1}\mid X_T=x, Y_T=y),$$

where both multiples $\dfrac{\mu^{\rho}_{x_0,y_0}(dx\,dy)}{dx\,dy}$ and $E_{x_0,y_0}(\rho^{-1} \mid X_T = x, Y_T = y)$ are positive (a.s. for the second one). For the second term this is because $0 < \rho^{-1} < \infty$ a.s. For the first one there is an explicit representation of this density, see (17) below. So, (16) can be rewritten *equivalently* as

$$\inf_{(x_0,y_0),(x_1,y_1)\in B_R} \int\limits_{B_R} \left(\frac{\mu_{x_0,y_0}(dx\,dy)}{dx\,dy} \wedge \frac{\mu_{x_1,y_1}(dx\,dy)}{dx\,dy} \right) dx\,dy \geq c > 0.$$

Let $L > 0$ and consider the densities,

$$\frac{\mu_{x_0,y_0}(dx\,dy)}{dx\,dy} := \frac{E_{x_0,y_0} 1(X_T \in dx,\, Y_T \in dy)}{dx\,dy}, \quad \frac{\mu^L_{x_0,y_0}(dx\,dy)}{dx\,dy} := \frac{E_{x_0,y_0} 1(X_T \in dx,\, Y_T \in dy)\, 1(\rho_T > L)}{dx\,dy}.$$

Clearly, the measure $\mu^L_{x,y}(dx\,dy)$ is absolutely continuous with respect to the Lebesgue measure $dxdy$, similarly to $\mu_{x,y}(dx\,dy)$. Moreover, ρ_T is a probability density (see the Proposition 3). So, we can use the following representations,

$$\frac{\mu_{x_0,y_0}(dx\,dy)}{dx\,dy} = \frac{E^{\rho}_{x_0,y_0} \rho^{-1} 1(X_T \in dx,\, Y_T \in dy)}{dx\,dy}, \; \frac{\mu^L_{x_0,y_0}(dx\,dy)}{dx\,dy} = \frac{E^{\rho}_{x_0,y_0} \rho^{-1} 1(X_T \in dx,\, Y_T \in dy)\, 1(\rho_T > L)}{dx\,dy}.$$

We estimate,

$$\frac{\mu_{x_0,y_0}(dx\,dy)}{dx\,dy} \geq L^{-1} \frac{E^{\rho}_{x_0,y_0} 1(X_T \in dx,\, Y_T \in dy)}{dx\,dy} - L^{-1} \frac{E^{\rho}_{x_0,y_0} 1(X_T \in dx,\, Y_T \in dy)\, 1(\rho_T > L)}{dx\,dy}\,.$$

Since here ρ is a probability density on Ω, the first term up to the multiple L^{-1} is a positive Gaussian density on R^2 under the probability measure P^{ρ}. In the other words,

$$\frac{E^{\rho}_{x_0,y_0} 1(X_T \in dx,\, Y_T \in dy)}{dx\,dy} =: p^{\rho}_{x_0,y_0}(x,y;T) = \frac{\sqrt{12}}{2\pi T^2} \exp\left(-\frac{1}{2} (x-x, y-y)(C_T^{-1})(x-x, y-y)^* \right). \quad (17)$$

In particular, the density p^{ρ}_T is uniformly bounded by the value $\sqrt{3}/T^2$. Next, with any L, the second term is also a (sub-probability) density, $p^L_{x_0,y_0}(x,y;T)$, which is dominated by p. Let us choose the constant L so large that

$$L^{-1} \sup p^{\rho}_T \leq 1.$$

Then, the lower bound for our density does not exceed one, so that the operation "minimum with one" disappears under the integral, and we may estimate,

$$\inf_{(x_0,y_0),(x_1,y_1)\in B_R} \int_{B_R} \left(\frac{\mu_{x_0,y_0}(dx\,dy)}{\mu_{x_1,y_1}(dx\,dy)} \wedge 1 \right) \mu_{x_1,y_1}(dx\,dy)$$
$$\geq L^{-1} \left(\inf_{(x,y),(x',y')\in B_R} p_{x,y}(x',y';T)\, |B_R| - 2 \sup_{(x,y)\in B_R} P^{\rho}_{x,y}(\rho_T > L) \right).$$

Here, clearly, $\inf\limits_{(x,y),(x',y')\in B_R} p_{x,y}(x',y';T)\, |B_R| = \pi R^2 \inf\limits_{(x,y),(x',y')\in B_R} p_{x,y}(x',y';T) > 0$. and this value does not depend on L. The second term admits the following bound due to Bienaimé–Chebyshev,

$$\sup_{(x_0,y_0)\in B_R} P^{\rho}_{x_0,y_0}(\rho_T(x,y) \geq L) \leq L^{-1} \sup_{(x_0,y_)\in B_R} E^{\rho}_{x_0,y_0} \rho_T(x,y).$$

Hence, in order to complete the proof of the Lemma, it *suffices* to show that

$$E\rho_T = 1, \quad \& \quad \sup_{(x_0,y_0)\in B_R} E^{\rho}_{x_0,y_0} \rho_T < \infty, \quad (18)$$

at least, for $T > 0$ small enough. Both inequalities in (18) have been established in the Lemma 2 above. The Lemma 9 is proved.

6 Proof of Theorems 1 and 2

Proof. The plan for the proof is to use the Lemmas 7 and 9 and the calculus from [14], with a natural replacement of polynomial inequalities by exponential ones. Both Theorems require the same calculus.

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A Multidimensional Comparison Theorem for Solutions of the Skorokhod Problem in a Wedge with Applications to Control of a Group of Identical Particles

Svetlana ANULOVA (*Institute for Control Science Moscow Russia)*

Pathwise comparison theorems for solutions of SDE remain a fresh subject, see [1], where 1-dimensional reflected processes are studied. We consider comparison with respect to a partial ordering of the Euclidean space.

Let K (resp. C) be a convex cone of full dimension in $\mathbb{R}^d$ with the vertex in 0 and a finite number of faces. Denote $\pi : \mathbb{R}^d \to K$ the orthogonal projection on K and Φ the Skorokhod operator for K (normal reflection on the boundary). Define an order $\preceq$: for $x^1, x^2 \in \mathbb{R}^d$ $x^1 \preceq x^2$ if $x^2 - x^1 \in C$, and for $y^1, y^2 : [0,\infty) \to \mathbb{R}^d$ $y^1 \preceq y^2$ means: $y^1(0) \preceq y^2(0)$ and $y^2 - y^1$ has a locally bounded variation such that

$$\frac{d(y^2 - y^1)}{d\,\mathrm{var}(y^2 - y^1)}(t) \in C \text{ a.e. for } t \in [0,\infty).$$

Theorem 1. *If π is monotonic, then Φ is monotonic (both with respect to $\preceq$).*

Example 1. For $d = 1$ and $K = C = [0,\infty)$ we have for any y^1, y^2 with $y^2 - y^1 \geq 0$ and nondecreasing: $\Phi(y^2) \geq \Phi(y^1)$.

As corollary we obtain the solution to the following control problem (cf. [2]). Ikeda and Watanabe were the first to use comparison theorems in control theory, see §2 ch.VI [3].

Let $(\Omega, \mathcal{F}, F, \mathbf{P})$ be a standard stochastic basis, $W : [0,\infty) \to \mathbb{R}^d$ a continuous gaussian F-martingale with independent identically distributed coordinates, $u = \{u(t) \in [0,1]^d,\ \sum_1^d u_i(t) \leq 1,\ t \in [0,\infty)\}$ a control policy (cf. [3]), $U = \{u\}$. Define $u^* : \mathbb{R}^d \to \mathbb{R}^d$:

$$u_i^*(x) = \begin{cases} 1, & \text{if } i = \min\{\arg\min\{x_j, j = 1,\dots,d\}\}; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2. *For any $y \in \mathbb{R}^d$ and $u \in U$ there exists an extension of the original stochastic basis $(\Omega', \mathcal{F}', F', \mathbf{P}')$ and a continuous gaussian (F')-martingale W', $\langle W' \rangle = \langle W \rangle$ on it such that for*

$$y^u(t) = y + \int_0^t u(s)ds + W(t),\ t \in [0,\infty),$$

and

$$y^*(t) = y + \int_0^t u^*(y^*(s))ds + \tilde{W}(t),\ t \in [0,\infty),$$

holds $\min_i y_i^*(t) \geq \min_i y_i^u(t)$ provided y^u exists.

The objects we have introduced describe a group of independent identical particles $W_i, i = 1, \ldots, d$, with controlled drifts. The theorem produces the optimal policy to hold the group as high as possible.

Proof of Theorem 1. Approximate Φ with its discretization in time with step δ Φ_δ. It is monotonic and

$$\lim_{\delta \to 0} \Phi_\delta = \Phi.$$

Proof of Theorem 2 for $d = 2$. The two coordinates of the controlled process are identic, thus we reduce the problem to the control of the pair

$$\bar{y}^u(t) = (\min\{y_1^u(t), y_2^u(t)\},\ \max\{y_1^u(t), y_2^u(t)\},\ t \geq 0).$$

The Ito formule shows, that this process is $\Phi(\bar{y}^u)$ for $K = \{x \in \mathbb{R}^2 : x_1 \leq x_2\}$ and

$$\bar{y}^u(t) = y + \int_0^t \bar{u}(s)ds + \bar{W}(t),\ t \in [0, \infty),$$

where $\bar{W}$ (resp. $\bar{u}$) is a new F-martingale (resp. drift) obtained from W (resp. u). Obviously, $\frac{d(\bar{y}^u - \bar{y}^*)}{dt} \in M$, where M is a cone,

$$M = \{\lambda_1(-1, 0)^T + \lambda_2(-1, 1)^T, \lambda_1, \lambda_2 \geq 0\}.$$

Unfortunately, Φ is not monotonic with respect to the partial order generated by M:

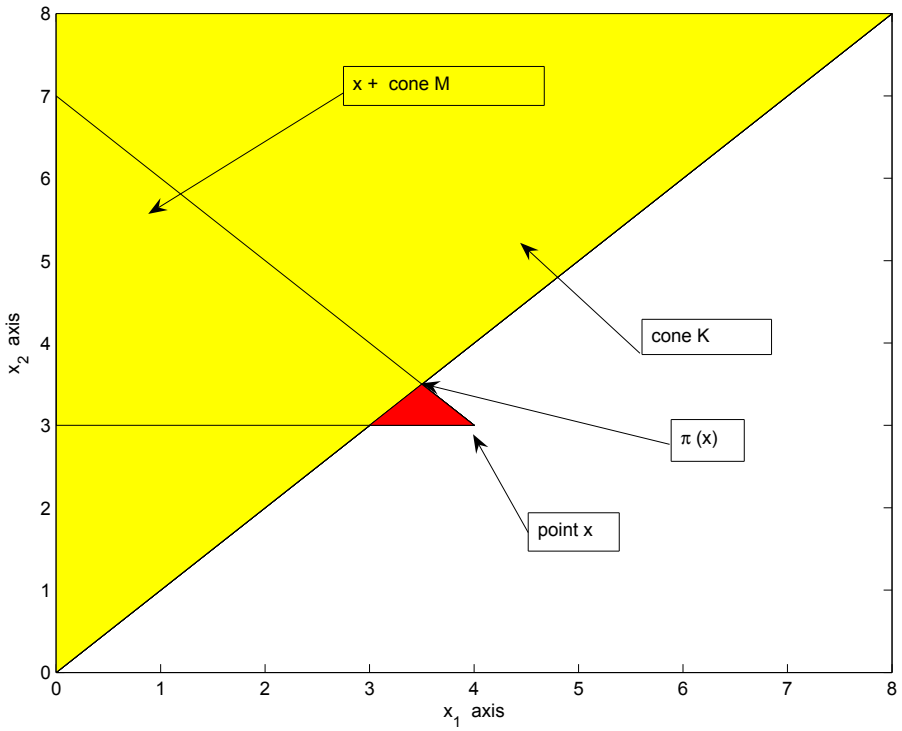

FIGURE 1. $\pi(x + M) \not\subseteq \pi(x) + M$

But if we take another cone

$$C = \{\lambda_1(-1,-1)^T + \lambda_2(-1,1)^T, \lambda_1, \lambda_2 \geq 0\},$$

the assumptions of Theorem 1 are satisfied. Thus

$$\Phi(\bar{y}^*) \preceq \Phi(\bar{y}^u),$$

in particular, $\Phi(\bar{y}^*)_1 \geq \Phi(\bar{y}^u)_1$, what means the assertion of the theorem for the process in K. In order to return in the plane we have to extend $(\Omega, \mathcal{F}, F, \mathbf{P})$ and construct a new martingale $W^{'}$ by randomizing $\bar{W}$.

Proof of Theorem 1 for $d > 2$. Define a mapping $f : \mathbb{R}^d \to \mathbb{R}^d$:

$$\{f_i(x), i = 1, \ldots, d\} = \{x_i, i = 1, \ldots, d\}$$

and $f_1(x) \leq f_2(x) \leq \ldots f_d(x)$. Then define a new F-martingale $\bar{W}$:

$$d\bar{W}(t) = \frac{\partial f}{\partial x}(y^u(t))dW(t)$$

and solve the equation for $f(y^*)$ with $\bar{W}$. It will be an equation in

$$K = \{x \in \mathbb{R}^d : x_1 \leq x_2, \ldots \leq x_d\}$$

for a martingale with drift and normal reflection on the boundary. Now apply Theorem 1 with a cone C, which is a rotated orthant. The first edge of this orthant is $(-1, 1, 0, \ldots, 0)^T$, the $d-$th is $(-1, \ldots, -1)^T$. At this stage we have, roughly speaking, a process $f(y^*)$. In order to "extract" y^* we have to extend $(\Omega, \mathcal{F}, F, \mathbf{P})$ and construct a new martingale $W^{'}$ by randomizing $\bar{W}$. That is, we take a product of the original stochastic basis and a canonical space of $d-$dimensional continuous functions and endow it with conditional probability equal to the distribution of y^* given $f(y^*)$ and independent from all the rest.

Remark 1. The theorem implies: for $T \in [0, \infty)$ and real functions G_1, G_2 monotonic in the second argument with respect to $\preceq$

$$\int_0^T G_1(t, W^u(t))dt + G_2(T, W^u(T))$$

is maximal for the policy u^* in probability.

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IPU RAN Moscow, Profsoyuznaya, 65

e-mail: anulovas@ipu.rssi.ru

Landau-Zener phenomenon in 2D lattices via Dirichlet-to-Neumann map

N. Bagraev [1], G. Martin[2], B. Pavlov[2,3]

[1] A.F. Ioffe Physico-Technical Institute, St. Petersburg, Russia.

[2] New Zealand Institute of Advanced Study, Massey University, New Zealand.

[3] V.A. Fock Institute for Physics of St.-Petersburg University, Petrodvorets, Russia.

Transport properties of periodic lattices are defined by the structure of the corresponding Bloch eigenfunctions. In the 1d case the Bloch eigenfunctions are found based on the transfer matrix constructed of the solutions of the relevant Cauchy problem. This approach fails in 2d, and, generally in the multi-dimensional case, because the Cauchy problem for the multi-dimensional Schrödinger equation is ill-posed. The approach based on "tight binding" ideas (Liner Combination of Atomic Orbitals - LCAO , see [8]) gives a reasonably good qualitative coincidence with experiment, but stays on a shaky mathematical basement. We develop an alternative approach to study of transport properties of quantum periodic lattices, based on Dirichlet-to-Neumann map, and suggest, on the base of corresponding 2d Landau-Zener effect, an interpretation of high-mobility of the charge carriers in bi-layer 2d periodic structures.

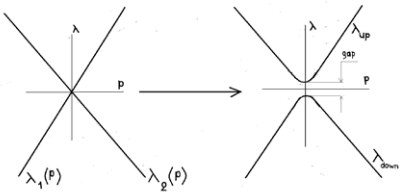

Figure 1: One dimensional Landau-Zener effect.

Landau-Zener effect is the transformation of the intersection of terms $\lambda_1(p), \lambda_2(p)$, see Fig. (1) into quasi-intersection. It was observed first, see [1], in one-dimensional lattices, with use of the transfer-matrix as a main spectral tool for study of corresponding space or time- periodic structures, see [2]. It was noticed that the interaction of terms $\lambda_s(p)$ in solid-state quantum problems implies pseudo-relativistic properties of the corresponding particles / quasi-particles. Fresh interest for quasi-relativism in solid state physics arose in connection with discovery of high mobility of charge carriers in graphen, see for instance [3]. Recent discovery of quasi-relativistic behavior of terms in man-made bi-layer periodic quasi-2d lattices, see [4], allows to conjecture that the weak interaction of 2d periodic lattices may be used as a source of various artificial structures with useful transport properties. Study of the Landau-Zener transformation of 2d terms requires an adequate analytic machinery. In [12] the Dirichlet-to-Neumann map was selected as an appropriate tool to substitute the transfer-matrix in analysis of perturbations of the two-dimensional terms. The standard DN-map is a linear transformation of the boundary "potential" $\psi\Big|_\Gamma$, $\Gamma \subset \partial\Omega$ into the "boundary current" $\frac{\partial\psi}{\partial n}\Big|_\Gamma$ of the solution $\psi$of the homogeneous Schrödinger equation on the domain Ω, with scaled spectral variable $\lambda = \frac{2mE}{\hbar^2}$.

$$-\Delta\psi + V\psi = \lambda\psi,\ \psi\ \ \mathcal{DN}(\lambda) : \psi\Big|_\Gamma \longrightarrow \frac{\partial\psi}{\partial n}\Big|_\Gamma.$$

It our talk we consider a modified version of DN-map, restricted by an orthogonal projection P_+ onto a contact subspace E_+ of $L_2(\Gamma)$. We ignore the spin of electron and initially assume that the one-electron wave functions on the neighboring romboidal periods, see Fig. 2, communicate with each other via relatively narrow connecting channels, which filter the evanescent waves off, see an extended analysis of the filtering in [5]. We simplify the spectral problem via replacement of the matching condition on Γ in closed channels by the partial zero boundary condition

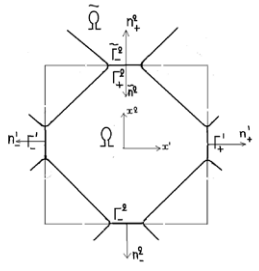

Figure 2: A detail of a square lattice with romboidal periods. The connecting leads are not shown.

on the slots Γ :

$$P_+ \left[\psi_\Omega - \psi_{\Omega'}\right]\Big|_{\Gamma_{\Omega,\Omega'}} = 0,$$

$$P_+ \left[\frac{\partial\psi_\Omega}{\partial n} + \frac{\partial\psi_\Omega}{\partial n'}\right]\Big|_{\Gamma_{\Omega,\Omega'}} = 0, \qquad (1)$$

$$P_-\psi\Big|_\Gamma = 0, \text{ with } P_- = \sum_{m>l,\Gamma_s} e^m\rangle\langle e^m \qquad (2)$$

We consider the Schrödinger operator $-\Delta * +V(x)* =: L$ on the periodic 2d lattice with periods connected by the open channels only, with normals $n' = -n$. The Schrödinger operator with above boundary conditions is self-adjoint and can be analyzed based on quasi-periodic problem on a period, with the partial matching boundary condition substituted by the quasi-periodicity on the pairs of opposite slots $\Gamma^s_\pm = \{x^s = \pm 1\}$ of the period Ω:

$$P_+\left[\psi_\Omega\Big|_{\Gamma^s_-}\right] = e^{-2ip_s a}P_+\left[\psi_\Omega\Big|_{\Gamma^s_+}\right],$$

$$P_+\left[\frac{\partial\psi_\Omega}{\partial n}\Big|_{\Gamma^s_-}\right] = -e^{-2ip_s a}P_+\left[\frac{\partial\psi_\Omega}{\partial n}\Big|_{\Gamma^s_+}\right], \quad (3)$$

where the differentiation is done with respect to the outward normals on the boundary of the relevant periods, see Fig. (1). Hereafter we assume that the width $2a = 2$ of the period is equal to 2, $\delta/2 << 1$, and the entrance subspace E^s_+ of the open channel attached to each slot $\Gamma^s_\pm$ is one-dimensional and spanned by $\frac{2}{\delta}\sin\frac{\pi y}{\delta} = e^s$, $P^s_+ = e^s\rangle\langle e^s$ on each slot. The electrons with the boundary data on $\Gamma^s = \partial\Omega \cap \partial\Omega'$ from E^s belong to both periods and form a covalent bond between the blocks Ω, Ω', see [13]. We use the relative intermediate DN-map $\mathcal{DN}_\Gamma$ associated with spectral/boundary problems with nonzero data on the slots. Assuming that the neighboring periods are connected by the leads of certain width δ, denote by $E^s_\pm$ the entrance subspaces of cross-section eigenfunctions of the open and closed channels respectively, and by $P^s_\pm$ the corresponding projections. Correspondingly the role of the conductivity band is played by the first spectral band in the leads $\Delta_1 = [\pi^2\,\delta^{-2}, 4\pi^2\,\delta^{-2}]$, with Fermi-level sitting on it: $\Lambda \in \Delta_1$. We consider the *intermediate boundary problem* with the partial boundary data and introduce the corresponding DN-map, see [10],by formal setting the exponential in the closed channels as $K_- = \infty$ and correspondingly choosing the zero boundary conditions on the bottom sections of closed channels.

Then the corresponding partial DN-map $\mathcal{DN}^\Lambda$ is defined as a restriction of the standard DN-map onto the slots Γ with subsequent framing by the projections onto $E_+ = \sum_{s=1,\,sgn=\pm} E^s_\pm$:

$$\mathcal{DN}^\Lambda = P_+\mathcal{DN}P_+ = \sum_{s,t=1,2,\,sgn,sgn'=\pm} P^s_{sgn}\Big|_{\Gamma^s_{sgn}}\mathcal{DN}P^t_{sgn'}\Big|_{\Gamma^t_{sgn'}}.$$

The corresponding intermediate DN-map $\mathcal{DN}^\Lambda$ is defined by the matrix elements of the standard DN-map of the period in the decomposition of the cross-section space $E = E_+ + E_-$ into an orthogonal sum of the entrance subspaces of the open and closed channels. We characterize the period Ω on given spectral interval Δ_T by the rational expression

$$\mathcal{DN}^\Lambda(\lambda) = \sum_{r=1}^n \frac{Q_r}{\lambda - \lambda_r} + P_+KP_+,\ \lambda_r \in \Delta_T, \qquad (4)$$

where $\frac{Q_r}{\lambda-\lambda_r} =$

$$\sum_{s,sgn;t,sgn'} e^s_{sgn}\rangle\frac{\langle e^s_{sgn}, \frac{\partial\psi_r}{\partial n}\rangle\,\langle\frac{\partial\psi_r}{\partial n}, e^t_{sgn'}\rangle}{\lambda-\lambda_r}\langle e^t_{sgn'},$$

$$\sum_{s,t,sgn,sgn'} e^s_{sgn}\rangle\langle e^s_{sgn}Ke^t_{sgn'}\rangle\,\langle e^t_{sgn'} = P_+KP_+. \qquad (5)$$

Here λ_r are the eigenvalues of the Schrödinger operator on the essential spectral interval Δ_T and PKP - the restriction of the regular part of the DN-map onto the open channels of the leads. The term P_+KP_+ contains the contribution to the DN - map from the complementary spectral subspace.

The spectral structure of the Schrödinger operator on the 2d periodic lattice is established based on study of the quasi-periodic spectral problem on the period Ω, which is represented via comparison of the projections of the boundary values and the boundary currents of the solutions of the Schrödinger equation $L\psi = \lambda\psi$ on the opposite slots:

$$P_+\begin{pmatrix}\psi^1_-\\ \psi^1_+\\ \psi^2_-\\ \psi^2_+\end{pmatrix} = P_+\begin{pmatrix}e^{-2ip_1}\psi^1_+\\ \psi^1_+\\ e^{-2ip_2}\psi^2_+\\ \psi^2_+\end{pmatrix} = \psi^1_+\,\nu^1 + \psi^2_+\,\nu^2,$$

$$P_+\begin{pmatrix}\psi'^1_-\\ \psi'^1_+\\ \psi'^2_-\\ \psi'^2_+\end{pmatrix}) = P_+\begin{pmatrix}-e^{-2ip_1}\psi'^1_+\\ \psi'^1_+\\ -e^{-2ip_2}\psi'^2_+\\ \psi'^2_+\end{pmatrix} = \psi'^1_+\,\mu^1 + \psi'^2_+\,\mu^2, \qquad (6)$$

where

$$\nu^1 = e^1 \begin{pmatrix} e^{-2ip_1} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \nu^2 = e^2 \begin{pmatrix} 0 \\ 0 \\ e^{-2ip_2} \\ 1 \end{pmatrix}, \mu^1 = e$$

and

$$\psi^s_+ = \langle \psi \Big|_{\Gamma^s_+}, e^s \rangle, \ \psi'^s_+ = \langle \frac{\partial \psi}{\partial n}\Big|_{\Gamma^s_+}, e^s \rangle.$$

Then the quasi-periodicity condition implies the equation:

$$\mathcal{DN}^\Lambda[\psi^1_+ \nu^1 + \psi^2_+ \nu^2] = \psi'^1_+ \mu^1 + \psi'^2_+ \mu^2, \quad (7)$$

with scalar coefficients ψ^s_+, ψ'^2_+. Notice that $\langle \nu^s, \mu^t \rangle = 0$, which implies

$$\langle \nu^1 \mathcal{DN} \nu^1 \rangle \psi^1_+ + \langle \nu^1 \mathcal{DN} \nu^2 \rangle \psi^2_+ = 0,$$
$$\langle \nu^2 \mathcal{DN} \nu^1 \rangle \psi^1_+ + \langle \nu^2 \mathcal{DN} \nu^2 \rangle \psi^2_+ = 0 \quad (8)$$

The condition of existence of the non-trivial Bloch function is represented in the determinant form:

$$\det \begin{pmatrix} \langle \nu^1, \mathcal{DN}_{11} \nu^1 \rangle & \langle \nu^1, \mathcal{DN}_{12} \nu^2 \rangle \\ \langle \nu^2, \mathcal{DN}_{21} \nu^1 \rangle & \langle \nu^2, \mathcal{DN}_{22} \nu^2 \rangle \end{pmatrix} = 0, \quad (9)$$

where $\langle \nu^s, \mathcal{DN}_{st} \nu^t \rangle =$

$$\sum_{r=1}^{n} \sum_{sgn, sgn'} \frac{\langle \nu^s \frac{\partial \psi_r}{\partial n} \rangle_{\Gamma^s_{sgn}} \langle \frac{\partial \psi_r}{\partial n}, \nu^t \rangle_{\Gamma^t_{sgn'}}}{\lambda - \lambda_r} + \langle \nu^s, K \nu^t \rangle. \quad (10)$$

We aim on the spectral analysis of a double periodic 2-d lattice with romboidal periods Ω^u, Ω^d playing the roles of basements of the upper and the lower cones of the two-storey joint period, see Fig.(2). Assume that first and the second storeys are connected by the link constructed in a form of a double cone with the slot Γ_0 dividing the upper and lower cones and a tunneling boundary condition on it defined by a real antisymmetric matrix $\mathbf{B}$: $P^\perp_0 \psi^u \Big|_{\Gamma_0} = 0, \ P^\perp_0 \psi^d \Big|_{\Gamma_0}$:

$$\begin{pmatrix} P_0 \frac{\partial \psi^u}{\partial n^u}\Big|_{\Gamma^u_0} \\ P_0 \frac{\partial \psi^d}{\partial n^d}\Big|_{\Gamma^d_0} \end{pmatrix} = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} P_0 \psi^u \Big|_{\Gamma^u_0} \\ P_0 \psi^d \Big|_{\Gamma^d_0} \end{pmatrix}. \quad (11)$$

with the outward normals $n^u = -n^d$,and an orthogonal 1d projection $P_0 = e^0 \rangle \langle e^0$ onto the open channels of the link. This tunneling boundary condition, with large β, emulates the potential barrier for the charge carriers, because implies $\psi^u \approx 0 \approx \psi^d$. If the slots of the upper and lower periods are equipped with the matching boundary conditions on the contact with the neighboring periods, then the Schrödinger operator on the whole lattice, with a real, bounded and piecewise continuous periodic potential is selfadjoint, and the corresponding dispersion equation can be derived from the Bloch condition on a single period, via comparison of the boundary values $\vec{\psi} = \left(\vec{\psi}^1_u, \vec{\psi}^2_u, \psi^u_0, \psi^d_0, \vec{\psi}^2_d, \vec{\psi}^1_d\right)$, of the wave-functions on the slots $\Gamma^s_u, \Gamma_0, \Gamma^t_d$ of the upper and lower periods and the balance of the corresponding boundary currents $\vec{\psi'} = \left(\vec{\psi'}^1_u, \vec{\psi'}^2_u, \psi'^u_0, \psi'^d_0, \vec{\psi'}^2_d, \vec{\psi'}^1_d\right)$ with the tunneling boundary condition. Indeed, the intermediate DN-map is obtained via framing of the standard DN-map by projections P^s_+, P^0_+ onto the open channels of the slots $\Gamma^s_{u,d}$ and ones of the link Γ^0. Imposing the quasi-periodic boundary conditions on the slots $\Gamma^s_{sgn}(u)$, $\Gamma^s_{sgn}(d)$ and the tunneling boundary conditions on Γ^u_0, Γ^d_0, we obtain the linear system for the variables $\psi_+ = \left(\psi^1_+ u, \psi^2_+ u, \psi^u_0, \psi^d_0, \psi^2_+ d, \psi^1_+ d\right)$, similar to (9). Existence of a non-trivial solution of this linear system is guaranteed by an appropriate determinant condition. Denote

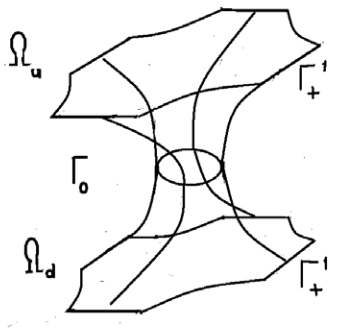

Figure 3: Detail of the double square lattice with romboidal periods

$$\mathcal{DN}^{u,d} :=$$
$$\begin{pmatrix} \langle \nu^1_{u,d} \mathcal{DN}^u_{11} \nu^1_{u,d} \rangle & \langle \nu^1_{u,d} \mathcal{DN}^{u,d}_{12} \nu^2_{u,d} \rangle & \langle \nu^1, \mathcal{DN}^{u,d}_{10} \rangle \\ \langle \nu^2_{u,d} \mathcal{DN}^{u,d}_{21} \nu^1_{u,d} \rangle & \langle \nu^2_{u,d} \mathcal{DN}^{u,d}_{22} \nu^2_{u,d} \rangle & \langle \nu^2, \mathcal{DN}^{u,d}_{20} \rangle \\ \langle \mathcal{DN}^{u,d}_{01}, \nu^1_u \rangle & \langle \mathcal{DN}^{u,d}_{02}, \nu^2_{u,d} \rangle & \mathcal{DN}^{u,d}_{00} \end{pmatrix},$$

$$\mathcal{DN}_T^{u,d} =:$$

$$\begin{pmatrix} \langle \nu_{u,d}^1 \mathcal{DN}_{11}^{u,d} \nu_{u,d}^1 \rangle & \langle \nu_{u,d}^1 \mathcal{DN}_{12}^{u,d} \nu_{u,d}^2 \rangle \\ \langle \nu_{u,d}^2 \mathcal{DN}_{21}^{u,d} \nu_{u,d}^1 \rangle & \langle \nu_{u,d}^2 \mathcal{DN}_{22}^{u,d} \nu_{u,d}^2 \rangle \end{pmatrix} \quad (12)$$

In particular, if $\beta \to \infty$, the linear system for $\vec{\psi}, \vec{\psi}'$ derived as (9) splits into a pair of independent blocks, corresponding to the upper and lower period, with the dispersion equations $\det \mathcal{DN}_T^u = 0$ and $\det \mathcal{DN}_T^d = 0$ similar to ones we obtained in previous section. If β is large, then the intersection of terms $\det \mathcal{DN}_T^u \det \mathcal{DN}_T^d = 0$ is transformed into a quasi-intersection. The transport properties for

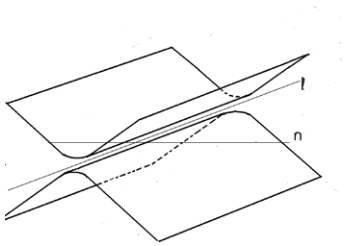

Figure 4: Two-dimensional Landau-Zener effect.

large β are defined by the second derivatives of λ with respect to the component p_n of the quasi-momentum p, orthogonal to the intersection l of the tangent planes of the dispersion surfaces of the upper and lower layers of the double lattice.

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Random Trees and SPDE Approximation

Yuri Bakhtin
Georgia Institite of Technology, Atlanta, GA, USA
e-mail: bakhtin@math.gatech.edu

In this talk I consider Boltzmann–Gibbs distributions on rooted plane trees with bounded braching.

The first result, see [1], is a Large Deviation Principle for the branching type of the tree. Its immediate consequence is a Law of Large Numbers that states that as the size of the tree grows, the branching frequences converge in probability to an explicitly computed limiting branching type, and the deviation rate is exponential. This result is interesting and has some implications for the RNA secondary structure analysis, see [2], but it does not take into account the geometry of the tree.

The next result, see [3], addresses this issue, although the limiting procedure is different. We show that as the order of the tree grows to infinity, these trees obey a certain "thermodynamic" limit theorem: for each natural n the distributions of the root's neighbourhood of depth n in the random tree converge to a limiting distribution. These distributions are, of course, consistent with each other for different values of n which allows to consider the limiting infinite tree and restate the theorem in terms of the weak convergence to this infinite random tree.

The limiting infinite tree possesses several curious properties. Although it is not a classical branching process, it is a Markov process on "generations". This allows to study scaling limits, and we show that the Markov random tree obeys a functional limit theorem: appropriately rescaled sizes of generations in the limiting tree admit an approximation by a diffusion process with explicitly computed characteristics.

That result does not take into account the way the generations are connected to each other. However, one can study finer structure of the random tree by partitioning some generation into a finite number of subsets and observing the progeny of each of them. This also leads to a functional limit theorem under the same scaling.

One can also wonder if there is a limiting object on one probability space that serves all possible partitions at once. The possibility of this was conjectured in [3]. It turns out that the answer is "yes", and one can introduce a Stochastic PDE w.r.t. a Brownian sheet such that the solution to this SPDE serves as a scaling limit for the fine structure of the random trees. The solution defines stochastic dynamics on monotone maps, and the most interesting and unexpected property of these maps is that they have jumps that one has to take into account when studying the scaling limits.

In the last part of the talk, I will define a new type of continuum random trees as the solutions of the aforementioned SPDE, state the scaling limit theorem for the random trees, and discuss the approriate topology in which the weak convergence holds.

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Exclusion type processes in continuum

Michael Blank*†

Abstract

We introduce and study a new class of exclusion type discrete time particle processes in continuum. Ergodic averages of particle velocities are obtained and their connections to other statistical quantities, in particular to the particle density (the so called Fundamental Diagram) is analyzed rigorously. The main technical tool is a "dynamical" coupling construction applied in a nonstandard fashion: instead of proving the existence of the successful coupling (which even might not hold) we use its presence/absence as an important diagnostic tool. Despite that this approach cannot be applied to lattice systems directly, it allows to obtain new results for the lattice systems embedding them to the systems in continuum.

1 Introduction

In 1970 Frank Spitzer introduced the (now classical) simple exclusion process as a Markov chain that describes nearest-neighbor random walks of a collection of particles on the one-dimensional infinite[1] integer lattice. Particles interact through the hard core exclusion rule, which means that at most one particle is allowed at each site. This seemingly very particular process appears naturally in a very broad list of scientific fields starting from various models of traffic flows [12, 9, 7, 2, 3], molecular motors and protein synthesis in biology, surface growth or percolation processes in physics (see [13, 5] for a review), and up to the analysis of Young diagrams in Representation Theory [6].

From the point of view of the order of particle interactions there are two principally different types of exclusion processes: with synchronous and asynchronous updating rules. In the latter case at each moment of time a.s. at most one particle may move and hence only a single interaction may take place. This is the main model considered in the mathematical literature (see e.g. [11] for a general account and [1, 8] for recent results), and indeed, the assumption about the asynchronous updating is quite natural in the continuous time setting. The synchronous updating means that *all* particles are trying to move simultaneously and hence an arbitrary large (and even infinite) number of interactions may occur at the same time. This makes the analysis of the synchronous updating case much more difficult, but this is what happens in the discrete time case.[2] This case is much less studied, but still there are a few results describing ergodic properties of such processes [2, 3, 4, 7, 9, 12].

Our aim is to introduce and study the synchronous updating version of the exclusion process in continuum. Note that recently some other interacting particle processes were generalized from lattice to continuum case (see e.g. [13]).

A configuration $x := \{x_i\}_{i\in\mathbb{Z}}$ is a bi-infinite sequence of real numbers $x_i \in \mathbb{R}$ interpreted as centers of particles represented by balls of radius $r \geq 0$ (see Fig. 1) and ordered with respect to their positions (i.e. $\ldots \leq x_{-1} \leq x_0 \leq x_1 \leq \ldots$). To emphasize the dependence on the radius $r \geq 0$ we shall use the notation $x(r)$ and drop it only if $r = 0$, i.e. $x \equiv x(0)$. We say that a configuration $x(r)$ is *admissible* if

$$x_i(r) + r \leq x_{i+1}(r) - r \quad \forall i \in \mathbb{Z}$$

(the corresponding balls may only touch each other) and denote by X the space of admissible configurations.

For a finite subset of integers I and a collection $C := \{C_i\}_{i\in I}$ of open intervals the subset $\mathcal{C}_{I,C} := \{x \in X : \quad x_i \in C_i \quad \forall i \in I\}$ is called a finite *cylinder*.[3] We endow the space of admissible configurations X by the σ-algebra $\mathcal{B}$ generated by the finite cylinders defining a topology in this space.

*Russian Academy of Sci., Inst. for Information Transm. Problems, e-mail: blank@iitp.ru

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[1] or finite with periodic boundary conditions

[2] if one do not consider some "artificial" updating rules like a sequential or random updating.

[3] In general the cylinder $\mathcal{C}_{I,C}$ might be empty for nonempty sets I, C.

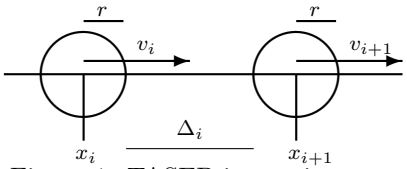

Figure 1: TASEP in continuum.

The dynamics will be defined as follows. For a trivial configuration consisting of a single particle located at time $t \geq 0$ at $x_0^t \in \mathbb{R}$ (i.e. $x^t \equiv \{x_0^t\}$) the dynamics is defined as

$$x_0^{t+1} := x_0^t + v_0^t,$$

and thus v_0^t is considered as a local velocity at time t, i.e this is simply a random walk on $\mathbb{R}$. To generalize this trivial setting for an infinite configuration $x(r) \in X$ we again interpret a (be-infinite on $i \in \mathbb{Z}$) sequence $\{v_i^t\}_{i,t}$ as *local velocities* for particles in $x^t(r)$ performing random walks conditioned to the order preservation and the hard core exclusion rule.

To simplify presentation we restrict ourselves here to the case of nonnegative local velocities. The point is that the formulations in the general case are becoming much more involved, but the results and arguments work with only very slight changes.

Since only nonnegative local velocities are considered the hard core exclusion rule means that the admissibility condition breaks down for the i-th particle at time $t \in \mathbb{Z}_+$ if and only if the inequality

$$x_i^t(r) + v_i^t + r \leq x_{i+1}^t(r) - r$$

does not hold. If this happens we say that there is a *conflict* between the particles i and $i+1$, and to resolve it one applies a *normalizing* construction

$$v_i^t \to \mathcal{N}(v_i^t, x^t(r)).$$

After the normalization the positions of particles are calculated according to the rule

$$x_i^{t+1}(r) := x_i^t(r) + \mathcal{N}(v_i^t, x^t(r)) \quad \forall i.$$

The normalization may be done in a number of ways and we restrict ourselves the *weak normalization* under which the conflicting velocity is modified to allow the particle to move as far as possible. In terms of *gaps*

$$\Delta_i(x^t) \equiv \Delta_i^t := x_{i+1}^t - x_i^t - 2r$$

between particles in the configuration x^t the normalization can written as follows:

$$\mathcal{N}(v_i^t, x^t) := \begin{cases} v_i^t & \text{if } v_i^t \leq \Delta_i^t \\ \Delta_i^t & \text{otherwise} \end{cases} .$$

Observe that any two particle configurations $x(r),\ \acute{x}(\acute{r})$ having the same sequence of gaps $\Delta := \{\Delta_i\}$ may be transformed to each other by a one-to-one map

$$\acute{x}_i(\acute{r}) = \varphi(x_i(r)) := x_i(r) - 2i(r - \acute{r}) \ \forall i \in \mathbb{Z}.$$

Since the normalization procedures that we consider depend only on the gaps between particles it is enough to study the case $r = 0$. On the other hand, if $r = 1/2$ and $x_i^0(r) \in \mathbb{Z}\ \forall i \in \mathbb{Z}$ and $v_i^t \in \mathbb{Z}\ \forall i \in \mathbb{Z}, t \geq 0$ then $x_i^t(r) \in \mathbb{Z}\ \forall i \in \mathbb{Z}, t \geq 0$ and hence we get a lattice particle system. Thus our results lead to a completely new approach to the analysis of lattice systems as well. Note however that in the case $r = 0$ an arbitrary number of particles may share the same spatial position which is prohibited in the lattice case.

Of course, without some specific assumptions on the structure of local velocities $\{v_i^t\}_{i,t}$ no interesting results are possible. We assume that $v_i^t \in [0, v] \quad \forall i \in \mathbb{Z},\ t \in \mathbb{Z}_{\geq} := \mathbb{Z}_+ \cup \{0\}$ and one of the following seemingly opposite assumptions holds:

(a) $v_i^t \equiv v_0^t \quad \forall i \in \mathbb{Z},\ t \in \mathbb{Z}_{\geq}$ and $\exists\ \bar{v}(\gamma) := \lim\limits_{t\to\infty} \frac{1}{t} \sum\limits_{s=0}^{t-1} \min(v_0^s, \gamma) \quad \forall \gamma > 0$ (a.s.);

(b) $\{v_i^t\}$ are i.i.d. (both in i and t) random variables.

Note that the intersection between the sets of local velocities satisfying the assumptions (a) and (b) contains an important case of pure deterministic velocities: $v_i^t \equiv v \quad \forall i \in \mathbb{Z},\ t \in \mathbb{Z}_{\geq}$. As we shall show properties of systems with local velocities satisfying to the assumption (a) are close to the pure deterministic setting. Therefore we refer to the setting (a) as *deterministic*[4] and to the setting (b) as *random*.

It is of interest that in the seemingly simplest deterministic setting $v_i^t \equiv v\ \forall i \in \mathbb{Z},\ t \in \mathbb{Z}_{\geq}$ the behavior of the corresponding deterministic dynamical system describing the dynamics of particle configurations is far from being trivial. We prove (Theorem 3) that this system is chaotic in the sense that its topological entropy is positive (and even infinite).

To emphasize that under dynamics no creation or annihilation of particles may take place this sort of systems is called *diffusive driven systems* (DDS) instead of a more general object – *interacting particle systems* (IPS).

The main technical tool in our analysis is a (somewhat unusual) "dynamical" coupling construction. Despite that various couplings are widely used in the analysis of IPS, applications of our approach is very different from usual. In particular, we do not prove the existence of the so called successful coupling (which even might not hold) but instead use its presence/absence as an important diagnostic tool. Remark also that typically one uses the coupling argument to prove the uniqueness of the invariant measure and to derive later other results from this fact. In our case there might be a very large number of ergodic invariant measures or no invariant measures at all (recall the trivial example of a single particle performing a skewed random walk). This indicates that there is another important statistical quantity – average particles velocity that can be computed at least in this example. The dynamical coupling will be used directly to find connections between the average particle velocities and other statistical features of the systems under consideration, in particular with the corresponding particle densities.

It is worth note that all approaches used to study lattice versions of DDS are heavily based on the combinatorial structure of particle configurations. This structure has no counterparts in the continuum setting under consideration. In particular the particle – vacancy symmetry is no longer applicable in our case. This explains the need to develop a fundamentally new techniques for the analysis of DDS in continuum. This techniques cannot be applied directly in the lattice case. Nevertheless, the embedding of lattice systems to the continuum setting allows to obtain (indirectly) new results for the lattice systems as well.

2 Basic properties of DDS

Here we shall study questions related to densities and velocities of DDS.

By the *density* $\rho(x, I)$ of a configuration $x \in X$ in a bounded segment $I = [a, b] \in \mathbb{R}$ we mean the number of particles from x whose centers x_i belong to I divided by the Lebesgue measure $|I| > 0$ of the segment I. If for any sequence of *nested* bounded segments $\{I_n\}$ with $|I_n| \stackrel{n\to\infty}{\longrightarrow} \infty$ the limit

$$\rho(x) := \lim_{n\to\infty} \rho(x, I_n)$$

is well defined we call it the *density* of the configuration $x \in X$. Otherwise one considers upper and lower (with respect to all possible collections of nested intervals I_n) particle densities $\rho_{\pm}(x)$.

Lemma 1 *The upper/lower densities $\rho_{\pm}(x^t)$ are preserved by dynamics, i.e. $\rho_{\pm}(x^t) = \rho_{\pm}(x^{t+1})\ \ \forall t$.*

By the (average) *velocity* of the i-th particle in the configuration $x \in X$ at time $t > 0$ we mean

$$V(x, i, t) := \frac{1}{t} \sum_{s=0}^{t-1} \mathcal{N}(v_i^s, x^s) \equiv (x_i^t - x_i^0)/t.$$

If the limit

$$V(x, i) := \lim_{t\to\infty} V(x, i, t)$$

is well defined we call it the (average) *velocity* of the i-th particle. Otherwise one considers upper and lower particle velocities $V_{\pm}(x, i)$.

[4] In this case v_0^t might be a trajectory of a deterministic chaotic map $f : [0, 1] \to [0, 1]$, e.g. $v_0^{t+1} := vf^t(v_0^t/v)$, as well as a realization of a true random Markov chain.

Lemma 2 *Let $x \in X$ then $|V(x,j,t) - V(x,i,t)| \overset{t\to\infty}{\longrightarrow} 0$ a.s. $\forall i,j \in \mathbb{Z}$.*

Corollary 3 *The upper and lower particle velocities $V_{\pm}(x,i)$ do not depend on i (but might be random).*

The proof of this result shows that in the deterministic setting the gaps between particles cannot become much larger than their initial values. The following result demonstrates that under some mild additional assumptions (which definitely hold for high particle densities) large gaps will disappear with time.

Lemma 4 *Let $x \in X$ and we consider only the pure deterministic setting (i.e. $v_i^t \equiv v$). Assume that $\forall i,t\ \exists j > t:\ \Delta_j(x^t) < v$. Then $\forall i\ \exists t_i < \infty:\ \Delta_i(x^t) < 2v\ \forall t \geq t_i$.*

3 Ergodic properties

Lemma 5 *The supremum of $|W_{ij}^t| := x_i^t - \acute{x}_j^t$ taken over all mutually paired particles in the coupled (see Section 4) process $(x^t, \acute{x}^t)$ is uniformly bounded by v for any $t \in \mathbb{Z}_{\geq}$.*

Under our assumptions the (standard) successful coupling[5] needs not hold (e.g. in the deterministic setting when two equally distributed initial configurations are shifted against each other). Therefore one cannot apply directly Lemma 5 to compare particle velocities. Nevertheless we show that the absence of coupling is not a serious obstacle and it can be used as a diagnostic tool.

Theorem 1 *Let the density $\rho(x)$ of a configuration $x \in X$ be well defined. Then the set of limit points as $t \to \infty$ of the sequence $\{V(x,t)\}_{t\in\mathbb{Z}_{\geq}}$ depends only on $\rho(x)$.*

Theorem 2 *(Fundamental Diagram) In the deterministic setting*

$$V(x) = \lim_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} \min(1/\rho, v_0^s) = \begin{cases} v & \text{if} \quad \rho(x) \leq 1/v \\ 1/\rho(x) & \text{otherwise} \end{cases} \qquad \text{if} \quad v_0^t \equiv v.$$

Remark 6 This result looks very similar to the one known for the deterministic version of the lattice TASEP (see [12, 2]), however the latter case is characterized by the following feature: if the density is large enough particles inevitably form dense clusters without vacancies inside (static traffic jams). The proof of the above result actually shows that the "typical" behavior of high density configurations in continuum is different: they do form particle clusters, but these clusters are not staying at rest but are moving at a constant velocity as an "echelon". It is of interest that in order to imitate such behavior a number of complicated lattice models were developed.

Remark 7 The construction used in the proof is especially striking in that the same family of uniformly spatially distributed configurations allows to study the limit dynamics in the deterministic setting for all configurations having densities. Note that this argument cannot be applied directly in the lattice version of DDS. Nevertheless since the "lattice configurations" are included in DDS under consideration the result holds as well, which implies completely new results for a lattice TASEP with long jumps.

In the deterministic setting (i.e. $v_i^t \equiv v\ \ \forall i,t$) the DDS is defined by a deterministic map $T_v : X \to X$ from the set of admissible configurations into itself. Our aim is to show that this map is chaotic in the sense that its topological entropy is infinite.[6]

We refer the reader to [10] for detailed definitions of the topological and metric entropies for deterministic dynamical systems and their properties that we use here. To avoid difficulties related to the noncompactness of the phase space we define the topological entropy of a map T_v (notation $h_{\text{top}}(T_v)$) as the supremum of metric entropies of this map taken over all probabilistic invariant measures.

Theorem 3 *The topological entropy of the pure deterministic exclusion process in continuum is infinite.*

[5] when a.a. particles are eventually becoming paired.

[6] Normally one says that a map is chaotic if its topological entropy is positive, so infinite value of the entropy indicates a very high level of chaoticity.

The proof of this result is based on a similar result for the action of a shift-map in continuum $\sigma_v : X \to X$ defined as

$$(\sigma_v x)_i := x_i + v \quad i \in \mathbb{Z}, x \in X.$$

Lemma 8 *The topological entropy of the shift-map in continuum σ_v is infinite.*

The idea of the proof is to construct an invariant subset of X on which the map σ_v is isomorphic to the full shift-map in the space of sequences with a countable alphabet. The result follows from the observation that the topological entropy of the full shift-map $\sigma^{(n)}$ with the alphabet consisting of n elements is equal to $\ln n$.

4 Coupling

Technically one of the main ingredients of the above mentioned results is special "dynamical" coupling construction.

Recall that a coupling of two Markov chains x^t and y^t acting on the space X is an arrangement of a pair of processes on a common probability space to facilitate their direct comparison, namely this is a pairs process (x^t, y^t) defined on the direct product space $X \times X$ satisfying the assumptions

$$P((x^t, y^t) \in A \times X) = P(x^t \in A) \quad \text{and} \quad P((x^t, y^t) \in X \times A) = P(y^t \in A),$$

i.e. the projections of the pairs process behave as in the individual processes.

Let us discuss the specific coupling between two copies $x^t, \acute{x}^t$ of Markov chains we consider throughout the paper. Typically in continuous time interacting particle systems one uses (see e.g. [11]) an *equal* coupling (pairing) when particles sharing the same sites in the processes $x^t, \acute{x}^t$ are considered to be paired and all choices of their velocities are assumed to be identical. This sort of coupling works rather well for continuous time systems when only a single particle may move at a given moment of time. In the discrete time case the situation is much more complicated since an arbitrary number of particles may move simultaneously and thus it is possible that the particles of the processes $x^t, \acute{x}^t$ pass each other and never share the same positions. In fact, this difficulty is not really crucial and can be cured under some simple technical assumptions. A more important obstacle is that if a pair is created and only one of its members is blocked at time t by an unpaired particle, then due to the simultaneous motion of the blocking unpaired particle and the non-blocked particle in the pair the following situation may happen: $\overset{\bullet}{\bullet}{}^{\circ} \longrightarrow {}^{\circ}{}_{\circ}{}^{\circ}$. Thus the old pair will be destroyed but no new pair will be created under the equal pairing construction. Here we use a diagrammatic representation for coupled configurations where paired particles are denoted by black circles and unpaired ones by open circles and use the upper line of the diagram for the x-particles (i.e. particles from the x-process) and the lower line for the $\acute{x}$-particles.

To deal with this obstacle we introduce a *dynamical* [7] coupling, a very preliminary version of which was described in [4] for the lattice case and was inspired by the idea proposed by L. Gray in the case the simplest discrete time lattice TASEP (unpublished). It is worth mention also the coupling proposed for the lattice continuous time case by O. Angel (see [1, 8]). Lemma 5 shows an important advantage of the dynamical coupling with respect to the Angel's construction: the former guarantees that the distances between mutually paired particles are uniformly bounded, while in the Angel's construction the distances may grow to infinity.

Let us give an informal description of the dynamical coupling of the processes $x^t, \acute{x}^t$. At $t = 0$ all particles are assumed to be unpaired and the coupling consists of a gradual pairing of close enough particles belonging to the opposite processes. Two unpaired particles from different configurations form a new pair if the segment between them is less or equal to v and it does not contain any other unpaired particles. Once particles are paired all choices of their velocities in the coupled process are identical. A member of a pair may be swapped with an unpaired particle from the same process if the latter approaches another member of the pair closer than it. It is convenient to think about the coupled process as a "gas" of single (unpaired) particles and "dumbbells" (pairs). A previously paired particle may inherit the role of the unpaired one from one of its neighbors. Our aim is to get rid of the unpaired particles and in order to keep track of their positions we shall refer to them as x- and $\acute{x}$-*defects* depending on the process they belong.

[7] The word "dynamical" is meant to emphasize that the mutual arrangement of particles in pairs may change with time under dynamics in distinction to the conventual equal coupling (where the particles have coinciding positions).

The formal description of the dynamical coupling can be done in terms of the resolution of x-triples (or) and minimal pairs of defects (or) in the coupled process as follows:

(1) each x-triple is recursively resolved: ,

(2) each $\acute{x}$-triple is recursively resolved: ,

(3) each minimal pair of defects is resolved:  .

During the time when two particles are paired all choices of their velocities in the coupled process are assumed to be identical. Therefore the particles from the same pair move synchronously until either the admissibility condition breaks down for only one of the particles (which basically means that its movement is blocked by another particle) or an unpaired particle comes close enough to one of the members of the pair. This construction clearly defines a Markovian coupling between two copies of the Markov chain describing our DDS.

One of the most important properties of the dynamical coupling construction is that once being created at time t_0 a pair of particles remains present for any moment of time $t \geq t_0$, however at different moments of time the roles of the pair's members may be played by different particles. Indeed, a pair breaks down only if one of its members is replaced by an unpaired particle, and hence the pair as a whole survives.

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ТОРИЧЕСКАЯ ТОПОЛОГИЯ

В. М. БУХШТАБЕР, Т. Е. ПАНОВ

Аннотация. В обзоре излагаются методы и основные результаты новой активно развивающейся области исследований — *торической топологии*. В этих исследованиях активное участие принимают сотрудники, аспиранты и студенты кафедры высшей геометрии и топологии мехмата МГУ.

1. Введение в предмет исследования

Теория действий тора имеет длинную историю развития и образует важную область алгебраической топологии. За последние 15 лет на стыке эквивариантной топологии, алгебраической и симплектической геометрии, комбинаторики, коммутативной и гомологической алгебры возникла новая область исследований — *торическая топология*, которая быстро привлекла внимание большого числа исследователей и активно развивается в настоящее время.

В центре внимания торической топологии находятся действия тора, пространства орбит которых несут богатую комбинаторную структуру. В ней решаются задачи на основе изучения алгебраических, комбинаторных и топологических свойств таких действий, естественно возникающих в различных направлениях исследований. Благодаря торической топологии фундаментальные результаты ряда областей математики получили новое развитие и нашли неожиданные замечательные приложения.

Первоначальный импульс этому развитию придала *торическая геометрия* — теория *торических многообразий* в алгебраической геометрии. Эта теория устанавливает взаимно однозначное соответствие между комплексными алгебраическими многообразиями с действием комплексного тора, имеющим плотную орбиту, и комбинаторными объектами, называемыми *веерами*. При помощи вееров алгебро-геометрические свойства торических многообразий полностью переводятся на язык комбинаторики. Торическая геометрия предоставляет богатый источник явных примеров алгебраических многообразий и имеет яркие приложения в таких областях, как теория особенностей и математическая физика. Пространство орбит $2n$-мерного неособого проективного торического многообразия по действию компактного тора T^n представляет собой выпуклый n-мерный простой многогранник P.

В симплектической геометрии, после появления теоремы выпуклости Атьи–Гиллёмина–Стернберга [At82] и формулы Дуистермаата–Хекмана [DH82] в начале 1980-х годов, активно изучались гамильтоновы действия групп. В работе Делзанта [De88] было показано, что в случае действия тора размерности, равной половине размерности многообразия, образ отображения моментов определяет многообразие с точностью до эквивариантного симплектоморфизма. В симплектической геометрии, как и в торической геометрии, различные геометрические конструкции имеют комбинаторную интерпретацию в терминах многогранников.

Имеется тесная взаимосвязь между алгебраическими и симплектическими многообразиями с действием тора: проективное вложение неособого торического многообразия определяет симплектическую форму и отображение моментов. Образом отображения моментов является многогранник, двойственный к вееру. Как в алгебраической, так и в симплектической ситуации, действие компактного тора локально изоморфно стандартному действию тора T^n на $\mathbb{C}^n$ покоординатными вращениями. Факторпространство многообразия по такому действию тора представляет собой многообразие с углами, которое несёт комбинаторную структуру, отражающую структуру частично упорядоченного множества стационарных подгрупп. Это позволяет полностью восстановить многообразие и действие. Замечательно, что такой подход работает и в обратном направлении: в терминах топологических инвариантов пространства с действием тора удаётся интерпретировать и доказывать весьма тонкие комбинаторные результаты топологически. Оказалось, что данная специфика алгебраических торических многообразий имеет чисто топологическую природу, что вызвало глубокое проникновение идей и методов торической и симплектической геометрии в алгебраическую топологию с начала 1990-х годов.

Дальнейшие исследования выявили ряд важных классов многообразий с действием тора, происхождение которых восходит к торическим или симплектическим многообразиям. Эти более общие многообразия как правило не являются алгебраическими или симплектическими, но в то же время обладают важнейшими топологическими свойствами их алгебраических или симплектических предшественников. Таким образом, была существенно расширена область приложений методов торической топологии в комбинаторике и коммутативной алгебре. Опишем некоторые из этих классов.

Подход Дэвиса–Янушкиевича [DJ91] к изучению торических многообразий топологическими методами привёл к появлению *квазиторических многообразий*. Этот класс многообразий определяется двумя условиями: действие тора локально выглядит как стандартное представление T^n в комплексном пространстве $\mathbb{C}^n$, а пространство орбит Q является комбинаторным простым многогранником. Оба условия выполнены для действия тора на неособом проективном торическом многообразии. Работы Бухштабера–Рэя [БР98], [BR01] показали, что квазиторические многообразия играют важную роль в теории комплексных кобордизмов — классической области алгебраической топологии [St68]. В отличие от торических многообразий, квазиторические многообразия могут быть не комплексными и даже не почти комплексными, однако они всегда допускают стабильно комплексную структуру, которая определяется в чисто комбинаторных терминах — при помощи так называемой *характеристической функции*, сопоставляющей каждой гиперграни многогранника некоторый примитивный вектор целочисленной решётки. Характеристическая функция играет роль веера, сопоставляемого торическому многообразию в алгебраической геометрии.

Комбинаторный подход к изучению гамильтоновых действий тора привёл к понятию *ГКМ-многообразий*. Согласно [GZ99], компактное $2n$-мерное многообразие M с эффективным действием тора T^k, $k \leqslant n$, называется ГКМ-многообразием, если множество неподвижных точек конечно, M обладает инвариантной почти комплексной структурой, и веса представлений тора T^k в касательных пространствах к неподвижным точкам попарно линейно независимы. Эти многообразия были названы в честь Горески, Коттвица и Макферсона, которые впервые ввели их в [GKM98]. Там же было показано, что «1-остов» такого многообразия M, т.е. множество точек, имеющих стационарную

подгруппу коразмерности не больше 1, может быть описано при помощи графа с метками (Γ, α). Этот граф, называемый *графом весов* (или *ГКМ-графом*), позволяет вычислять важные топологические инварианты многообразия M, такие как его числа Бетти или кольцо эквивариантных когомологий. Изучение таких графов приобрело самостоятельный комбинаторный интерес благодаря работам Гиллёмина–Зары [GZ99] и других. Отметим, что в топологии идея сопоставления графа с метками многообразию с действием окружности использовалась начиная с 1970-х годов, см., например, работу Мусина [My80].

Стенли был одним из первых, кто осознал большой потенциал торических действий для комбинаторных приложений, использовав их для доказательства *гипотезы Макмюллена* о числах граней симплициальных многогранников и *гипотезы о верхней границе* для триангуляций сфер. Его результаты и методы легли в основу известной монографии [St96] и предопределили дальнейшие приложения коммутативной алгебры и гомологических методов в комбинаторной геометрии.

Многие идеи Стенли нашли топологические применения; так оказалось, что *кольцо граней* (или *кольцо Стенли–Риснера*) $\mathbb{Z}[\mathcal{K}]$ симплициального комплекса $\mathcal{K}$ является важной составляющей в вычислении кольца когомологий квазиторического многообразия M. В [DJ91] показано, что *эквивариантные когомологии* многообразия M изоморфны кольцу граней $\mathbb{Z}[\mathcal{K}_P]$ симплициального комплекса $\mathcal{K}_P$, двойственного к границе простого многогранника P. Кольцо обычных когомологий $H^*(M)$ получается из $\mathbb{Z}[\mathcal{K}_P]$ факторизацией по идеалу, порождённому некоторыми линейными формами, в точности как и в случае торических многообразий.

С появлением кольца граней стало ясно, что многие тонкие комбинаторные свойства комплексов $\mathcal{K}$ можно интерпретировать алгебраически. Изучение колец граней получило самостоятельное развитие и привело к новому классу *колец Коэна–Маколея*, имеющих геометрическую природу. В частности, возникло новое топологическое понятие *симплициального комплекса Коэна–Маколея* $\mathcal{K}$, для которого $\mathbb{Z}[\mathcal{K}]$ является кольцом Коэна–Маколея. Подробное изложение этих понятий можно найти в монографии [BH98], где также подчёркивается важность гомологического подхода. Например, в [St96] и [BH98] рассматриваются размерности биградуированных компонент векторных пространств $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$, называемые *алгебраическими числам Бетти* кольца $\mathbf{k}[\mathcal{K}]$, для любого поля $\mathbf{k}$. Эти числа являются весьма тонкими инвариантами: они зависят от комбинаторики $\mathcal{K}$, а не только от топологии его реализации $|\mathcal{K}|$, и полностью определяют «обычные» топологические числа Бетти для $|\mathcal{K}|$. Теорема Хохстера [Ho77] выражает алгебраические числа Бетти через когомологии полных подкомплексов в $\mathcal{K}$.

Более подробно ознакомиться с основными этапами развития торической топологии можно по монографии [БП04] и недавнему обзору Бухштабера–Рэя [BR08]. Среди других работ по торической топологии сотрудников и аспирантов кафедры высшей геометрии и топологии выделим [Ба03], [ББП04], [До01], [Ер08].

2. Торические и квазиторические многообразия

Рассмотрим выпуклый n-мерный многогранник с m гипергранями в евклидовом пространстве $\mathbb{R}^n$, заданный как пересечение m полупространств:

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon (\boldsymbol{a}_i, \boldsymbol{x}) + b_i \geqslant 0 \quad \text{при } 1 \leqslant i \leqslant m\}, \tag{2.1}$$

где $\boldsymbol{a}_i \in \mathbb{R}^n$ — некоторые векторы и $b_i \in \mathbb{R}$. Многогранник P называется *простым*, если ограничивающие его гиперплоскости находятся в общем положении в каждой его вершине; далее мы будем рассматривать лишь простые многогранники.

Многогранник (2.1) можно задать одним матричным неравенством

$$A_P\boldsymbol{x} + b_P \geqslant 0,$$

где A_P — матрица размера $m \times n$ со строками $\boldsymbol{a}_i$, а b_P — столбец из чисел b_i; неравенство считается покоординатным. Тогда аффинное отображение

$$i_P\colon \mathbb{R}^n \to \mathbb{R}^m; \quad \boldsymbol{x} \mapsto A_P\boldsymbol{x} + b_P$$

отождествляет P с пересечением *положительного ортанта* $\mathbb{R}^m_{\geqslant}$ и n-мерной плоскости $i_P(\mathbb{R}^n)$. Ортант $\mathbb{R}^m_{\geqslant}$ является пространством орбит *стандартного* (покоординатного) действия тора T^m на комплексном пространстве $\mathbb{C}^m$; в качестве проекции на пространство орбит возьмем отображение

$$\mu\colon \mathbb{C}^m \to \mathbb{R}^m_{\geqslant}; \quad (z_1, \ldots, z_m) \mapsto (|z_1|^2, \ldots, |z_m|^2).$$

Теперь определим пространство $\mathcal{Z}_P$ из коммутативной диаграммы

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{\;i_Z\;} & \mathbb{C}^m \\ \big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ P & \xrightarrow{\;i_P\;} & \mathbb{R}^m_{\geqslant}. \end{array} \tag{2.2}$$

По построению, $\mathcal{Z}_P$ является T^m-инвариантным подмножеством в $\mathbb{C}^m$ с пространством орбит P, а i_Z является T^m-эквивариантным вложением.

Теорема 2.1. *Пространство $\mathcal{Z}_P$ является T^m-инвариантным гладким вещественным $(m+n)$-мерным подмногообразием в $\mathbb{C}^m$ с тривиальным нормальным расслоением.*

Выбрав вещественную $(m-n) \times m$-матрицу $D = (d_{ki})$ ранга $(m-n)$, такую что $DA_P = 0$, можно задать $\mathcal{Z}_P$ как полное пересечение вещественных квадрик в $\mathbb{C}^m \cong \mathbb{R}^{2m}$:

$$\sum_{i=1}^{m} d_{ki}(|z_i|^2 - b_i) = 0, \quad 1 \leqslant k \leqslant m-n.$$

Мы называем $\mathcal{Z}_P$ *момент-угол многообразием* многогранника P (название связано с тем, что $\mathcal{Z}_P$ является поверхностью уровня для *отображения моментов* в симплектической конструкции торических многообразий [БП04, §9.2]).

Действие тора T^m на $\mathcal{Z}_P$ не является свободным: вершины многогранника имеют максимальные (n-мерные) стационарные подгруппы. Во многих случаях удаётся найти $(m-n)$-мерную подгруппу в T^m, действующую на $\mathcal{Z}_P$ свободно. Важнейшие примеры возникают, когда многогранник P является *целочисленным*, т.е. имеет вершины в точках целочисленной решётки $\mathbb{Z}^n \subset \mathbb{R}^n$. В этом случае векторы $\boldsymbol{a}_i$ в (2.1) можно выбрать целочисленными и примитивными; тогда отображение A_P происходит из эпиморфизма решёток $\mathbb{Z}^m \to \mathbb{Z}^n$, который задаёт эпиморфизм торов $T^m \to T^n$. Обозначим его ядро через $K(P)$.

Лемма 2.2. *Пусть для каждой вершины многогранника P набор из n векторов $\boldsymbol{a}_i$, ортогональных к гиперграням, содержащим эту вершину, образует базис целочисленной решётки. Тогда $K(P)$ является $(m-n)$-мерным тором, действующим на $\mathcal{Z}_P$ свободно.*

Соответствующее фактор-многообразие $V_P = \mathcal{Z}_P/K(P)$ (размерности $2n$) называется *торическим многообразием*, соответствующим целочисленному

многограннику P. Оно является неособым проективным алгебраическим многообразием с действием алгебраического тора $(\mathbb{C}^\times)^n$, имеющим плотную орбиту [Да78], [Fu93]. Компактный тор $T^n = T^m/K(P)$ является максимальной компактной подгруппой в $(\mathbb{C}^\times)^n$.

Торические подгруппы в T^m, действующие на $\mathcal{Z}_P$ свободно, можно также получать из следующей более общей конструкции. Пусть Λ — целочисленная $m \times n$-матрица, строки которой удовлетворяют условию на векторы $\boldsymbol{a}_i$ из леммы 2.2. Тогда ядро $K(\Lambda)$ соответствующего отображения торов $T^m \to T^n$ также действует на $\mathcal{Z}_P$ свободно. Фактор-многообразие $M = M(P, \Lambda) = \mathcal{Z}_P/K(\Lambda)$ называется *квазиторическим многообразием*, задаваемым данными (P, Λ). Торические многообразия получаются как частный случай при $\Lambda = A_P$. Действие тора $T^n = T^m/K(\Lambda)$ на M обладает двумя свойствами, которые привели Дэвиса и Янушкевича к понятию квазиторического многообразия (см. Введение). Можно доказать, что любое многообразие с действием тора T^n, удовлетворяющим этим условиям, получается из предыдущей конструкции как факторпространство момент-угол многообразия.

Следующая конструкция показывает, что на каждом квазиторическом многообразии имеется *стабильно комплексная структура*.

Пусть $F_1, \ldots, F_m$ — гиперграни многогранника P и $\pi\colon M \to P$ — проекция на пространство орбит квазиторического многообразия. Тогда $M_i = \pi^{-1}(F_i)$ является ориентируемым подмногообразием в M коразмерности два, называемым *характеристическим подмногообразием*. Тем самым определено вещественное 2-мерное ориентируемое расслоение ρ_i над M, ограничение которого на M_i совпадает с нормальным расслоением вложения $M_i \subset M$.

Теорема 2.3 ([DJ91, BR01])**.** *Имеет место изоморфизм вещественных $2m$-мерных расслоений*

$$\mathcal{T}M \oplus \mathbb{R}^{2(m-n)} \cong \rho_1 \oplus \ldots \oplus \rho_m,$$

где $\mathcal{T}M$ — касательное расслоение, а $\mathbb{R}^{2(m-n)}$ — тривиальное $2(m-n)$-мерное расслоение над M.

Так как выбор ориентации в вещественном 2-мерном расслоении эквивалентен заданию на нём комплексной структуры, стабильное касательное расслоение к M допускает комплексную структуру. Выбор этой структуры становится однозначным, если зафиксировать ориентацию самого M и всех характеристических подмногообразий M_i. Такой набор ориентаций называется *полиориентацией*. Для каждого полиориентированного квазиторического многообразия M определён его класс $[M] \in \Omega_U$ в кольце комплексных кобордизмов.

Теорема 2.4 ([BPR07])**.** *Каждый класс комплексных кобордизмов размерности > 2 содержит квазиторическое многообразие (непременно связное), стабильно комплексная структура которого задаётся некоторой полиориентацией, а следовательно согласована с действием тора.*

Данный результат можно рассматривать как решение квазиторического аналога известной проблемы Хирцебруха о классах кобордизма, представляемых связными неособыми алгебраическими многообразиях.

Следствие 2.5. *Каждый класс комплексных кобордизмов размерности > 2 представляется фактор-пространством полного пересечения вещественных квадрик по свободному действию тора.*

3. Момент-угол комплексы и многообразия

Теория момент-угол комплексов является одним из основных инструментов приложений торической топологии и объединяет методы комбинаторной геометрии, гомологической алгебры и эквивариантной топологии.

В предыдущем разделе мы сопоставили каждому геометрическому простому многограннику (2.1) гладкое момент-угол многообразие $\mathcal{Z}_P$ с действием тора T^m, получаемое как полное пересечение вещественных квадрик в $\mathbb{C}^m$. Можно показать, что $\mathcal{Z}_P$ отождествляется с факторпространством $P \times T^m / \sim$ по некоторому отношению эквивалентности, откуда вытекает, что топологический тип многообразия $\mathcal{Z}_P$ определяется лишь комбинаторной структурой многогранника P. Эта последняя конструкция многообразия $\mathcal{Z}_P$ впервые появилась в [DJ91] и была мотивирована конструкциями Винберга [Ви71] для групп Кокстера. Также в [DJ91] было получено обобщение конструкции $\mathcal{Z}_P$ на произвольные конечные симплициальные комплексы $\mathcal{K}$ с m вершинами (при этом простой многогранник P соответствует симплициальному комплексу $\mathcal{K}_P$ — границе двойственного многогранника). Получаемые пространства $\mathcal{Z}_\mathcal{K}$ мы и называем *момент-угол комплексами*. В [DJ91] им отводилась лишь вспомогательная роль при изучении квазиторических многообразий, но вскоре стало ясно, что момент-угол комплексы имеют самостоятельное большое значение.

Пусть $\mathcal{K}$ — конечный абстрактный симплициальный комплекс на множестве $[m] = \{1, \dots, m\}$. В [БП99] нами была предложена другая конструкция момент-угол комплекса $\mathcal{Z}_\mathcal{K}$. Рассмотрим единичный комплексный полидиск

$$(\mathbb{D}^2)^m = \big\{(z_1, \dots, z_m) \in \mathbb{C}^m \colon |z_i|^2 \leqslant 1, \quad i = 1, \dots, m\big\}.$$

С каждым симплексом $\sigma \in \mathcal{K}$ свяжем подмножество

$$B_\sigma = \big\{(z_1, \dots, z_m) \in (\mathbb{D}^2)^m \colon |z_i|^2 = 1 \text{ при } i \notin \sigma\big\}$$

и определим *момент-угол комплекс*

$$\mathcal{Z}_\mathcal{K} = \bigcup_{\sigma \in \mathcal{K}} B_\sigma \subset (\mathbb{D}^2)^m,$$

где объединение берётся в полидиске $(\mathbb{D}^2)^m$. По построению, $\mathcal{Z}_\mathcal{K}$ является T^m-инвариантным подпространством, содержащим стандартный тор $\mathbb{T}^m \subset (\mathbb{D}^2)^m$.

Пример 3.1. Если $\mathcal{K} = \partial(\Delta^{m-1})$ — граница $(m-1)$-мерного симплекса, то $\mathcal{Z}_\mathcal{K} = \partial((\mathbb{D}^2)^m) \cong S^{2m-1}$.

Предложение 3.2.

1. *Пусть $\mathcal{K} = \mathcal{K}_P$ — граница симплициального многогранника, двойственного к простому многограннику P. Тогда соответствующий момент-угол комплекс T^m-эквивариантно гомеоморфен момент-угол многообразию $\mathcal{Z}_P$.*
2. *Если $\mathcal{K}$ является симплициальным разбиением $(n-1)$-мерной сферы, то $\mathcal{Z}_\mathcal{K}$ является (замкнутым) T^m-многообразием.*
3. *Если $\mathcal{K}$ является симплициальным разбиением $(n-1)$-мерного многообразия, то дополнение $\mathcal{Z}_\mathcal{K} \setminus \mathbb{T}^m$ до стандартного тора $\mathbb{T}^m \subset (\mathbb{D}^2)^m$ является открытым T^m-многообразием.*

Предложение 3.3. *Сопоставление $\mathcal{K} \mapsto \mathcal{Z}_\mathcal{K}$ задаёт функтор из категории симплициальных комплексов и симплициальных отображений в категорию пространств с действием тора и эквивариантных отображений.*

Одним из наших основных результатов о момент-угол комплексах является вычисление их колец когомологий в терминах комбинаторики симплициальных комплексов. Напомним, что *кольцом граней* (или *кольцом Стенли–Риснера*) симплициального комплекса $\mathcal{K}$ называется градуированное факторкольцо

$$\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \dots, v_m]/\mathcal{I},$$

где $\deg v_i = 2$, а идеал $\mathcal{I}$ порождён мономами $v_{i_1} \cdots v_{i_k}$, где $\{i_1, \dots, i_k\} \notin \mathcal{K}$.

Теорема 3.4. *Имеют место функториальные по $\mathcal{K}$ изоморфизмы градуированных алгебр*

$$H^*(\mathcal{Z}_\mathcal{K}) \cong \operatorname{Tor}^*_{\mathbb{Z}[v_1,\dots,v_m]}\big(\mathbb{Z}[\mathcal{K}], \mathbb{Z}\big) \cong H\big[\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d\big].$$

Здесь последняя часть формулы обозначает алгебру когомологий дифференциальной градуированной алгебры $\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}]$, где образующие u_i внешней алгебры имеют степень 1, а дифференциал задан на образующих следующим образом: $du_i = v_i$, $dv_i = 0$.

Второй изоморфизм в предыдущей теореме основан на стандартном вычислении Tor-алгебры при помощи комплекса Кошуля. Доказательство изоморфизма между когомологиями момент-угол комплекса и Tor-алгеброй основан на построении клеточного разбиения пространства $\mathcal{Z}_\mathcal{K}$ (при котором каждый диск $\mathbb{D}^2$ разбивается на три клетки) и анализе умножения в клеточных коцепях при помощи специальной клеточной аппроксимации диагонального отображения $\Delta\colon \mathcal{Z}_\mathcal{K} \to \mathcal{Z}_\mathcal{K} \times \mathcal{Z}_\mathcal{K}$, функториальной относительно отображений симплициальных комплексов. При этом показано, что биградуировка в Tor-модулях имеет явную геометрическую реализацию, обусловленную введённой в $\mathcal{Z}_\mathcal{K}$ биградуированной клеточной структурой. Детали см. в [БП04, §8.1].

Теорема 3.4 даёт достаточно эффективное описание кольца $H^*(\mathcal{Z}_\mathcal{K})$ и легко применяется для конкретных вычислений с симплициальными комплексами. В случае комплексов с большим числом вершин для вычисления размерностей биградуированных компонент Tor-модулей можно привлечь известные пакеты компьютерных программ (Macaulay2, Bistellar и др.). Кроме того, применение теоремы Хохстера позволяет свести вычисление к когомологиям полных подкомплексов в $\mathcal{K}$:

Теорема 3.5. *Имеют место изоморфизмы групп*

$$H^k(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{\omega \subset [m]} \widetilde{H}^{k-|\omega|-1}(\mathcal{K}_\omega),$$

где $\mathcal{K}_\omega$ — полный подкомплекс в $\mathcal{K}$ (ограничение $\mathcal{K}$ на подмножество $\omega \subset [m]$).

Тем самым конструкция момент-угол комплексов позволила применить методы эквивариантной топологии для изучения комбинаторики симплициальных комплексов и алгебраических свойств их колец граней, придавая новое, геометрическое, измерение «комбинаторной коммутативной алгебре». В частности, вычисление когомологий момент-угол комплексов позволило топологически интерпретировать гомологические инварианты колец граней, такие как Tor-алгебры и алгебраические числа Бетти.

Несмотря на простоту конструкций момент-угол комплексов и многообразий, их топология достаточно сложна. Это видно уже из вычислений (на основе теоремы 3.4) когомологий момент-угол комплексов, соответствующих комплексам $\mathcal{K}$ с небольшим числом вершин. Оказалось, что в алгебрах рациональных когомологий момент-угол комплексов существуют нетривиальные произведения Масси [Ба03]. В некоторых случаях (например, для границ многоугольников или остовов симплексов) удаётся явно описать топологический тип пространства $\mathcal{Z}_\mathcal{K}$ (см. пример 3.7), но всяких раз такое описание использует весьма тонкий анализ различных конструкций момент-угол комплексов.

Важным аспектом теории момент-угол комплексов является их тесная взаимосвязь с конфигурациями координатных подпространств и их дополнениями. Эти пространства играют важную роль в алгебраической геометрии, теории особенностей и теории шарнирных механизмов.

Координатное подпространство $\mathbb{C}^m$ можно задать в виде

$$L_\omega = \{(z_1, \dots, z_m) \in \mathbb{C}^m \colon z_{i_1} = \dots = z_{i_k} = 0\}, \tag{3.1}$$

где $\omega = \{i_1, \dots, i_k\} \subset [m]$.

Для каждого симплициального комплекса $\mathcal{K}$ на множестве $[m]$ рассмотрим *конфигурацию комплексных координатных подпространств* $\mathcal{A}(\mathcal{K}) = \{L_\omega \colon \omega \notin \mathcal{K}\}$ и её *дополнение* в $\mathbb{C}^m$:

$$U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\omega \notin \mathcal{K}} L_\omega.$$

Сопоставление $\mathcal{K} \mapsto U(\mathcal{K})$ определяет взаимно однозначное соответствие между симплициальными комплексами на множестве $[m]$ и дополнениями координатных конфигураций в $\mathbb{C}^m$, сохраняющее отношение вложения.

Теорема 3.6. *Для любого симплициального комплекса $\mathcal{K}$ на множестве $[m]$ имеется T^m-эквивариантная деформационная ретракция*

$$\mathcal{Z}_\mathcal{K} \hookrightarrow U(\mathcal{K}) \xrightarrow{\simeq} \mathcal{Z}_\mathcal{K}.$$

Наличие гомотопической эквивалентности $U(\mathcal{K}) \simeq \mathcal{Z}_\mathcal{K}$ позволяет применять наши результаты о момент-угол комплексах в теории конфигураций. В частности, мы получаем решение известной задачи об описании кольца когомологий дополнения конфигурации координатных подпространств. Отметим, что другие известные результаты о когомологиях дополнений конфигураций координатных подпространств не описывают мультипликативной структуры (как общая теорема Горески–Макферсона [GM88]), либо дают лишь описание произведения двух данных коциклов в комбинаторных терминах (как результат де Лонгвилле [dL00]). Наш результат о момент-угол комплексах даёт исчерпывающее глобальное описание кольца когомологий дополнения конфигурации координатных подпространств. Результаты Горески–Макферсона (в части координатных конфигураций) и де Лонгвилле сводятся к частным случаям нашего результата при помощи двойственности Александера.

Пример 3.7. Пусть $\mathcal{K}$ представляет собой набор из m точек. Тогда $\mathcal{Z}_\mathcal{K}$ гомотопически эквивалентно дополнению

$$U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{1 \leqslant i < j \leqslant m} \{z_i = z_j = 0\}$$

всех координатных плоскостей коразмерности два. Кольцо граней имеет вид

$$\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \dots, v_m]/(v_i v_j,\ i \neq j).$$

Пространство коциклов в алгебре $\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}]$ имеет базис из мономов

$$v_{i_1} u_{i_2} u_{i_3} \cdots u_{i_k}, \quad k \geqslant 1 \text{ и } i_p \neq i_q \text{ при } p \neq q.$$

Пространство $(k{+}1)$-мерных кограниц порождено элементами вида $d(u_{i_1} \cdots u_{i_k})$. Вычисляя размерности этих пространств, получаем

$$\begin{aligned} &\dim H^0(\mathcal{Z}_\mathcal{K}) = 1, \quad \dim H^1(\mathcal{Z}_\mathcal{K}) = H^2(U(K)) = 0, \\ &\dim H^{k+1}(\mathcal{Z}_\mathcal{K}) = mC_{m-1}^{k-1} - C_m^k = (k-1)C_m^k, \quad 2 \leqslant k \leqslant m, \end{aligned}$$

а умножение в когомологиях дополнения $U(\mathcal{K})$ тривиально.

В частности, при $m = 3$ получаем

$$H^*(U(\mathcal{K})) \cong H^*(S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4),$$

и можно показать, что этот изоморфизм колец когомологий индуцирован гомотопической эквивалентностью пространств. Более того, дополнение $U(\mathcal{K})$ из этого примера гомотопически эквивалентно букету сфер для любого m [ГТ04].

4. Новые области приложений

Момент-угол комплексы нашли приложения в теории действий алгебраических групп, а именно, при построении *множеств типа Кемпфа–Несс* для действий алгебраического тора на квазиаффинных многообразиях. В классической ситуации действий алгебраических групп на аффинных многообразиях понятие множества Кемпфа–Несс позволяет заменить категорный фактор на факторпространство по действию максимальной компактной подгруппы. В [Па08] показано, что момент-угол комплекс $\mathcal{Z}_\mathcal{K}$ играет роль множества Кемпфа–Несс для класса действий алгебраического тора на квазиафинных многообразиях (дополнениях конфигураций координатных подпространств), возникающих в подходе Батырева–Кокса к торическим многообразиям на основе геометрической теории инвариантов. Таким образом, наши результаты о момент-угол комплексах применимы и к вычислению когомологий этих «торических» множеств Кемпфа–Несс. В случае неособых проективных торических многообразий соответствующие множества Кемпфа–Несс могут быть описаны как полные пересечения вещественных квадрик в комплексном пространстве.

Возвращаясь к нашему описанию момент-угол многообразий $\mathcal{Z}_P$ как полному пересечению вещественных квадрик, отметим область приложений, открытую в [BM06]. В этой работе был рассмотрен достаточно общий класс полных пересечений вещественных квадрик в $\mathbb{C}^m$, называемых *линками* (условия, накладываемые на уравнения квадрик обеспечивают неособость их пересечения). В [BM06] показано, что все линки допускают структуру некэлеровых комплексных многообразий (в случае линков нечётных размерностей необходимо взять произведение с окружностью), тем самым обобщая известные серии некэлеровых многообразий Хопфа и Калаби–Экмана. Можно показать, что класс линков совпадает с классом момент-угол многообразий $\mathcal{Z}_P$, соответствующих простым многогранникам. Тем самым открываются новые взаимосвязи между торической топологией и комплексной геометрией.

За последние 10 лет появились различные конструкции широкого класса простых многогранников, обобщающих замечательные серии пермутоэдров, ассоциэдров (многогранников Сташефа), циклоэдров (многогранников Бота–Таубса). Например, каждому связному простому графу (т.е. графу без петель и кратных ребер) с $(n+1)$ вершиной сопоставляется простой n-мерный многогранник, так что пермутоэдру соответствует полный граф, ассоциэдру — путь, а циклоэдру — цикл (см. например, [PRW07]).

Благодаря конструкции момент-угол многообразия $\mathcal{Z}_P$ и квазиторического многообразия в виде $M^{2n} = \mathcal{Z}_P/K(\Lambda)$ эти результаты позволили ввести явные примеры новых классов многообразий и бесспорно будут способствовать развитию взаимосвязей между комбинаторикой, теорией графов и алгебраической топологией (см. [Bu08]).

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Механико-математический факультет МГУ
E-mail address: buchstab@mi.ras.ru
E-mail address: tpanov@mech.math.msu.su

M. V. Burnashev

Institute for Information Transmission Problems
Russian Academy of Sciences

NOISY FEEDBACK IMPROVES CHANNEL RELIABILITY FUNCTION [1]

For the information transmission a binary symmetric channel is used. There is also another noisy binary symmetric channel (feedback channel), and the transmitter observes without delay all the outputs of the forward channel via that feedback channel. The transmission of a exponential number of messages (i.e. the transmission rate is positive) is considered. The achievable decoding error exponent for such a combination of channels is investigated. It is shown that if the crossover probability of the feedback channel is less than a certain positive value, then the achievable error exponent is better than the best known lower bound for the error exponent of the no-feedback channel.

Results can be generalized for a wide class of memoryless channels.

The binary symmetric channel BSC(p) with crossover probability $0 < p < 1/2$ (and $q = 1 - p$) is considered. It is assumed that there is the feedback BSC(p_1) channel, and the transmitter observes (without delay) all outputs of the forward BSC(p) channel via that noisy feedback channel. No coding is used in the feedback channel (i.e. the receiver simply re-transmits all received outputs to the transmitter). In words, the feedback channel is "passive".

The overall transmission time n and $M = e^{Rn}$ equiprobable messages $\{\theta_1, \ldots, \theta_M\}$ are given. After the moment n, the receiver makes a decision $\hat{\theta}$ on the message transmitted.

Denote $E(R, p)$ the best decoding error exponent of BSC(p) without feedback, and by $F(R, p, p_1)$ the best error exponent of BSC(p) with the noisy BSC(p_1) feedback channel.

Since the Shannon's paper [1] it has been known that even the noiseless feedback does not increase the capacity $C(p) = \ln 2 - h(p)$ of the BSC(p) (or any other memoryless channel). However, the feedback can improve the best error exponent $E(R, p)$. In the case of BSC with noiseless feedback (i.e. when $p_1 = 0$) investigations of the function $F(R, p) = F(R, p, 0)$ have been actively studied since Dobrushin [2], Horstein [3] and Berlekamp [4]. In particular, Dobrushin [2] has proved the "sphere-packing" upper bound for the function $F(R, p)$ (for a wide class of discrete channels). Some characteristics of a number of efficient transmission methods have been also investigated [1–10]. In particular, it was shown [4, 7] that $F(R, p) > E(R, p)$ for sufficiently small rates R.

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The case of noisy feedback was not investigated. It was not even known whether such feedback can improve the error exponent $E(R,p)$ of the no-feedback channel. The reason was that all known transmission methods in the noiseless feedback case [1–10] demanded an ideal mutual understanding (mutual coordination) between the transmitter and the receiver. If we try to apply any of the transmission methods from [1–10] to a noisy feedback case, we find that the transmitter and the receiver rather quickly loose their mutual coordination. As a result, it implies a very bad decoding error performance.

A certain progress has been done in the recent paper [10] (see also [11]), where the case of nonexponential (on n) number of messages M was considered (i.e. the transmission rate $R = 0$). It was shown that if the crossover probability p_1 of the feedback channel BSC(p_1) is less then the certain positive value $p_0(p)$, then $F(0,p,p_1) > E(0,p)$ (i.e. it is possible to improve the best error exponent $E(0,p)$ of BSC(p) without feedback). For the proof the transmission method with one "switching" moment was used.

In this paper we show that similar result holds for some positive rates R as well.

More exactly, combining Dobrushin [2] upper and Elias [5] lower bounds we get that $F(R,p) = E(R,p)$ for all rates $R_{\text{crit}}(p) \le R \le C(p)$, where

$$R_{\text{crit}}(p) = \ln 2 - h\left(\frac{\sqrt{p}}{\sqrt{p}+\sqrt{q}}\right).$$

Therefore, at most, we may have $F(R,p,p_1) > E(R,p)$ only when $R < R_{\text{crit}}(p)$.

T h e o r e m. *There exists a positive function* $p_0(R,p)$ *such that if* $R < R_{\text{crit}}(p)$ *and* $p_1 < p_0(R,p)$, *then* $F(R,p,p_1) > E(R,p)$.

Some estimates for the functions $p_0(R,p)$ and $F(R,p,p_1)$ will also be presented.

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Gibbs point field models for extraction problems in image analysis *

X. Descombes, E. Zhizhina

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1 Probabilistic approach in image analysis.

The basic idea of probabilistic approach in image analysis, see e.g. [1, 2], was to rewrite an image processing procedure in the language of statistical physics using concepts of statistical ensembles, equilibrium and non-equilibrium dynamics. Under this view, images are considered as configurations of a Gibbs field. The implicit assumption behind the probabilistic approach in image analysis is that, for a given problem, there exists a Gibbs field such that its ground states represent regularized solutions of the problem. Thus, the crucial step in the probabilistic approach is the choice of a proper configuration space and the choice of a distribution, or equivalently, in the case of the Gibbs random fields approach, the choice of an energy function $H(X)$. The energy function contains usually few types of terms. One of them arises from the observable image (a data driven term) and has the form of an external field term. Others are due to generic or prior knowledge on the structure of images. Prior terms in the energy function are specified by potentials associated with local interactions of neighboring variables. Thus, each variable directly depends only on its neighborhood, although from a global point of view, all variables are mutually dependent through the combination of successive local interactions.

Recently, there has been again growing interest in the applications of Gibbs point fields and Markov point processes to inverse problems of image processing such as feature extraction, object detection, surface reconstruction, stereo matching. All these problems related with consideration of strong geometrical constraints in a priori potential. In this paper we present a new multiple birth and death algorithm constructed as approximation of a stochastic Glauber type dynamics. We discuss results of its implementation on the example of two extraction problems.

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2 Gibbs fields models for feature extraction problems. The general setting.

We discuss here stochastic algorithms in the framework of the Gibbs fields approach for feature extraction problems. These problems become critical in remote sensing with the development of high resolution sensors for which the object geometry is well defined. Marked point processes framework is found very proper for extraction problems, since it is difficult to incorporate strong non-local geometrical constraints in the potential in lattice based models. Random sets of objects are represented in the models by marked point configurations in continuous space. Features of the objects, such as shape and/or size are described by a mark, and locations of the objects by a point configuration.

If we denote by $\Gamma := \Gamma(V)$ the set of all point configurations from a finite volume $V \subset R^2$, by S a space of marks (a spin space) and by π_z the Poisson measures with activity z, $z > 0$, then the marked configuration space $\hat{\Gamma}$ of the model is:

$$\hat{\Gamma} := \left\{ \hat{\gamma} = (\gamma, \sigma_\gamma), \gamma \in \Gamma,\ \sigma_\gamma = \{\sigma_x(\gamma)\}_{x\in\gamma} = \{\sigma_x\}_{x\in\gamma},\ \sigma_x \in S \right\}.$$

A reference measure μ_0 on $\hat{\Gamma}$ can be written as $d\mu_0(\hat{\gamma}) \ = \ d\omega(\sigma_\gamma)\ d\pi_z(\gamma)$, where $d\omega(\sigma_\gamma) = \prod_{x\in\gamma} d\omega(\sigma_x)$ is the conditional (under given configuration γ for positions of marks) free marks measure equals to the product of the free mark measures ω over all points from the configuration γ. The probability distribution on the configuration space $\hat{\Gamma}$ is defined then as a Gibbs reconstruction μ_β of the reference measure μ_0 with the energy function $H(\hat{\gamma})$ involving both objects positions and their marks. To find global minimizers of the energy function, one can consider various stochastic dynamics with a given stationary Gibbs measure under the annealing procedure.

Here we will discuss two models for extraction problems (a random disc model and a random point model), both of them can be described as pure point models without marks, in this case $\hat{\Gamma} = \Gamma$. We consider an equilibrium birth-and-death dynamics with the stationary Gibbs measure μ_β given by the following generator

$$(L_\beta\ f)(\gamma) \ = \ \sum_{x\in\gamma} e^{\beta E(x,\gamma\backslash x)}(f(\gamma\backslash x) - f(\gamma)) \ + \ z \int_V (f(\gamma \cup y) - f(\gamma))\ dy, \qquad (1)$$

defined in the functional space $f(\gamma) \in L^2(\Gamma, \mu_{\beta,z})$, where $E(x, \gamma\backslash x) \ = \ H(\gamma) - H(\gamma\backslash x)$.

3 Approximation process

In this section we present the mathematical background of our algorithm, which consists of two main steps:
1) the construction of the approximation process and the proof of the convergence of the approximation to the continuous time process as the discretization step tends to zero, and
2) the proof of the convergence of the corresponding evolution of measures under the annealing regime to a measure concentrated on the global minima of the energy function with a minimal number of points in the configuration.

We define a discrete time approximation $T_{\beta,\delta}(n), n = 0, 1, 2, \ldots$ of the continuous time birth-and-death process generated by (1). It is a Markov chain on the same space $\Gamma(V)$ with the transition operator $P_{\beta,\delta}$ ($T_{\beta,\delta}(n) = P^n_{\beta,\delta}$) of the form:

$$(P_{\beta,\delta} f)(\gamma) \;=\; \sum_{\gamma_1 \subseteq \gamma} \prod_{x \in \gamma_1} \frac{1}{1 + a_x \delta} \prod_{x \in \gamma \setminus \gamma_1} \frac{a_x \delta}{1 + a_x \delta} \tag{2}$$

$$\Xi_\delta^{-1} \sum_{k=0}^{\infty} \int\limits_{V^k} \frac{(z\delta)^k}{k!} f(\gamma_1 \cup y_1 \cup \ldots \cup y_k)\, dy_1 \ldots dy_k,$$

where $\Xi_\delta \;=\; \Xi_\delta(V, z, \delta)$ is the normalizing factor, $a_x = a_x(\gamma) = e^{\beta E(x, \gamma \setminus x)}$.

Let $\mathcal{L} = B(\Gamma(V))$ be a Banach space of bounded functions on $\Gamma(V)$ with a norm

$$\|F\| \;=\; \sup_{\gamma \in \Gamma(V)} |F(\gamma)|,$$

and by $\mathcal{B}$ we denote a family of measures with a bounded density w.r.t. the Poisson measure π_z (and hence also w.r.t. the Gibbs measure μ_β).

Theorem 3 (Convergence of the approximations) [3]. *For each $F \in \mathcal{L}$*

$$\|T_{\beta,\delta}(\left[\frac{t}{\delta}\right])F \;-\; T_\beta(t)F\|_{\mathcal{L}} \;=\; \sup_\gamma |(T_{\beta,\delta}(\left[\frac{t}{\delta}\right])F)(\gamma) \;-\; (T_\beta(t)F)(\gamma)| \;\to\; 0, \tag{3}$$

as $\delta \to 0$ for all $t \geq 0$ uniformly on bounded intervals of time.

Let $S_{\beta,\delta}(n)$ be an adjoint to $T_{\beta,\delta}(n)$ semigroup acting on measures, such that for any $\nu \in \mathcal{B}$:

$$\langle S_{\beta,\delta}(n)\nu, F\rangle = (p_\nu, T_{\beta,\delta}(n)F)_{\mu_\beta} \;\text{ with }\; p_\nu = \frac{d\nu}{d\mu_\beta}.$$

Let $n_0 \in N \cup \{0\}$ be the minimal number of points in configurations $\bar\gamma$ minimizing the energy function $H(\gamma)$. Then the Gibbs distributions μ_β converge weakly as $\beta \to \infty$ to a distribution μ_∞ on $\Gamma(V)$ of the form

$$\mu_\infty \;=\; \sum_{\bar\gamma : |\bar\gamma| = n_0} C_{\bar\gamma} \delta_{\bar\gamma} \text{ if } n_0 > 0, \;\text{ and }\; \mu_\infty \;=\; \delta_{\{\emptyset\}} \text{ if } n_0 = 0. \tag{4}$$

Here $\delta_{\bar\gamma}$ is the unit measure concentrated on the configuration $\bar\gamma$, and $\sum\limits_{\bar\gamma : |\bar\gamma| = n_0} C_{\bar\gamma} \;=\; 1$.

Theorem 4 (Convergence in the annealing regim) [3]. *Let $F \in B(\Gamma(V))$ and an initial measure $\nu \in \mathcal{B}$. Then under relation $\delta\, e^{\beta b} \;<\; const$ with $b = \sup_{\gamma \in \Gamma_d(V)} \sup_{x \in \gamma} E(x, \gamma \setminus x)$ we have*

$$\lim_{\beta \to \infty,\; t \to \infty,\; \delta \to 0} \langle F \rangle_{S_{\beta,\delta}(\left[\frac{t}{\delta}\right])\nu} \;=\; \langle F \rangle_{\mu_\infty}, \tag{5}$$

where $\langle F \rangle_{S_{\beta,\delta}(\left[\frac{t}{\delta}\right])\nu} \;=\; \langle S_{\beta,\delta}(\left[\frac{t}{\delta}\right])\nu, F\rangle$.

4 A new Multiple Birth and Death algorithm.

The main idea behind our algorithm is to use the continuous time stochastic dynamics generated by (1) and then to take the transition operator of the discrete time approximation process (2) as a base of stochastic iterative steps of the algorithm. The algorithm simulating the process is defined as follows:

- **Computation of the birth map:** To speed up the process, we consider instead of z a non homogeneous birth rate $B(s),\ s \in V$ to favor birth where the data term is strong. This non homogeneous birth rate refers to a non homogeneous reference Poisson measure.

- **Main program:** initialise the inverse temperature parameter $\beta = \beta_0$ and the discretization step $\delta = \delta_0$ and alternate birth and death steps. Birth step is taken with density $\delta B(s)$ w.r.t. the Lebesgue measure on V. Death step: for each point from the configuration, the death probability is defined as follows:

$$D(x) = \frac{\delta a_x}{1 + \delta a_x}.$$

  Decrease the temperature and the discretization step by a given factor and go back to the birth step.

5 Results

5.1 Application to birds detection

We consider a model of partially overlapping discs $\{d_{x_1}, \ldots, d_{x_k}\}$ of the same radius r with a hard core distance $\epsilon_0 < r$ between any two elements, lying in a bounded domain $V \subset R^2$. Then Γ is the configuration space of the centers of the discs. The energy function is a sum of data and a priori terms

$$H(\gamma) = \alpha \sum_{x \in \gamma} H_1(x) + \sum_{\{x,y\} \subset \gamma} H_2(x, y),$$

where α is a weighting parameter. The second term represents prior knowledge on the discs configuration and it is defined by pair interactions (repulsion on small distances) between neighboring discs. A data term is added for each object to fit the disc configuration onto the data, it is a sum of local energy functions associated with each object. For a given object, the local energy depends on a statistical test between the pixel distribution inside of the projection of the disc on the lattice and the pixel distribution in the neighborhood of the disc. The higher the contrast between the interior of the object and its neighboring ring, the lower the energy.

The fragment of the initial image of flamingo colony and the result image of detected birds are given on figure 1.

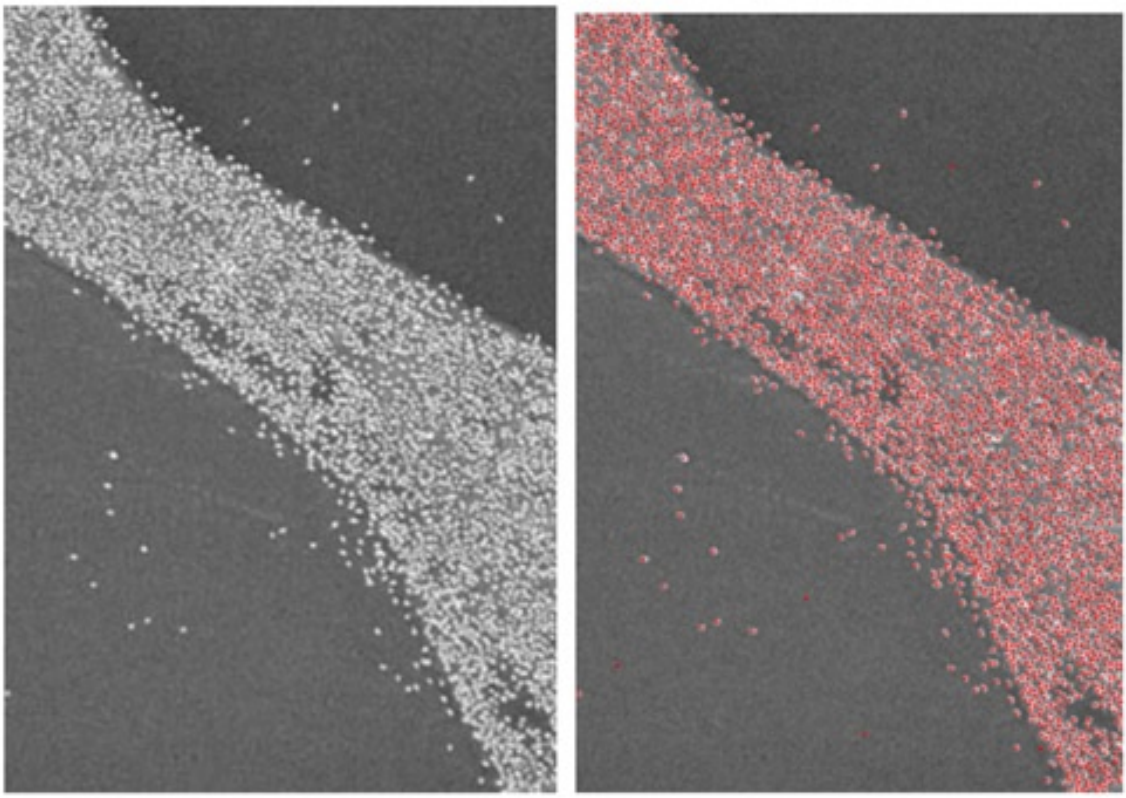

Figure 1: Fragment of the image of the bird population, Station Biologique Tour du Valat (left); image of detected birds (right)

5.2 Application to road network extraction problem

Different approaches for a fully automatic road network extraction from satellite images have been proposed recently, and Gibbs fields models among them. We present here a new model with simple objects (points) but with a complicated energy including the interaction energy of a special form. We propose here a new preprocessing procedure to extract from the input image a significant information in the form of another pixel-wise marked image $\mathcal{P}$, where each pixel p has two marks: an angle and a contrast. The value of contrast n_p describes how much pixel p is likely to belong to the road, and then angle θ_p defines a local direction of the road for relevant pixels. The results $\mathcal{P}$ of this preprocessing procedure is used in the data driven energy and in the interaction energy terms.

The energy function is a sum of three terms: the data term, a priori term and an interaction term. The prior knowledge models the high connectivity and the low curvature of a road network, the priori potential is used for reconstruction of hidden parts of the roads as well as for junction detection. The interaction energy is generated by pair potential depending on preprocessing image $\mathcal{P}$. It is a repulsive energy on short distances to prevent accumulation of points in the configuration and an attractive energy on fitting distances. In exceptional situations (junctions or parallel roads), that can be seen from the preprocessing image, the repulsive energy is vanishing. The data term contributes in the energy through a sum of local energy functions at each point of the configuration using preprocessing data. Then we exploit algorithms based on approximations of stochastic birth and death dynamics embeded into a simulated annealing regime, see Sect. 4 above.

Figure 2: Aerial image and obtained result (the point configuration)

6 Conclusion

Thus, the main advantages of marked point process in image analysis are in their geometrical adaptativity and generality. Any geometrical properties can easily be introduced into the model through the object geometry. Different types of objects (trees, roads, buildings, etc.) can be considered within the same model but with appropriate interactions. Moreover, interactions between points allows to model some prior information on the object configuration, and the data are taken into account at the object level, thus improving robustness of the algorithms.

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Combinatorial species and cluster expansions

William G. Faris
University of Arizona

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Abstract

This paper will survey recent progress on clarifying the connection between enumerative combinatorics and cluster expansions. The combinatorics side concerns species of combinatorial structures and the associated exponential generating functions. Cluster expansions, on the other hand, are supposed to give convergent expressions for measures on infinite dimensional spaces, such as those that occur in statistical mechanics. There is a kind of dictionary between these two subjects that sheds light on each of them. In particular, it gives insight into convergence results for cluster expansions.

1 Enumerative combinatorics

1.1 Combinatorial species

We begin with a general framework for constructing combinatorial structures in a systematic way. Each instance of such a construction is called a "species" of structures. The theory is explained in detail in the book of Bergeron, Labelle, and Leroux [1]; here we can only give an outline.

The combinatorial structures under consideration are built over colored sets. These are defined as follows. There is a set $\mathcal{P}$ that serves as a fixed palette of colors. A *colored set* is a function $a : U \to \mathcal{P}$, where U is a finite set. If j is a point in U, then $a(j)$ is the color of j. The colored sets form the objects of a category $\mathbf{B}$. The morphisms in this category are bijections of the underlying sets that preserve the colors.

In combinatorics the underlying set U is often called a set of labels, and the coloring is an additional structure that is imposed on a label set. We shall see that this structure also occurs in physics. In this case the interpretation of the set $\mathcal{P}$ is as a fixed set of locations. A set U is a set of particles, and a function $a : U \to \mathcal{P}$ is a particle configuration, that is, an assignment of particles to locations.

We need another category $\mathbf{E}$ with objects that consist of weighted sets. There is a fixed commutative ring R; for instance this could be the real numbers or the complex numbers. A *weighted set* is a finite set H together with a weight

function $\mathrm{wt} : H \to R$. A morphism of weighted sets is a bijection that preserves the weights. The basic requirement on the category is that the weight function behaves well on disjoint unions and on cartesian products. The weight function on a disjoint union must agree with the weights on the individual parts. The weight function on a cartesian product must assign to each ordered tuple the product of the weights of the components.

If H is a weighted set, then the total weight of H is the sum of the weights of the points in H. The total weight of a sum (disjoint union) is obviously the sum of the total weights of the parts. The total weight of a product (cartesian product) is the product of the total weights of the factors.

There are many examples of such categories, but the one of most use in the following is the category of sets of graphs over colored sets. First, one has a function t that assigns to each ordered pair of colors p, q in $\mathcal{P}$ an element $t(p, q)$ in the ring R. This function is fixed once and for all. It is required to be symmetric.

Consider a colored set $a : U \to \mathcal{P}$ with underlying set U. We think of U as a vertex set. Then a graph g with vertex set U is identified with a set of two-element subsets $\{i, j\}$ of U. These are the edges of the graph g. The weight of a graph g is

$$\mathrm{wt}(g) = \prod_{\{i,j\} \in g} t(a(i), a(j)). \tag{1}$$

An object in the category is a set of graphs g with fixed colored vertex set U. A morphism is a color preserving bijection of vertices that carries one set of graphs into another set of graphs.

In this category the product is constructed as follows. Say that for each U in some indexed family we have a set of graphs. We can take the various vertex sets U to be disjoint. Then an element of the product is a graph on the disjoint union of the sets U that comes from an indexed family of graphs on the individual parts.

In the physics application the palette of colors $\mathcal{P}$ represents a set of locations. The colored set $a : U \to \mathcal{P}$ represents a particle configuration. A graph g is a collection of two-element sets of particles that are regarded as interacting. The interaction between two particles i, j depends on their locations $a(i), a(j)$ and is given by $t(a(i), a(j))$. The interaction for the entire collection of pairs is the product of these individual pair interactions, that is, it is the weight of the graph. The total weight of a set of graphs is of course the sum of the weighs of the individual graphs.

A combinatorial *species* is a functor F from the category $\mathbf{B}$ of colored sets to the appropriate category $\mathbf{E}$ of weighted sets. Thus for every colored set $a : U \to \mathcal{P}$ there is a corresponding weighted set $F[a]$.

There are several species of interest to us. The species G associates to each colored set $a : U \to \mathcal{P}$ the set $G[a]$ of all $2^{\binom{n}{2}}$ graphs with vertex set U. The weight of a graph is as given above.

Even more important is the species C of connected graphs. A graph is connected if there exists a vertex i that is connected by a path to every other

vertex. (Then every vertex is connected by a path to every other vertex.) A connected graph on an n element vertex set has at least $n-1$ edges. The number of connected graphs in $C[a]$ is somewhat smaller but almost as large as the number of graphs in $G[a]$.

A final example in this series is the species T of trees. A graph is a tree if there is a vertex i that is connected by a unique path to every other vertex. (Then every vertex is connected by a unique path to every other vertex.) A tree on an n element vertex set has $n-1$ edges. It is a minimal connected graph. The number of trees in $T[a]$ is only n^{n-2}.

1.2 Operations on species

There are various important operations on species. If F is a species, then for each color p there is another species $F_p^\bullet$. Then $F_p^\bullet[a]$ for $a : U \to \mathcal{P}$ consists of all ordered pairs consisting of a point in U of color p and an element of $F[a]$. Thus this species incorporates a *distinguished point* of color p. As examples we have $G_p^\bullet$ and $C_p^\bullet$ and $T_p^\bullet$.

Another important operation is the *combinatorial exponential*. If F is a species, then $E \circ F$ is a new species. The value of this species on $a : U \to \mathcal{P}$ is given by

$$(E \circ F)[a] = \sum_{\Gamma} \prod_{V \in \Gamma} F[a_V]. \tag{2}$$

Here Γ ranges over partitions of U into disjoint non-empty sets V. The colored set $a_V : V \to \mathcal{P}$ is given by restriction. The sum is disjoint union, and the product is cartesian product.

One of the most famous examples of the combinatorial exponential is the relation

$$G = E \circ C. \tag{3}$$

This says that for every connected graph on U there is a partition Γ of U with a connected graph on each set V in the partition.

Another important operation is the *combinatorial convolution*. The convolution of two species F, G is defined by

$$(F * G)[a] = \sum_{V \subseteq U} F[a_V] \times G[a_{U \setminus V}]. \tag{4}$$

In other words, one splits the underlying set in all possible ways. This gives a disjoint union of a cartesian product corresponding to the two parts in the splitting.

Here is an example. We have

$$G_p^\bullet = C_p^\bullet * G. \tag{5}$$

This says that a graph together with a distinguished point of color p corresponds to a connected graph with a distinguished point of color p on a subset together with a graph on the complement.

1.3 Exponential generating functions

A key idea in the theory of species is the *exponential generating function*. A species is a functor F from the category of colored sets to the category of weighted sets. If a is a colored set, then we write the corresponding weighted set of combinatorial structures as $F[a]$. The sum of weights of $F[a]$ is written $f(a)$. The exponential generating function is a function of many variables, one for each possible color. Thus we use a variable w_p for each color $p \in \mathcal{P}$. For each n let U_n be a set with n points. The exponential generating function is written

$$F(w) = \sum_{n=0}^{\infty} \sum_{a:U_n \to \mathcal{P}} \frac{1}{n!} f(a) \prod_{i \in U_n} w_{a(i)}. \tag{6}$$

The operation of choosing a distinguished point has a simple expression in terms of exponential generating functions. It is

$$F_p^\bullet(w) = w_p \frac{\partial}{\partial w_p} F(w) \tag{7}$$

The operation of taking the combinatorial exponential is also simple; we have

$$(E \circ F)(w) = \exp(F(w)). \tag{8}$$

The combinatorial convolution is easy; in this case

$$(F * G)(w) = F(w)G(w). \tag{9}$$

As an example, note that $G(w) = \exp(C(w))$, and $G_p^\bullet(w) = w_p(\partial/\partial w_p)G(w) = C_p^\bullet(w)G(w)$, as one would expect from the combinatorial convolution.

1.4 Combinatorial fixed point equations

The next topic is combinatorial fixed point equations. One case where this is straightforward is for rooted trees. Let E_1^p be the species that indicates one point sets of color p. In other words, it produces a single point for each such set, and the empty set otherwise. The rooted tree equation is actually a family of equations, one for each color p. Thus we should think of $T_p^\bullet$ as a family of species. For each color p, one can construct a single species $T^{\bullet p}$ as follows. Take $T^{\bullet p}[a]$ to be the set of all trees (of whatever color) on the underlying set U, with the weight of each tree with root of color q multiplied by $t(p, q)$. Then the fixed point equation is

$$T_p^\bullet = E_{1p} * (E \circ T^{\bullet p}). \tag{10}$$

This says that a rooted tree with root of color p consists of a single point of color p, together with a structure on the complement of this point. This structure consists of a partition of the tree into disjoint non-empty sets V. On each set V in the partition there is a tree with a root of some color q. These trees get the additional weight $t(p, q)$.

On the level of exponential generating functions, the rooted tree equation is

$$T_p^\bullet(w) = w_p \exp(\sum_q t(p,q) T_q^\bullet(w)). \tag{11}$$

Unfortunately, the equations for rooted graphs and for rooted connected graphs are more complicated. The problem is that we need to designate entire subsets rather than individual points.

For the case of graphs, define $G^{\#p}[a]$ to be the set of ordered pairs consisting of a subset W of the underlying set U and a graph. The weight of this ordered pair is the weight of the graph times $\prod_{j \in W} t(p, a(j))$. The combinatorial equation is

$$G_p^\bullet = E_{1p} * G^{\#p}. \tag{12}$$

This says that every graph with a designated point of color p consists of a point of that color plus a graphical structure on the complement. This structure also must incorporate the edges that connect the designated point to some designated subset of the complement.

On the level of exponential generating functions, the rooted graph equation is

$$G_p^\bullet(w) = w_p G((1 + t_p)w). \tag{13}$$

Here $t_p w$ denotes the variables $t(p,q)w_p$, as q ranges over the colors.

For the case of connected graphs, define $C_+^{\#p}[a]$ to be the set of ordered pairs consisting of a non-empty subset W of the underlying set U and a connected graph. The weight of this ordered pair is the weight of the connected graph times $\prod_{j \in W} t(p, a(j)$. The combinatorial equation is

$$C_p^\bullet = E_{1p} * (E \circ C_+^{\#p}). \tag{14}$$

This says that every connected graph with a designated point of color p consists of a point of that color plus a structure on the complement. The complement is partitioned into disjoint non-empty subsets V. One each such subset V there is a connected graph. The structure also must incorporate the edges that connect the designated point to some designated non-empty subset of V.

On the level of exponential generating functions, the rooted connected graph equation is

$$C_p^\bullet(w) = w_p \exp(C((1 + t_p)w) - C(w)). \tag{15}$$

This may be converted into an equation for a rooted connected graph fixed point. We use

$$C_p^\bullet(w) = w_p \frac{\partial}{\partial w_p} C(w) \tag{16}$$

to get

$$C_p^\bullet(w) = w_p \exp(\sum_q t(p,q) \int_0^1 C_q^\bullet((1 + s t_p)w)\, ds) \tag{17}$$

Let us look at the rooted tree fixed point equation (11) in more detail. Take each vertex weight $w_p \geq 0$. Let the edge weights $0 \leq t(p,q)$ be positive. The tree fixed point equation for $z_p = T_p^\bullet(w)$ is

$$z_p = w_p \exp(\sum_q t(p,q) z_q). \tag{18}$$

The Kotecký-Preiss condition is that there exists $0 \leq x_p < \infty$ such that

$$w_p \exp(\sum_q t(p,q) x_q) \leq x_p. \tag{19}$$

The tree fixed point equation has a least finite solution z if and only if the Kotecký-Preiss condition holds. In that case $z_p \leq x_p$ for all p, This follows from the Knaster-Tarski fixed point theorem.

2 Cluster expansions

2.1 The equilibrium discrete particle gas model

In the application to a discrete particle gas the terminology is somewhat different. The color palette $\mathcal{P}$ is a fixed set of particle locations. The vertex set U is a finite set of particles. The colored set $a : U \to \mathcal{P}$ is a particle configuration. The color variable in the exponential generating function $0 \leq w_p$ is the weight for particles at $p \in \mathcal{P}$ (the activity). Finally, there is a quantity $0 \leq 1 + t(p,q) \leq 1$ which is the Boltzmann factor for pair of particles at locations $p, q \in \mathcal{P}$. Thus $-1 \leq t(p,q) \leq 0$ is a measure of the interaction between the pair of particles.

The grand partition function

$$G(w) = \sum_{n=0}^{\infty} \sum_{a:U_n \to \mathcal{P}} \frac{1}{n!} g(a) \prod_{i \in U_n} w_{a(i)} \tag{20}$$

is the exponential generating function for graphs.

A particularly convenient quantity for convergence results is the density (expected number of particles at a location). The density at p is

$$n(p) = \frac{1}{G(w)} \, w_p \frac{\partial}{\partial w_p} G(w) = \frac{1}{G(w)} G_p^\bullet(w) = C_p^\bullet(w). \tag{21}$$

Thus the density $n(p)$ at p of the gas, regarded as a function of the local activity variables w, is the exponential generating function $C_p^\bullet(w)$ for rooted connected graphs with root of color p. This fundamental relation is at the heart of statistical physics.

2.2 Fixed points and convergence in the gas model

Here are two cluster expansion theorems. The first is a classic result; the second is relatively recent. See [3, 2] for references for these and earlier results.

Cluster expansion theorem (Kotecký-Preiss version). If there are $x_p \geq 0$ such that for each p we have the inequality

$$w_p \exp(\sum_q |t(p,q)|x_q) \leq x_p, \tag{22}$$

then the cluster expansion for the density in powers of w converges absolutely.

This theorem may be understand via a comparison of the rooted tree fixed point equation (11) with the rooted connected graph fixed point equation (17).

Cluster expansion theorem (Fernández-Procacci version). If there are $x_p \geq 0$ such that for each p we have the inequality

$$w_p \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n \to \mathcal{P}} \left[\prod_{\{i,j\}} (1 + t(a(i), a(j))) \right] \prod_{i \in U_n} \left(|t(p, a(i))| x_{a(i)} \right) \leq x_p, \tag{23}$$

then the cluster expansion for the density in powers of w converges absolutely.

The Fernández-Procacci result [3] is stronger. The factors $1 + t(a(i), a(j))$ are between 0 and 1. If one drops the product with these factors, the expression in the condition can only become larger. The result is

$$w_p \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n \to \mathcal{P}} \prod_{i \in U_n} \left(|t(p, a(i))| x_{a(i)} \right) = w_p \exp(\sum_q |t(p,q)|x_q). \tag{24}$$

Thus the iteration function in the Fernández-Procacci condition is majorized by the iteration function in the Kotecký-Preiss condition.

The combinatorial interpretation of the Fernández-Procacci condition is in terms of an enriched rooted tree fixed point equation. See [2] for a proof of their result based on an identity that relates connected graphs and trees.

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On the limit law of a conditioned random walk

Sergey G. Foss and Anatolii A. Puhalskii

Let $\xi_1, \xi_2, \dots$ be i.i.d. random variables with negative mean defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Suppose that $\mathbf{E}\exp(\lambda\xi_1) < \infty$ for some $\lambda > 0$ and that there exists $\gamma > 0$ with $\mathbf{E}\exp(\gamma\xi_1) = 1$. It is known that if, in addition, $\mathbf{E}\,\xi_1\exp(\gamma\xi_1) < \infty$, then the most likely way for the random walk $S_k = \sum_{i=1}^k \xi_i$ to reach a high level is to follow a straight line with a positive slope. We study the case where $\mathbf{E}\,\xi_1\exp(\gamma\xi_1) = \infty$. Assuming that the distribution $F(dx) = \exp(\gamma x)\mathbf{P}(\xi_1 \in dx)$ belongs to the domain of attraction of a spectrally positive stable law, we obtain a weak convergence limit theorem as $r \to \infty$ for the conditional distribution of the process $\big(r^{-1}\sum_{i=1}^{\lfloor t/(1-F(r))\rfloor}\xi_i,\, t \ge 0\big)$ stopped at the time when it reaches level 1 given that the latter event occurs. The limit is an increasing jump process. It is shown to be distributed as an increasing stable Lévy process stopped at the time when it reaches level 1 conditioned on the event this level is not overshot.

We now state the main results. Let $\alpha \in (0,1)$.

Theorem 1. *There exists a stochastic process $\widetilde{X} = (\widetilde{X}(t)\,, t \in \mathbb{R}_+)$ defined on a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{F}}, \widetilde{\mathbf{P}})$ with the following properties:*

1. *$\widetilde{X}$ is a pure-jump semimartingale with $\widetilde{X}(0) = 0$,*
2. *the $\widetilde{\mathbf{F}}$-predictable measure of jumps of $\widetilde{X}$ is of the form*

$$\nu([0,t], G) = \int_0^t \int_{G\setminus\{0\}} \mathbf{1}_{\{0<x<1-\tilde{X}(s)\}}\Big(1 - \frac{x}{1-\tilde{X}(s)}\Big)^{\alpha-1}\alpha x^{-\alpha-1}\,dx\,ds\,,\ G \in \mathcal{B}(\mathbb{R}).$$

.

The distribution of $\widetilde{X}$ is specified uniquely. In addition, $\widetilde{X}$ has increasing trajectories a.s., $\widetilde{X}(t) \in [0,1]$ a.s. for $t \in \mathbb{R}_+$, and $\widetilde{X}(t) = 1$ for all t large enough a.s.

Let $X = (X(t)\,, t \in \mathbb{R}_+)$ represent an increasing pure-jump stable Lévy process starting at zero with Lévy measure $\alpha x^{-\alpha-1}\,dx$. We also denote

$$\tau = \inf\{t:\, X(t) \ge 1\},$$

and let $\widehat{X}$ denote the process X stopped at τ: $\widehat{X}(t) = X(t\wedge\tau)$.

Theorem 2. *The conditional laws of $\widehat{X}$ given the events $X(\tau) \le 1+\epsilon$ weakly converge as $\epsilon \downarrow 0$ to the law of $\widetilde{X}$.*

Let, for $r > 0$,

$$\tau^{(r)} = \min\{n:\, S_n \ge r\},$$
$$X^{(r)}(t) = \frac{1}{r}\sum_{i=1}^{\lfloor t/(1-F(r))\rfloor}\xi_i,$$
$$\widehat{\tau}^{(r)} = \inf\{t:\, X^{(r)}(t) \ge 1\}.$$

We denote by $\widehat{X}^{(r)} = (\widehat{X}^{(r)}(t)\,, t \in \mathbb{R}_+)$ the process $X^{(r)}$ stopped at $\widehat{\tau}^{(r)}$, i.e., $\widehat{X}^{(r)}(t) = X^{(r)}(t \wedge \widehat{\tau}^{(r)})\,.$

Theorem 3. *Let the following conditions hold:*

1. *the righthand tail of the distribution function F is regularly varying at infinity with index $-\alpha$, where $\alpha \in (1/2, 1)$,*
2. *there exist $C > 0$ and $\rho \in (0, 1)$ such that, for all y great enough and all $x \in (\rho, 1)$,*

$$\frac{1 - F(yx)}{1 - F(y)} \leq 1 + C(1 - x)\,.$$

If, in addition, F is a nonlattice distribution, then, as $r \to \infty$, the conditional distributions of the $\widehat{X}^{(r)}$ given $\tau^{(r)} < \infty$ weakly converge to the distribution of $\widetilde{X}$. If, instead, F is a lattice distribution with span h, then, as $n \to \infty$, where $n \in \mathbb{N}$, the conditional distributions of the $\widehat{X}^{(nh)}$ given $\tau^{(nh)} < \infty$ weakly converge to the distribution of $\widetilde{X}$.

On Sums of Conditionally Independent Subexponential Random Variables

Serguei Foss[1,2] and Andrew Richards[2]

Finding the asymptotic tail behaviour of sums of heavy-tailed random variables is an important problem in finance, insurance and many other disciplines. The case when the random variables are independent and subexponentially distributed has been extensively studied and is well-understood. The key idea is that such a sum will exceed a high threshold because of a single, very large jump; following other authors we shall refer to this as *the principle of the single big jump*. However, for many practical purposes the independence assumption is too restrictive. In recent years, many authors have developed results in this area (see, for example, [1,2,4,9-13] and references therein). Denuit, Genest and Marceau [6] constructed bounds for these sums, but did not consider asymptotics. Goovaerts, Kaas, Tang and Vernic [8] considered the situation of dependent random variables with regularly varying tails; there have also been results on negative dependence for various classes of subexponential distributions.

Once we drop the requirement of independence, two questions naturally arise. First, *what kind of behaviours can occur as the dependence between the random variables strengthens?* And secondly, *how far beyond the independent case does the principle of the single big jump still hold?* These questions are of real interest, both from theoretical and practical viewpoints.

Albrecher, Asmussen and Kortschak [1] consider the first question for the sum of two dependent random variables. Their approach, as for many authors, is to study the possible effects of the dependence by considering the copula structure. They demonstrate that many possible behaviours naturally occur, and that, in some specific cases the principle of the single big jump is insensitive to the strength of the copula structure. Other papers that concentrate on the copula structure include [2, 10]. Mitra and Resnick [13] investigate random variables belonging to the maximum domain of attraction of the Gumbel distribution and which are asymptotically independent. The results we present contain overlap with all these approaches, but we neither impose a particular dependence structure, nor a particular distribution for the random variables, beyond the necessary constraint that at least one be subexponential.

We consider the second question, and to establish conditions on the strength of the dependence which will preserve the results of the theory established for independent random variables; in particular, the principle of the single big jump. This principle is well known. However, we would like to examine it again from a probabilistic point of view by considering the sum of two identically distributed subexponential random variables X_1, X_2.

$$\begin{aligned}
&\mathbf{P}(X_1 + X_2 > x) = \mathbf{P}(X_1 \vee X_2 > x) + \mathbf{P}(X_1 \vee X_2 \leq x, X_1 + X_2 > x) \\
&= \mathbf{P}(X_1 > x) + \mathbf{P}(X_2 > x) - \mathbf{P}(X_1 \wedge X_2 > x) + \mathbf{P}(X_1 \vee X_2 \leq x, X_1 + X_2 > x) \\
&:= \mathbf{P}(X_1 > x) + \mathbf{P}(X_2 > x) - P_2(x) + P_1(x), \qquad (1)
\end{aligned}$$

where $X_1 \vee X_2 = \max(X_1, X_2)$ and $X_1 \wedge X_2 = \min(X_1, X_2)$. If $P_1(x)$ is negligible compared to $\mathbf{P}(X_1 > x)$, which in the independent case follows from the definition of subexponentiality, we

[1]Institute of Mathematics, Novosibirsk, Russia. Email: foss@math.nsc.ru

[2]School of Mathematics and Computer Sciences and the Maxwell Institute for Mathematical Sciences, Heriot-Watt University, Edinburgh, EH14 4AS, Scotland, UK. E-mail: S.Foss@hw.ac.uk and awr2@hw.ac.uk

shall say that we have the *principle of the big jump*. If in addition $P_2(x)$ is negligible compared to $\mathbf{P}(X_1 > x)$, as again is straightforward in the independent case, then we shall say that we have the *principle of the* **single** *big jump*. If the dependence is very strong, for instance if $X_1 = X_2$ a.s. (almost surely), then clearly the principle of the single big jump fails.

We consider sums of random variables that are conditionally independent on some sigma algebra. This is a fresh approach to studying the effect of dependence on subexponential sums and allows a great deal of generality (in particular, we need neither specify a particular subclass of subexponential distribution for which our results hold, nor assume the summands are identically distributed, nor specify any particular copula structure). We believe this is a fruitful line of enquiry, both practically and theoretically, as the range of examples we give illustrates.

Clearly, any sequence of random variables can be considered to be conditionally independent by choosing an appropriate sigma algebra on which to condition. This is an obvious observation, and in itself not really helpful. However, there are practical situations where a conditional independence structure arises naturally from the problem. As an example, consider a sequence of identical random variables $X_1, X_2, \ldots, X_n$, each with distribution function F_β depending on some parameter β that is itself drawn from a different distribution. The X_i are independent once β is known: this is a typically Bayesian situation. It is natural to view the X_i as conditionally independent on the sigma algebra generated by β. We suppose the X_i to have subexponential (unconditional) distribution F and ask under what conditions the distribution of the sum follows the principle of the single big jump.

A distribution function F supported on the positive half-line is subexponential if and only if

$$\overline{F^{*2}}(x) := \int_0^x \overline{F}(x-y)F(dy) + \overline{F}(x) \sim 2\overline{F}(x).$$

It is known, and may be easily checked, that a distribution supported on the positive half-line is subexponential if and only if the following two conditions are met:

1. F is long-tailed. That is, there exists a non decreasing function $h(x) < x/2$, tending to infinity, such that $\overline{F}(x + h(x)) \sim \overline{F}(x)$, $x \to \infty$. So, F is *h-insensitive*.
2. For such $h(x)$,
$$\int_{h(x)}^{x-h(x)} \overline{F}(x-y)F(dy) = o(\overline{F}(x)).$$

We work in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let X_i, $i = 1, 2, \ldots$, be non-negative random variables with distribution function (d.f.) F_i. Let F be a subexponential reference distribution concentrated on the positive half-line and h be a function satisfying the long-tailed condition. Let $\mathcal{G}$ be a σ-algebra, $\mathcal{G} \subset \mathcal{F}$. We make the following assumptions about the dependence structure of the X_i's:

(D1) $X_1, X_2, \ldots$ are conditionally independent given $\mathcal{G}$. That is, for any collection of indices $\{i_1, \ldots, i_r\}$, and any collection of sets $\{B_{i_1}, \ldots, B_{i_r}\}$, all belonging to $\mathcal{F}$, then $\mathbf{P}(X_{i_1} \in B_{i_1}, \ldots, X_{i_r} \in B_{i_r}|\mathcal{G})) = \mathbf{P}(X_{i_1} \in B_{i_1}|\mathcal{G})\mathbf{P}(X_{i_2} \in B_{i_2}|\mathcal{G}) \ldots \mathbf{P}(X_{i_r} \in B_{i_r}|\mathcal{G})$.

(D2) For each $i \geq 1$, $\overline{F}_i(x) \sim c_i\overline{F}(x)$, with at least one $c_i \neq 0$, and for all $i \geq 1$ there exists $c > 0$ and $x_0 > 0$ such that $\overline{F}_i(x) \leq c\overline{F}(x)$ for all $x > x_0$.

(D3) For each $i \geq 1$ there exists a non-decreasing functions $r(x)$ and an increasing collection of sets $B_i(x) \in \mathcal{G}$, with $B_i(x) \to \Omega$ as $x \to \infty$, such that

$$\mathbf{P}(X_i > x|\mathcal{G})\mathbf{1}(B_i(x)) \leq r(x)\overline{F}(x)\mathbf{1}(B_i(x)) \quad \text{almost surely.} \tag{2}$$

and, as $x \to \infty$, uniformly in i,

(i)

$$\mathbf{P}(\overline{B}_i(h(x))) = o(\overline{F}(x)) \tag{3}$$

(ii)

$$r(x)\overline{F}(h(x)) = o(1), \tag{4}$$

(iii)

$$r(x)\int_{h(x)}^{x-h(x)} \overline{F}(x-y)F(dy) = o(\overline{F}(x)). \tag{5}$$

We have the following result.

Proposition. Let X_i, $i = 1, 2, \ldots$ satisfy conditions (D1), (D2) and (D3) for some subexponential F concentrated on the positive half-line and which is h-insensitive. Let τ be an independent counting random variable such that $\mathbf{E}(e^{\gamma\tau}) < \infty$ for some $\gamma > 0$ Then

$$\mathbf{P}(X_1 + \cdots + X_\tau > x) \sim \mathbf{E}\left(\sum_{i=1}^{\tau} c_i\right)\overline{F}(x).$$

We formulate further results and give a wide range of examples of collections of random variables, some satisfying the principle of the single big jump, some not, and we suggest that these examples are of independent interest in and of themselves. We also discuss a number of related problems.

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MICROSCOPIC THEORY OF ISOTHERMAL ELASTICITY: HYPERBOLIC SCALING BEYOND SHOCKS VIA COMPENSATED COMPACTNESS. DEDICATED TO THE MEMORY OF ROLAND LVOVICH DOBRUSHIN

JÓZSEF FRITZ, TU BUDAPEST

Historical Notes and Remarks: The mathematical theory of hydrodynamic limits has been initiated by Roland Lvovich Dobrushin and co-workers [1,2,4,5]. As motivated by the principles of statistical physics, in the first period hyperbolic scaling of some simple mechanical systems (hard rods and harmonic oscillators) was investigated; the study of deterministic models with a more realistic interaction is still out of question. Results of H. Rost [16] and F. Rezakhanlou [15] are based on the specific structure of attractive, one-component systems; models with two conservation laws do not allow an effective coupling. In contrast to diffusive scaling, in the case of hyperbolic problems a direct strong compactness argument, the Two Blocks Lemma of Guo - Papanicolau - Varadhan [13] is not available because entropy production does not vanish. Assuming smoothness of the macroscopic solution, the method of H.-T. Yau [20] works in fairly general situations, but in a regime of shocks a synthesis of probabilistic and advanced PDE techniques is required. As far as I understand, *compensated compactness* is the only tool that works also in the case of microscopic systems with two conservation laws. Unfortunately, it is restricted to one space dimension, and uniqueness of the limit is a formidable open problem. Of course, verification of the *strong ergodic hypothesis,* which means a description of all translation invariant stationary states of the microscopic system, can not be avoided. That is why we consider random perturbations of a Hamiltonian dynamics, namely those of the *anharmonic chain.*

We are going to derive the following couple of conservation laws,

$$\partial_t \pi(t,x) = \partial_x S'(\rho(t,x))\,, \qquad \partial_t \rho(t,x) = \partial_x \pi(t,x) \tag{1}$$

for $\pi, \rho \in \mathbb{R}$, $t \geq 0$, $x \in \mathbb{R}$. This *p-system* has a direct physical interpretation: π and ρ are the velocity and deformation (strain) of an elastic medium in a thermal equilibrium with total energy $\chi := \pi^2/2 + S(\rho)$. In our case the stress $S' : \mathbb{R} \mapsto \mathbb{R}$ is the derivative of a smooth convex function, gas dynamics (in Lagrangian coordinates) is obtained when S has a singularity at zero. This second, most interesting problem is out of the range of our tools.

The Anharmonic Chain: The Hamiltonian of coupled oscillators on $\mathbb{Z}$ reads as

$$H(\omega) := \frac{1}{2} \sum_{k \in \mathbb{Z}} \left(p_k^2 + V(q_{k+1} - q_k) + V(q_{k-1} - q_k) \right)\,,$$

where $p_k\,, q_k \in \mathbb{R}$ are the momentum and position at site $k \in \mathbb{Z}$, $\omega := (p_k, q_k)_{k \in \mathbb{Z}}$. In terms of the deformation $r_k := q_{k+1} - q_k$, the equations of motion read as

$$\dot{p_k} = V'(r_k) - V'(r_{k-1})\,, \qquad \dot{r_k} = p_{k+1} - p_k\,, \quad k \in \mathbb{Z}\,, \tag{2}$$

where $V(y) = y^2/2 + U(y)$ such that U, U', U'' are bounded; this is the condition of the logarithmic Sobolev inequality of [14].

Since H, $P := \sum p_k$ and $R := \sum r_k$ are all preserved by the evolution, we have a three-parameter family $\lambda_{\beta,\pi,\gamma,}$ of translation invariant stationary product measures with local densities $\exp\left(-\beta(p_k - \pi)^2/2 - \beta V(r_k) + \gamma r_k - F(\beta,\gamma)\right)$, where $\beta > 0$ denotes the inverse temperature, $\pi, \gamma \in \mathbb{R}$, and F is the normalization.

Although (2) is a direct lattice approximation of the p-system with $S = V$, convergence does not take place because of several reasons. We have (at least) three conservation laws, therefore a triplet, the compressible Euler equations are expected to govern macroscopic behavior. In the paper [5] by R. L. Dobrushin and coworkers a full description of stationary states and that of the associated conserved quantities of the harmonic chain are given. It turned out that there is a huge class of extra stationary measures and conservation laws, therefore hydrodynamic limit of the harmonic chain results in a continuum of macroscopic equations. The anharmonic chain is much more difficult, there is no real hope to verify any version of the ergodic hypothesis. Moreover, an approximation scheme like (2) is not a stable one, it has to be regularized somehow, see [17].

The Small Viscosity Limit: Let $u(t,x) := (v, \rho)$ and $f := -(S', v)$, then (1) becomes $\partial_t u + \partial_x f(u) = 0$, and its viscous approximation reads as

$$\partial_t u_\sigma + \partial_x f(u_\sigma) = \sigma \partial_x^2 u_\sigma \,, \tag{3}$$

where $\sigma \geq 0$ may be a matrix, too. The small viscosity limit, i.e. $\sigma \to 0$ is a popular, although not the most powerful approximation scheme. In many cases it is possible to show that, at least along subsequences, u_σ converges to a weak solution, a locally integrable function satisfying

$$\int_0^\infty \int_{-\infty}^\infty \left(\psi_t' \cdot u + \psi_x' \cdot f(u)\right)\, dx\, dt + \int_{-\infty}^\infty \psi(0,x) \cdot u(0,x)\, dx = 0$$

for all compactly supported test functions $\psi : \mathbb{R}^2 \mapsto \mathbb{R}^2$. Usually (3) admits bounded, *positively invariant regions* implying existence of bounded solutions for bounded initial values, see the pioneering paper [3] by R. DiPerna. Ten years later J. Shearer [19] and Serre - Shearer [18] managed to prove existence of L^p solutions for $p < 2$ in this way.

Discretized versions of (3) are also available, but the regular stationary states are killed by such numerical schemes: viscosity results in a relaxation to an evolution at temperature zero. Random perturbations have to be used to get a fully developed hydrodynamic behavior. Then the microscopic evolution is generated by an operator $\mathcal{L} = \mathcal{L}_0 + \sigma \mathcal{S}$, where $\mathcal{L}_0$ is the Liouville operator of the Hamiltonian part (2) of the process, while $\mathcal{S}$ is symmetric (reversible) with respect to the preferred equilibrium states. For example, we can do random exchange of velocities across neighboring sites such that all actions are independent of each other. This mechanism admits three conservation laws, thus the product measures $\lambda_{\beta,\gamma,z}$ are all stationary states.

The Problem of Stationary States: The anharmonic chain with physical viscosity belongs to the Ginzburg - Landau category, it is given by the following set of stochastic differential equations:

$$\begin{aligned} dp_k = (V'(r_k) - V'(r_{k-1}))\, dt + \sigma\,(p_{k+1} + p_{k-1} - 2p_k)\, dt \\ + \sqrt{2\sigma}\,(dw_k - dw_{k-1})\,, \quad dr_k = (p_{k+1} - p_k)\, dt\,, \end{aligned}$$

where $\sigma > 0$ and $\{w_k\,, k \in \mathbb{Z}\}$ is a family of independent Wiener processes. Total energy is not preserved any more because a thermal equilibrium is maintained by the noise. The product measures $\lambda_{\pi,\gamma} := \lambda_{1,\pi,\gamma}$ are stationary, and a converse statement is also true [8]. Indeed, assuming that V is quadratically bounded and convex at infinity, we prove that every translation invariant stationary measure of finite specific entropy is a superposition of such product measures. The normalization (free energy) reads simply as

$$F(\gamma) := \log \int_{-\infty}^{\infty} \exp(\gamma x - V(x))\, dx\,,$$

some expectations are calculated as $\lambda_{\pi,\gamma}(p_k) = \pi\,,\ \lambda_{\pi,\gamma}(r_k) = F'(z) = \rho\,,$ and $\lambda_{\pi,\gamma}(V'(r_k)) = \gamma\,,$ while $S(\rho) := \sup_\gamma\{\gamma\rho - F(z)\}\,.$ Since $\gamma = S'(\rho) = \lambda_\rho(V')$ if $\rho := \lambda_\rho(r_k) = F'(\gamma)\,,$ we expect that the p-system (1) is governing macroscopic behavior of the model. Indeed, the relative entropy method of H.-T. Yau [20] applies when the macroscopic solution is a classical one. Similarly, in the case of random exchange of velocities, the method of [7] yields a description of translation invariant stationary measures, thus following Yau, we can derive the set of compressible Euler equations in a smooth regime.

Main Result: Since no version of the model is attractive, in a regime of shocks a very strong artificial viscosity should be added to the equations of motion. We consider a Ginzburg - Landau type stochastic system mimicking the viscous approximation.

$$\begin{aligned} dp_k &= (V'(r_k) - V'(r_{k-1}))\, dt + \sigma(\varepsilon)\,(p_{k+1} + p_{k-1} - 2p_k)\, dt \\ &\quad + \sqrt{2\sigma(\varepsilon)}\,(dw_k - dw_{k-1})\,, \\ dr_k &= (p_{k+1} - p_k)\, dt + \sigma(\varepsilon)\,(V'(r_{k+1}) + V'(r_{k-1}) - 2V'(r_k))\, dt \\ &\quad + \sqrt{2\sigma(\varepsilon)}\,(d\tilde{w}_{k+1} - d\tilde{w}_k) \end{aligned}$$

where $\{w_k\}$ and $\{\tilde{w}_k\}$ are independent families of independent Wiener processes. The condition $\varepsilon\sigma(\varepsilon) \to 0$ is natural, $\varepsilon\sigma^2(\varepsilon) \to +\infty$ is needed to suppress extreme fluctuations in the system. Conservation of total energy is violated by the noise, thus $\lambda_{\pi,\gamma}\,,\ \pi, \gamma \in \mathbb{R}$ is the family of stationary product measures.

At a level $\varepsilon > 0$ of scaling, $\mu_{0,\varepsilon}$ is the initial distribution, and $M_{n,\varepsilon}$ denotes the joint density of the variables $\omega^{(n)} := \{(p_k(0), r_k(0)) : |k| \le n\}$ with respect to $\lambda := \lambda_{0,0}\,.$ Since we can not prove uniqueness of the hydrodynamic limit, our main hypothesis on the initial distribution is an entropy bound:

$$S_n(\mu_{0,\varepsilon}|\lambda) := \int M_{n,\varepsilon} \log M_{n,\varepsilon}\, d\lambda \le Cn \tag{4}$$

for all $\varepsilon > 0$ and $n \in \mathbb{N}$ with the same constant $C\,.$ Under this condition every translation invariant stationary state is a superposition of product measures $\lambda_{\gamma,z}\,.$

The empirical process, $u_\varepsilon(t,x) = (\pi_\varepsilon, \rho_\varepsilon)$ is now defined by $\pi_\varepsilon(t,x) := p_k(t/\varepsilon)$ and $\rho_\varepsilon(t) := r_k(t/\varepsilon)$ if $|\varepsilon k - x| < \varepsilon/2\,;$ P_ε denotes its distribution. We interpret u_ε as Lebesgue density of a measure. The construction of the effective, slowly increasing entropy pairs of Shearer [19] and Serre [18] requires an implicit condition on the macroscopic flux: S''' has at most one root. Of course, $S'''(0) = 0$ if V is symmetric.
Theorem: P_ε is a tight family, and its limit distributions are concentrated on a set of weak solutions to the p-system.

The notion of weak convergence above changes from step to step of the argument. We start with the Young measure of the block-averaged process, and at the end we get strong convergence in the local $L^p(\mathbb{R}^2_+)$ space for $p < 2$.

Energy, and Entropy Pairs of the P-system: Additional conservation laws play a crucial role in the study of hyperbolic systems. For example, total energy $H := \int (v^2/2 + S(\rho))\, dx$ is constant along classical solutions to (1), and

$$\partial_t H(u) = \sigma \int (\pi \partial_x^2 \pi + S'(\rho)\partial_x^2 \rho)\, dx = -\sigma \int \left((\partial_x \pi)^2 + S''(\rho)(\partial_x \rho)^2\right)\, dx\,.$$

for viscid solutions. More generally, a couple $\{h(u), J(u)\}$ is called a Lax entropy pair if formally $\partial_t h + \partial_x J = 0$, i.e. $\nabla J = \nabla h f'$, where f' is the Jacobian of the flux. $J = -\pi S'(\rho)$ in the previous case of $h = H$, in general $J'_\pi(\pi, \rho) = -h'_\rho(\pi, \rho)$ and $J'_\rho(\pi, \rho) = -S''(\rho)h'_\pi(\pi, \rho)$. In the viscous approximation we have

$$\partial_t h(u) = \sigma \partial_x^2 h(u) - \sigma \left(h''_{\pi\pi}(u)(\partial_x^2 \pi)^2 + h''_{\rho\rho}(u)(\partial_x^2 \rho)^2 + 2h''_{\pi\rho}\partial_x \pi \partial_x \rho\right)\,. \qquad (5)$$

If h is convex then there is a negative term on the right hand side, but $\sigma \to 0$, thus we have no control of $\partial_x u$. Something else: Compensated Compactness is needed. The microscopic picture is similar but more complicated, the famous two-blocks estimate is the missing information in that case. Application of compensated compactness to stochastic systems has been proposed in [8], where asymmetric exclusions are discussed in details, some more interesting examples are treated in the papers [9,10,11].

The Microscopic Frame: The microscopic dynamics does not admit additional conservation laws, entropy pairs can only be recovered in terms of block averages $\hat{p}_{l,k}$ and $\hat{r}_{l,k}$ of size $l = l(\varepsilon)$ such that

$$\lim_{\varepsilon \to 0} \frac{l(\varepsilon)}{\sigma(\varepsilon)} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\varepsilon l^3(\varepsilon)}{\sigma(\varepsilon)} = +\infty\,.$$

Because of some technical reasons, besides the traditional arithmetic means $\bar{\xi}_{l,k}$, we introduce also the more smooth block averages

$$\hat{\xi}_{l,k} := \frac{1}{l^2} \sum_{j=-l}^{l} (l - |j|)\, \xi_{k+j} \quad \text{for} \quad \xi_k = p_k \text{ or } \xi_k = r_k\,.$$

As an example, cancelation of oscillations in case of total energy is demonstrated as follows:

$$\begin{aligned} \mathcal{L}_0 \sum_{k \in \mathbb{Z}} \left(\hat{p}^2_{l,k}/2 + S(\hat{r}_{l,k})\right) &= \sum_{k \in \mathbb{Z}} \Big(\hat{p}_{l,k}(\hat{V}'_{l,k} - \hat{V}'_{l,k-1}) + S'(\hat{r}_{l,k})(\hat{p}_{l,k+1} - \hat{p}_{l,k})\Big) \\ &= \sum_{k \in \mathbb{Z}} (\hat{p}_{l,k+1} - \hat{p}_{l,k})(S'(\hat{r}_{l,k}) - \hat{V}'_{l,k}) = R_\varepsilon\,, \end{aligned}$$

where $\hat{V}'$ denotes the block average of $V'_k := V'(r_k)$. In the scaling limit the microscopic time is speeded up as $t \to t/\varepsilon$, thus a fairly singular expression has been obtained. Nevertheless both factors on the right hand side can be estimated via LSI, the second one is the deviation of the microscopic flux from its macroscopic counterpart. Due to $\varepsilon\sigma^2(\varepsilon) \to +\infty$, it is possible to show that $R_\varepsilon \approx 0$ in a mean sense. Oscillations of other entropies are controlled in a similar way.

Stochastic Compensated Compactness: The empirical process is defined as $\hat{u}_\varepsilon(t,x) := (\hat{p}_{l,k}(t/\varepsilon), \hat{r}_{l,k}(t/\varepsilon))$ if $|\varepsilon k - x| < \varepsilon/2$. Since $\hat{u}_\varepsilon$ is bounded in a mean sense in $L^2(dt,dx)$, we have tightness of the distributions $\hat{\mathsf{P}}_\varepsilon$ of the Young measures Θ. These are defined as $d\Theta_\varepsilon := dt\,dx\,\theta^\varepsilon_{t,x}(du)$, where $\theta^\varepsilon_{t,x}$ is the Dirac mass at the actual value of $\hat{u}_\varepsilon(t,x)$. The Young family controls the asymptotic behavior of various functions of the empirical process.

Given an entropy pair (h,J), the associated entropy production is defined as

$$X_\varepsilon(\psi,h) := -\int_0^\infty \int_{-\infty}^\infty \big(h(\hat{u}_\varepsilon)\psi'_t(t,x) + J(\hat{u}_\varepsilon)\psi'_x(t,x)\big)\,dx\,dt\,,$$

where the test function ψ is compactly supported in the interior of $\mathbb{R}^2_+$.

An entropy pair (h,J) is *well controlled* if its entropy production decomposes as $X_\varepsilon(\psi,h) = Y_\varepsilon(\psi,h) + Z_\varepsilon(\psi,h)$, and we have two random functionals, $A_\varepsilon(\phi,h)$ and $B_\varepsilon(\phi,h)$ such that

$$|Y_\varepsilon(\psi\phi,h)| \le A_\varepsilon(\phi,h)\|\psi\|_{H^1}\,, \quad |Z_\varepsilon(\psi,h)| \le B_\varepsilon(\phi,h)\|\psi\|\,;$$

where $\|\cdot\|$ denotes the uniform norm, ψ and ϕ are compactly supported test functions, $\lim \mathsf{E}A_\varepsilon(\phi,h) = 0$ and $\limsup \mathsf{E}B_\varepsilon(\phi,h) < +\infty$ as $\varepsilon \to 0$.

Lemma St-Div-Curl: If (h_1,J_2) and (h_2,J_2) are well controlled entropy pairs, then distributions of the Young measure are tight, and

$$\theta_{t,x}(h_1J_2) - \theta_{t,x}(h_2J_1) = \theta_{t,x}(h_1)\theta_{t,x}(J_2) - \theta_{t,x}(h_2)\theta_{t,x}(J_1)$$

holds true almost surely with respect to any limit distribution of P_ε.

The proof of this lemma is not difficult, by means of the Skorohod embedding theorem it can be reduced to the original, deterministic version. The main problem is the verification of the conditions, LSI of [14] plays an essential role here.

As a consequence of the Div-Curl Lemma, it was shown by DiPerna [3] that oscillations of uniformly bounded approximate solutions die out, thus pointwise convergence takes place along subsequences. Unfortunately, it is not easy to find uniform bounds for stochastic models, therefore results of J. Shearer [19] and Serre - Shearer [18] are most useful for us. Starting from the energy inequalities implied by (3), Shearer derived (**??**) by means of an L^p theory of the Young measure. Then he has constructed two clever families of Lax entropy pairs such that (**??**) implies the Dirac property of $\theta_{t,x}$. Here $S''' > 0$ is assumed, while $S'''(0) = 0$ but $S'''(\rho) \neq 0$ if $\rho \neq 0$ in the case of [18].

Energy inequalities are not sufficient to control entropy production of the microscopic system because rapid oscillations are generated by the deterministic part of the evolution. In view of the Ito lemma, $m_\varepsilon(dt,x;h) := dh(\hat{u}_\varepsilon) - (1/\varepsilon)\mathcal{L}h(\hat{u}_\varepsilon)\,dt$ defines a martingale $m_\varepsilon(t,x,h)$ for each x such that letting

$$M_\varepsilon(\psi,h) := \int_{-\infty}^\infty \int_0^\infty \psi(t,x)\,m_\varepsilon(dt,x,h)\,dx$$

we have

$$X_\varepsilon(\psi,h) = \frac{1}{\varepsilon}\int_0^\infty \int_{-\infty}^\infty \psi(t,x)\mathcal{L}h(\hat{u}_\varepsilon)\,dx\,dt + M_\varepsilon(\psi,h)$$
$$+ \int_0^\infty \int_{-\infty}^\infty \psi(t,x)\left(J'_\pi(\hat{u}_\varepsilon)\nabla_\varepsilon\hat{\pi}_\varepsilon - J'_\pi(\hat{u}_\varepsilon)\nabla^*_\varepsilon\hat{\rho}_\varepsilon\right)\,dx\,dt + N_\varepsilon(\psi,h)\,,$$

where $\nabla_\varepsilon\varphi(x) := (1/\varepsilon)\big(\varphi(x+\varepsilon) - \varphi(x)\big)$, $\nabla^*_\varepsilon\varphi(x) := -\nabla_\varepsilon\varphi(x-\varepsilon)$ defines the adjoint of ∇_ε, finally N_ε is the numerical error due to this discretization of the space derivative. The crucial step of the proof is the replacement of $h'_\pi(\hat{u}_\varepsilon)\nabla^*_\varepsilon\hat{V}'$ appearing in $\mathcal{L}_0 h(\hat{u}_\varepsilon)$ with $h'_\pi(\hat{u}_\varepsilon)\nabla^*_\varepsilon S'(\bar{\rho}_\varepsilon)$. Nevertheless, a full non-gradient analysis is not needed because calculations are done in terms of mesoscopic block averages. The fundamental a priori bounds for the proof of the stochastic Div-Curl Lemma follow from the probabilistic entropy inequality and the associated logarithmic Sobolev inequality. The proof is then completed by a direct application of the results of J. Shearer and D. Serre, see also Chapter 9 of [17]

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On movable singularities of Garnier systems

R. R. Gontsov

Abstract

We study movable singularities of Garnier systems using the connection of the latter with isomonodromic deformations of Fuchsian systems.

§1. What is Painlevé VI equations and Garnier systems?

We start with the Painlevé VI ($\mathrm{P_{VI}}$) equation

$$\begin{aligned}\frac{d^2u}{dt^2} &= \frac{1}{2}\left(\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-t}\right)\left(\frac{du}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{u-t}\right)\frac{du}{dt}+\\ &+\frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha+\beta\,\frac{t}{u^2}+\gamma\,\frac{t-1}{(u-1)^2}+\delta\,\frac{t(t-1)}{(u-t)^2}\right),\end{aligned}\tag{1}$$

the second order ODE for a complex function $u(t)$, where $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ are constants.

However, simply giving the explicit equation seems to be the least helpful introduction to it. Perhaps, it is more convenient to look at $\mathrm{P_{VI}}$ as at

- the equation for an apparent (fifth) singularity of isomonodromic family of second order scalar Fuchsian equations with the four singularities $t,0,1,\infty$;
- the most general second order ODE with the Painlevé property;
- the equation controlling isomonodromic deformations of certain rank 2 Fuchsian systems with the four singularities $t,0,1,\infty$.

Let us recall the first two viewpoints in more details (the last one will appear in §3).

The monodromy of a linear differential equation

$$\frac{d^pu}{dz^p}+b_1(z)\frac{d^{p-1}u}{dz^{p-1}}+\ldots+b_p(z)u=0\tag{2}$$

with singularities $a_1,\ldots,a_n\in\overline{\mathbb{C}}$ (which are the poles of the coefficients) can be defined as follows. In a neighbourhood of a non-singular point z_0 we consider a basis $(u_1,\ldots,u_p)$ in the solution space of the equation (2). Analytic continuations of the functions $u_1(z),\ldots,u_p(z)$ along an arbitrary loop γ outgoing from z_0 and lying in $\overline{\mathbb{C}}\setminus\{a_1,\ldots,a_n\}$ transform the basis $(u_1,\ldots,u_p)$ into a (in general case different) basis $(\tilde u_1,\ldots,\tilde u_p)$. The two bases are related by means of a non-singular transition matrix G_γ corresponding to the loop γ:

$$(u_1,\ldots,u_p)=(\tilde u_1,\ldots,\tilde u_p)G_\gamma.$$

The map $[\gamma]\mapsto G_\gamma$ (which depends only on the homotopy class $[\gamma]$ of the loop γ) defines the representation

$$\chi:\pi_1(\overline{\mathbb{C}}\setminus\{a_1,\ldots,a_n\},z_0)\longrightarrow \mathrm{GL}(p,\mathbb{C})$$

of the fundamental group of the space $\overline{\mathbb{C}}\setminus\{a_1,\ldots,a_n\}$ in the space of non-singular complex matrices of size p. This representation is called the *monodromy* of the equation (2).

A singular point a_i of the equation (2) is said to be *regular* if any solution of the equation has a polynomial (with respect to $1/|z-a_i|$) growth near a_i. Linear differential equations with regular singular points only are called *Fuchsian*.

A. Poincaré [12] has established that the number of parameters determining a Fuchsian equation of order p with n singular points is less than the dimension of the space of representations χ, if $p > 2, n > 2$ or $p = 2, n > 3$ (see also [1], pp. 158–159). Hence in the construction of a Fuchsian equation with the given singularities and monodromy there arise so-called *apparent* singularities, at which the coefficients of the equation have poles but the solutions are single-valued meromorphic functions. In the case $p = 2$, $n = 4$ $(a_1, a_2, a_3, a_4 = t, 0, 1, \infty)$ the number of such singularities equals one. If we move a little the singularity $z = t$ so that the monodromy of the equation preserves (this is an *isomonodromy* property which is defined precisely in the next paragraph), the apparent (fifth) singularity $w(t)$ will move satisfying $\mathrm{P_{VI}}$ (this was first obtained by R. Fuchs [4]).

The equation (1) has three fixed singular points – $0, 1, \infty$. Its movable singularities (which depend on the initial conditions) can be poles only. In other words, any local solution of the equation defined in a neighbourhood of $t_0 \neq 0, 1, \infty$ can be extended to a meromorphic function on the universal cover of $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$. This is the *Painlevé property*. The statement on movable poles of the equation (1) is the following. In the case $\alpha \neq 0$ they can be simple only, and in the case $\alpha = 0$ their orders do not exceed two (see, for instance, [7], Ch. VI, §6).

The Garnier system $\mathcal{G}_n(\theta)$ depending on $n + 3$ complex parameters $\theta_1, \ldots, \theta_{n+2}, \theta_\infty$ is a completely integrable Hamiltonian system (see [8], Ch. III, §4)

$$\frac{\partial u_i}{\partial a_j} = \frac{\partial H_j}{\partial v_i}, \qquad \frac{\partial v_i}{\partial a_j} = -\frac{\partial H_j}{\partial u_i}, \qquad i, j = 1, \ldots, n, \tag{3}$$

with certain Hamiltonians $H_i = H_i(a, u, v, \theta)$ rationally depending on $a = (a_1, \ldots, a_n)$, $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$, $\theta = (\theta_1, \ldots, \theta_{n+2}, \theta_\infty)$. It was obtained by R. Garnier [5] as an extension of the first of the above three viewpoints to general case of $n + 3$ singularities $a_1, \ldots, a_n, 0, 1, \infty$[1]. Namely, $u_1(a), \ldots, u_n(a)$ are apparent singular points of a certain isomonodromic family of Fuchsian equations with singularities $a_1, \ldots, a_n, 0, 1, \infty$.

For $n > 1$ the Garnier system generically does not satisfy the Painlevé property. However, due to Garnier's theorem, the elementary symmetric polynomials $\sigma_i(u_1(a), \ldots, u_n(a))$, depending on local solutions of the Garnier system, extend to meromorphic functions $F_i(a)$ on the universal cover Z' of the space $(\mathbb{C} \setminus \{0, 1\})^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}$. Our addition to this theorem consists in some estimates for orders of irreducible components of the polar loci of the functions F_i (Theorem 2).

§2. Isomonodromic deformations of Fuchsian systems

Let us include a Fuchsian system

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i^0}{z - a_i^0} \right) y, \qquad B_i^0 \in \mathrm{Mat}(p, \mathbb{C}), \qquad \sum_{i=1}^{n} B_i^0 = 0, \tag{4}$$

of p equations with singularities $a_1^0, \ldots, a_n^0$ into a family

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i(a)}{z - a_i} \right) y, \qquad B_i(a^0) = B_i^0, \qquad \sum_{i=1}^{n} B_i(a) = 0, \tag{5}$$

of Fuchsian systems holomorphically depending on the parameter $a = (a_1, \ldots, a_n) \in D(a^0)$, where $D(a^0)$ is a disk of small radius centered at the point $a^0 = (a_1^0, \ldots, a_n^0)$ of the space $\mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}$.

[1] In the case $n = 1$ the Garnier system $\mathcal{G}_1(\theta_1, \theta_2, \theta_3, \theta_\infty)$ is an equivalent (Hamiltonian) form of $\mathrm{P_{VI}}$ (1), where $\alpha = \frac{1}{2}\theta_\infty^2$, $\beta = -\frac{1}{2}\theta_2^2$, $\gamma = \frac{1}{2}\theta_3^2$, $\delta = \frac{1}{2}(1 - \theta_1^2)$; see [11].

One says that the family (5) is *isomonodromic* (or it is an *isomonodromic deformation* of the system (4)), if for all $a \in D(a^0)$ the monodromies

$$\chi : \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}) \longrightarrow \mathrm{GL}(p, \mathbb{C})$$

of the corresponding systems are the same. This means that for every value a there exists a fundamental matrix $Y(z, a)$ of the corresponding system from (5) that has the same monodromy for all $a \in D(a^0)$. This matrix $Y(z, a)$ is called an *isomonodromic fundamental matrix*.

Is it always possible to include the system (4) into an isomonodromic family of Fuchsian systems? The answer is affirmative. Exactly, if the matrices $B_i(a)$ satisfy the *Schlesinger equation* [13]

$$dB_i(a) = -\sum_{j=1, j\neq i}^{n} \frac{[B_i(a), B_j(a)]}{a_i - a_j}\, d(a_i - a_j),$$

then the family (5) is isomonodromic (in this case it is called the *Schlesinger isomonodromic family*).

Due to Malgrange's theorem [9], for arbitrary initial conditions $B_i(a^0) = B_i^0$ the Schlesinger equation has a unique solution $\{B_1(a), \dots, B_n(a)\}$ in some disk $D(a^0)$, and the matrices $B_i(a)$ can be extended to the universal cover Z of the space $\mathbb{C}^n \setminus \bigcup_{i\neq j}\{a_i = a_j\}$ as meromorphic functions. Thus, the Schlesinger equation satisfies the Painlevé property. The polar locus $\Theta \subset Z$ of the extended matrix functions $B_1(a), \dots, B_n(a)$ is called the *Malgrange* Θ*-divisor*.

In what follows we will use the theorem of Bolibrukh[2] describing a general solution of the Schlesinger equation near the Θ-divisor in the case $p = 2$. For the polar locus $P \subset Z$ of a function f meromorphic on Z, and $a^* \in P$, let us denote by $\Sigma_{a^*}(f)$ the sum of orders of all irreducible components of $P \cap D(a^*)$.

Theorem 1. *If the monodromy of the two-dimensional family* (5) *is irreducible, then* $\Sigma_{a^*}(B_i) \geqslant 2 - n$ *for every* $a^* \in \Theta$ $(i = 1, \dots, n)$.

The following auxiliary lemma is a simplified version of Proposition 6.4.1 from [8].

Lemma 1. *Consider a two-dimensional Schlesinger isomonodromic family of the form*

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i(a)}{z - a_i}\right) y, \qquad \sum_{i=1}^{n} B_i(a) = K = \mathrm{diag}(\theta, -\theta), \quad \theta \in \mathbb{C},$$

and the function $b(a) = \sum_{i=1}^{n} b_i^{12}(a) a_i$, *where* $b_i^{12}(a)$ *are the upper-right elements of the matrices* $B_i(a)$ *respectively. Then the differential of the function* $b(a)$ *is given by the formula*

$$db(a) = (2\theta + 1) \sum_{i=1}^{n} b_i^{12}(a) da_i.$$

Proof. The differential $db(a)$ has the form

$$db(a) = \sum_{i=1}^{n} a_i db_i^{12}(a) + \sum_{i=1}^{n} b_i^{12}(a) da_i.$$

[2]This theorem was announced in [3], its particular case contains in [2], the proof can be found in [6].

To find the first of the two latter summands, let us use the Schlesinger equation for the matrices $B_i(a)$. Then we have

$$\sum_{i=1}^{n} a_i\, dB_i(a) = -\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} a_i \frac{[B_i(a),B_j(a)]}{a_i-a_j}\, d(a_i-a_j) = -\sum_{i=1}^{n}\sum_{j>i}^{n}[B_i(a),B_j(a)]d(a_i-a_j) =$$
$$= -\sum_{i=1}^{n}\Big[B_i(a),\sum_{j=1,j\neq i}^{n} B_j(a)\Big]da_i = -\sum_{i=1}^{n}[B_i(a),K]da_i.$$

The upper-right element of the latter matrix 1-form is equal to $\sum_{i=1}^{n} 2\theta b_i^{12}(a)da_i$, hence $\sum_{i=1}^{n} a_i db_i^{12}(a) = 2\theta\sum_{i=1}^{n} b_i^{12}(a)da_i$, and $db(a) = (2\theta+1)\sum_{i=1}^{n} b_i^{12}(a)da_i$. □

§3. Schlesinger isomonodromic deformations and Garnier systems

Let us recall the relationship between Schlesinger isomonodromic deformations and Garnier systems.

Consider a two-dimensional Schlesinger isomonodromic family

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n+2}\frac{B_i(a)}{z-a_i}\right) y, \qquad B_i(a^0) = B_i^0 \in sl(2,\mathbb{C}), \tag{6}$$

of Fuchsian systems with singular points $a_1,\dots,a_n$, $a_{n+1}=0$, $a_{n+2}=1$, $a_{n+3}=\infty$ which depends holomorphically on the parameter $a=(a_1,\dots,a_n)\in D(a^0)$, where $D(a^0)$ is a disk of small radius centered at the point a^0 of the space $(\mathbb{C}\setminus\{0,1\})^n\setminus\bigcup_{i\neq j}\{a_i=a_j\}$. Denote by $\pm\beta_i$ the eigenvalues of the matrices $B_i(a)$ respectively[3]. As follows from the Schlesinger equation, the matrix residue at the infinity is constant. We assume that it is a diagonalisable matrix, i. e., $\sum_{i=1}^{n+2} B_i(a) = -B_\infty = \mathrm{diag}(-\beta_\infty,\beta_\infty)$.

By Malgrange's theorem the matrix functions

$$B_i(a) = \begin{pmatrix} b_i^{11}(a) & b_i(a) \\ b_i^{21}(a) & b_i^{22}(a)\end{pmatrix}$$

can be extended to the universal cover Z' of the space $(\mathbb{C}\setminus\{0,1\})^n\setminus\bigcup_{i\neq j}\{a_i=a_j\}$ as meromorphic functions (holomorphic off the analytic subset Θ of codimension one).

Denote by $B(z,a)$ the coefficient matrix of the family (6). Since the upper-right element of the matrix B_∞ equals zero, for every fixed a the same element of the matrix $z(z-1)(z-a_1)\dots(z-a_n)B(z,a)$ is a polynomial $P_n(z,a)$ of degree n in z. We denote by $u_1(a),\dots,u_n(a)$ the roots of this polynomial and define the functions $v_1(a),\dots,v_n(a)$:

$$v_j(a) = \sum_{i=1}^{n+2}\frac{b_i^{11}(a)+\beta_i}{u_j(a)-a_i}, \qquad j=1,\dots,n.$$

Then the following statement takes place: *the pair* $(u(a),v(a)) = (u_1,\dots,u_n,v_1,\dots,v_n)$ *satisfies the Garnier system* (3) *with the parameters* $2\beta_1,\dots,2\beta_{n+2},2\beta_\infty-1$ (see [8], Cor. 6.2.2).

One can express the coefficients of the polynomial $P_n(z,a)$ in terms of the upper-right elements $b_i(a)$ of the matrices $B_i(a)$. Let

$$\sigma_1(a) = \sum_{i=1}^{n+2} a_i, \quad \sigma_2(a) = \sum_{1\leqslant i<j\leqslant n+2} a_ia_j, \quad \dots, \quad \sigma_{n+1}(a) = a_1\dots a_n$$

[3]An isomonodromic deformation also preserves the eigenvalues of the residue matrices $B_i(a)$.

be the elementary symmetric polynomials in $a_1, \dots, a_n$, $a_{n+1} = 0$, $a_{n+2} = 1$, and $Q(z) = \prod_{i=1}^{n+2}(z - a_i)$. Then

$$P_n(z,a) = \sum_{i=1}^{n+2} b_i(a)\frac{Q(z)}{z-a_i} =: b(a)z^n + f_1(a)z^{n-1} + \ldots + f_n(a)$$

(recall that $\sum_{i=1}^{n+2} b_i(a) = 0$). By the Viète theorem one has

$$\begin{aligned} b(a) &= \sum_{i=1}^{n+2} b_i(a)(-\sigma_1(a) + a_i) = \sum_{i=1}^{n+2} b_i(a)a_i = \sum_{i=1}^{n} b_i(a)a_i + b_{n+2}(a), \\ f_1(a) &= \sum_{i=1}^{n+2} b_i(a)\Big(\sigma_2(a) - \sum_{j=1, j\neq i}^{n+2} a_i a_j\Big) = -\sum_{1\leqslant i<j\leqslant n+2} (b_i(a) + b_j(a))a_i a_j. \end{aligned}$$

In the similar way,

$$f_k(a) = (-1)^k \sum_{1\leqslant i_1<\ldots<i_{k+1}\leqslant n+2} (b_{i_1}(a) + \ldots + b_{i_{k+1}}(a))a_{i_1}\ldots a_{i_{k+1}}$$

for each $k = 1, \dots, n$.

Alongside formulae for the transition from a two-dimensional Schlesinger isomonodromic family with $sl(2,\mathbb{C})$-residues to a Garnier system, there also exist formulae for the inverse transition (see [8]). This allows to suggest some addition to Garnier's theorem (which claims that the elementary symmetric polynomials $F_i(a) = \sigma_i(u_1(a), \dots, u_n(a))$ of solutions of a Garnier system are meromorphic on Z').

Theorem 2. *Let $(u(a), v(a))$ be a solution of the Garnier system (3) that corresponds to a two-dimensional Schlesinger isomonodromic family with irreducible monodromy, and $u_i(a) \not\equiv u_j(a)$ for $i \neq j$. Then for each function $F_i(a)$ and any point a^* of its polar locus one has*

$$\Sigma_{a^*}(F_i) \geqslant -n - 1.$$

Proof. Consider the family (6) with the irreducible monodromy, and the functions $b(a)$, $f_1(a), \dots, f_n(a)$ constructed by the residue matrices $B_i(a)$. By the Viète theorem, $F_i(a) = (-1)^i f_i(a)/b(a)$. Due to Theorem 1, for each function $f_i(a)$ and any point a^* of the Θ-divisor of the family (6) one has $\Sigma_{a^*}(f_i) \geqslant -n - 1$. Thus, to prove the estimate of Theorem 2, it is sufficient to prove that the function $b(a)$ is irreducible and does not vanish on the polar locus of the functions $f_i(a)$.

By Lemma 1 we have $db(a) = (-2\beta_\infty + 1)\sum_{i=1}^{n} b_i(a)da_i$.

i) In the case $\beta_\infty \neq 1/2$

$$\begin{aligned} b_i(a) &= -\frac{1}{\theta_\infty}\frac{\partial b(a)}{\partial a_i}, \qquad i = 1, \dots, n; \quad (\theta_\infty = 2\beta_\infty - 1 \neq 0) \\ b_{n+2}(a) &= b(a) - \sum_{i=1}^{n} b_i(a)a_i, \quad b_{n+1}(a) = -b_{n+2}(a) - \sum_{i=1}^{n} b_i(a). \end{aligned} \tag{7}$$

Thus, if the function $b(a)$ is holomorphic at a point $a' \in Z'$, so are the functions $b_i(a)$, $i = 1, \dots, n+2$, and hence, the functions $f_i(a)$.

If for some $a' \in \{b(a) = 0\}$ one has $db(a') \equiv 0$, then $\sum_{i=1}^{n} b_i(a')da_i \equiv 0$ and $b_1(a') = \ldots = b_n(a') = 0$. Taking into consideration the relations (7), one gets also $b_{n+2}(a') = 0$ and $b_{n+1}(a') = 0$. This contradicts the irreducibility of the monodromy of the family (6).

ii) In the case $\beta_\infty = 1/2$ one has $db(a) \equiv 0$ for all $a \in D(a^0)$, hence $b(a) \equiv \text{const} \neq 0$. Indeed, if $b(a) \equiv 0$, then $P_n(z, a)$ is a polynomial of degree $n-1$ in z, and $u_i(a) \equiv u_j(a)$ for some $i \neq j$. $\square$

Remark. M. Mazzocco [10] has shown that the solutions of the Garnier system (3), that correspond to two-dimensional Schlesinger isomonodromic families with *reducible* monodromy, are classical functions (in each variable, in sense of Umemura [14]) and can be expressed via Lauricella Hypergeometric equations. Thus, Theorem 2 can be applied, for example, to non-classical solutions of Garnier systems.

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Invariant Measures for Multivalued Mappings

Gorbachev A.N.

1. Let Ω be a space with σ-algebra of measurable sets, R is measurable multivalued mapping on Ω, Γ_R is graph of R.

Measure μ on $\Omega \times \Omega$ is called *invariant for* R, if $\mu(\Omega \times \Omega \setminus \Gamma_R) = 0$ and projections on the first and second component of product space $\Omega \times \Omega$ (mappings $\pi_1, \pi_2 : \Omega \times \Omega \to \Omega$, $\pi_1(t_1, t_2) = t_1$, $\pi_2(t_1, t_2) = t_2$) have the same action on measure μ [1]. In other words for any measurable set $A \subset \Omega$

$$\mu(\pi_1^{-1}A \cap \Gamma_R) = \mu(\pi_2^{-1}A \cap \Gamma_R).$$

If multivalued mapping R on Ω is parameterized by measurable one-valued mappings $r_1, r_2 : \Omega \to \Omega$, т.е. $\Gamma_R = (r_1(t), r_2(t)), t \in \Omega$, then we have

PROPOSITION. *For multivalued mapping R on the set Ω, that is parameterized by mappings r_1 и r_2, there exists an invariant measure μ if and only if there exists such measure η on Ω, that $r_{1*}\eta = r_{2*}\eta$ (i.e. for any measurable set $A \subset \Omega$ $\eta(r_1^{-1}A) = \eta(r_2^{-1}A)$).*

We call measure η (r_1, r_2)-*invariant* for multivalued mapping R.

Let k и l be coprimes. *Linear (k, l)-mapping* S^1 is multivalued mapping on S^1 with parametrization $r_1(t) = kt$, $r_2(t) = lt$.

2. Let $r_1, \ldots, r_n$ be one-valued continuous transformations of the compact space Ω with σ-algebra of measurable sets. We take the system of equations:

$$r_{1*}\nu = r_{2*}\nu = \ldots = r_{n*}\nu \qquad (I)$$

where η is probabilistic measure on Ω. In that case, when $n = 2$, any solution of *(I)* is by definition (r_1, r_2)-invariant measure for multivalued mapping, parameterized be mappings r_1 and r_2.

THEOREM **1.** *If semigroup $S = \langle r_1, \ldots, r_n \rangle$ is amenable and S satisfies cancellation law, then there exists sequence in group algebra A over S ($A = \{\sum_{i=1}^{n} \alpha_i s_i,\ s_i \in S\}$) such that its action on arbitrary finite measure on Ω has accumulation point – the solution of system (I). In addition, action of this sequence on any solution of system (I) is fixed.*

Semigroup S has amenable group of quotients $G = S^{-1}S$ that slides to S by the right shift [2].

Let us take the set $K = \{r_1^{-1}r_2, \ldots, r_1^{-1}r_n\} \subset G$. Then $\forall \varepsilon > 0\ \exists F_{\varepsilon,K} \subset G:\ \forall k \in K\ |F_{\varepsilon,K} \triangle kF_{\varepsilon,K}| < \varepsilon |F_{\varepsilon,K}|$. Let ν be an arbitrary finite measure on Ω. We build sequence $\nu_N = \frac{1}{|F_{\frac{1}{N},K}|} \sum\limits_{x \in F_{\frac{1}{N},K}} (xg_N)_*\nu$, where g_N shifts $F_{\frac{1}{N},K}$ in S. From the sequence ν_N we can take subsequence ν_{N_i} that is weakly convergent to some normed measure $\widetilde{\nu}$. In this case for all $j = 2, \ldots, n$: $r_{1*}\nu_N - r_{j*}\nu_N \to 0$ when $N \to 0$. Processing to limit by N_i we have $r_{1*}\widetilde{\nu} = r_{j*}\widetilde{\nu}$ for all $j = 2 \ldots n$. So $\widetilde{\nu}$ is the solution of system (I).

Now we show an example of such mappings $r_1, \ldots, r_n$ with semigroup $S = \langle r_1 \ldots r_n \rangle$ that there exists the solution of system (I) that is not S-invariant measure.

EXAMPLE. On $S^1 \simeq \mathbb{R}/\mathbb{Z}$ we take $r_i(t) = 2^{n-i}3^i t (\mathrm{mod} 1)$, $i = 0, \ldots, n$. Then we take invariant for the mapping $t \to \frac{3}{2}t$ measure η [3]. Then $T_{2*}\eta = T_{3*}\eta$, where $T_2 : t \to 2t$ and $T_3 : t \to 3t$. Hence $r_{i*}\eta = (\widetilde{r}_i T_2)_*\eta = (\widetilde{r}_i T_3)_*\eta = r_{i+1*}\eta$, where $\widetilde{r}_i(t) = 2^{n-i}3^i t (\mathrm{mod} 1)$. Thus η is solution of system (I). I.e. η is absolutely continuous relatively to Lebesgue measure [3] but doesn't match with it, so $r_{i*}\eta \neq \eta$ for all $i = 1, \ldots, n$, and then η is not S-invariant.

3. Let P be nonempty compact subset of locally convex space E, μ is probabilistic measure on P. It is said that point $p \in E$ *is represented by measure* μ, if $f(p) = \int_P f d\mu$ for any continuous linear functional f on E. *Extreme point* of the set P is called such point that doesn't split to the linear combination of other elements of P. The set of extreme points we denote by ex$P$.

THEOREM **2**. *There exists such multivalued mapping F on S^1, point $\hat{m} \in M$, where M is the set of invariant measures for F and two different measures on ex$M$, that represent $\hat{m}$.*

Let us take linear $(2,3)$-mapping F (with graph Γ_F) on $S^1 \simeq \mathbb{R}/\mathbb{Z}$. As the point $\hat{m}$ we take measure with constant density on Γ_F.

Now we take $A = \alpha_s$ и $\tilde{A} = \tilde{\alpha}_s$ — the sets of one-valued mappings S^1 on itself such, that their graphs are subsets of Γ_F and defines by the following ways:

$$\alpha_s :\ r_1(t) = 2(t+s);\ r_2(t) = 3(t+s),\ t \in [0, \frac{1}{2})$$

$$\tilde{\alpha}_s :\ r_1(t) = 2(t+s),\ t \in [0, \frac{1}{2});\ r_2(t) = \begin{cases} 3(t+s), & t \in [0, \frac{1}{4}) \cup [\frac{1}{3}, \frac{1}{2}) \\ 3(t+s) + \frac{1}{2}, & t \in [\frac{1}{4}, \frac{1}{3}) \end{cases}$$

For any $s \in [0,1)$ α_s и $\tilde{\alpha}_s$ is piecewise monotonic transformation with finite number of discontinuities and category C^2 in all intervals of

compliment, so there exist invariant measures for all of them μ_s and $\tilde{\mu}_s$ [3] (on their graphs). Moreover measures μ_s and $\tilde{\mu}_s$ are invariant for F and are ergodic for α_s и $\tilde{\alpha}_s$ respectively, so they are extreme points in the set of invariant measures for F. Futhermore, the measures μ_s и $\tilde{\mu}_s$ are pairwise distinct, because they have different supporters.

Then we take $\int\limits_0^1 \mu_s ds$ и $\int\limits_0^1 \tilde{\mu}_s ds$ $((\int\limits_0^1 \mu_s ds)(A) = \int\limits_0^1 \mu_s(A) ds$, $(\int\limits_0^1 \tilde{\mu}_s ds)(A) = \int\limits_0^1 \tilde{\mu}_s(A) ds)$, they both have constant density on Γ_F.

So, we've got two representations of point $\hat{m}$ with measures, that is concentrated on different subsets of ex$M$ ■

Let P be compact convex subset of locally convex space E, λ и μ be non-negative Borel measures on P. We define $\lambda \succ \mu$, if $\lambda(f) \geq \mu(f)$ for any function f in $C(P)$. We take the measures that are maximal concerning to this order.

According the Choquet-Meillet theorem [4] one compact convex set P is simplex if and only if for any p in P there exists unique maximal measure μ_p, that represents p. In respect that any measure on ex$P$, is maximal [4] we have:

THEOREM **3**. *There exists such multivalued mapping on S^1, that the set of its invariant measures is not semplexes.*

For built in theorem 2 set M and point $\hat{m}$ there exists at least two different representations by maximal measures.

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DLR thermodynamic formalism and properties of time and space means

B.M. Gurevich*and A.A. Tempelman

Let X be a metric space, $\tau = \{\tau_t, t \in \mathbb{Z}^d\}$ an action of the group $T = \mathbb{Z}^d$ on X by homeomorphisms, and $\mathcal{I}$ the set of all τ-invariant Borel probability measures on X. For each $\mu \in \mathcal{I}$, we denote by $h(\mu)$ the entropy of τ with respect to μ. In what follows we assume that the function $\mu \mapsto h(\mu)$ is upper semi-continuous on $\mathcal{I}$. (This is the case if τ is expansive.)

If $f \in C(X; \mathbb{R}^m)$ and $\mu \in \mathcal{I}$ we denote $\mu(f) = \int_X f d\mu$. For each function $f = (f_1, \dots, f_m) \in C(X; \mathbb{R}^m)$ and each $\beta = (\beta_1, \dots, \beta_m)$, we consider the pressure $P_f(\beta) := \sup_{\mu \in \mathcal{I}}[h(\mu) - \mu(\langle \beta, f \rangle)]$, where $\langle \beta, f \rangle := \sum_{i=1}^m \beta_i f_i \in C(X, \mathbb{R}^1)$. Denote by $\mathcal{E}_{f,\beta}$ the set of measures μ at which the supremum is attained ((f, β)-equilibrium measures), and put $\mathcal{E}_f = \bigcup_{\beta \in \mathbb{R}^m} \mathcal{E}_{f,\beta}$.

Let $\mathcal{T} = \{T_n\}$ be a sequence of finite subsets of T. For $f \in C(X, \mathbb{R}^m)$, $x \in X$, we consider the time mean

$$a_{f,\mathcal{T}}(x) := \lim_{n \to \infty} \frac{1}{|T_n|} \sum_{t \in T_n} f(\tau_t x),$$

provided the limit exists. We assume that $\mathcal{T}$ is a Følner sequence, i.e. $\lim_{n \to \infty} \frac{|T_n \triangle (T_n + t)|}{|T_n|} = 0$ for each $t \in T$; in this case the time means and the set of points x where they exist are τ-invariant. Let $M_{f,\mathcal{T}}$ be the set of all time means, $I_f := \{\mu(f), \mu \in \mathcal{I}\}$ the set of space means, and $E_f = \{\mu(f), \mu \in \mathcal{E}_f\}$ the set of equilibrium space means.

It is clear that I_f is a compact convex set. We first consider properties of E_f and the relationship between I_f and E_f.

Theorem 1. *The relative interior* $\mathrm{ri}(E_f)$ *of* E_f *coincides with the relative interior* $\mathrm{ri}(I_f)$ *of* I_f.

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This theorem is a refinement of Theorem 2 in [1].

Theorem 2. *Assume that the function P_f has the following property: if P_f is affine on a ray, then it is affine on the whole straight line containing this ray. Then E_f is convex and relatively open; moreover, it coincides with* $\mathrm{ri}(I_f)$.

Of course, the condition on P_f in Theorem 2 is fulfilled if this function is strictly convex. On the other hand, if the components of the vector function f are linearly dependent, P_f is constant on an affine manifold and satisfies the condition of Theorem 2 as well.

In what follows we assume that $X = S^T$, where S is a finite set, and $(\tau_t x)(s) = x(t+s)$ for $x \in X$, $s, t \in T$. In this case the above semi-continuity property of the entropy function holds. We also assume that $\mathcal{T}$ is an increasing sequence of parallelepipeds $T_n = \{t = (t_1, \dots, t_d) : 0 \le t_i < t_i^{(n)}\} \cap T$ such that $\lim\limits_{n\to\infty} t_i^{(n)} = \infty$, $i = 1, ..., m$, and $\dfrac{\max_i^{(n)}}{\min_i^{(n)}} < c < \infty$.

Theorem 3. *$M_{f,\mathcal{T}} = I_f$; so the set $M_{f,\mathcal{T}}$ is convex, compact, and does not depend on $\mathcal{T}$.*

Define the "standard metric" ρ_θ, $\theta \in (0,1)$, on X by $\rho(x,y) = \theta^m$ if $x \ne y$, $m = \min\{n : x|_{T_n} \ne y|_{T_n}\}$, and $\rho(x,y) = 0$ if $x = y$.

With this metric, we are interested in the Hausdorff dimension (denoted below by $\dim_\theta$) of the level set

$$X_{f,\alpha,\mathcal{T}} = \{x \in X : a_{f,\mathcal{T}}(x) = \alpha\}.$$

Theorem 3 implies $X_{f,\alpha,\mathcal{T}} \ne \emptyset$ for each $\alpha \in I_f$.

The following theorem generalizes Theorem 5.1 in [2].

Theorem 4. (1) *If $\alpha \in I_f$, then*

$$\dim_\theta(X_{f,\alpha,\mathcal{T}}) = \theta^{-1} \max_{\mu\in\mathcal{I},\mu(f)=\alpha} h(\mu) = \theta^{-1} \inf_{\beta\in\mathbb{R}^m} [P_f(\beta) + \langle\beta,\alpha\rangle].$$

(2) *If $\alpha \in E_f$, i.e. $\alpha = \mu_0(f)$ where $\mu_0 \in \mathcal{E}_f(\langle\beta_0,\alpha\rangle)$ with some $\beta_0 \in \mathbb{R}^m$, then*

$$\dim_\theta(X_{f,\alpha,\mathcal{T}}) = \theta^{-1} h(\mu_0) = \theta^{-1}[P_f(\beta_0) + \langle\beta_0,\alpha\rangle].$$

(3) *The function $\alpha \mapsto \dim_\rho^H(X_{f,\alpha,\mathcal{A}})$ is concave and upper semi-continuous on I_f, continuous on* $\mathrm{ri}(I_f)$, *and Lipschitz continuous on each closed subset of* $\mathrm{ri}(I_f)$; *moreover, the restriction of this function to each closed straight line interval in I_f is continuous.*

It should be mentioned that, for $d = 1$, some close results were earlier obtained by different methods (see, for instance [3],[4]), who applied different approaches. But when $d > 1$, the DLR approach seems to be the most relevant.

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SOME UPPER ESTIMATES ON THE NUMBER OF LIMIT CYCLES OF EVEN DEGREE LIÉNARD EQUATIONS IN THE FOCUS CASE

GRISHA KOLUTSKY

Abstract. We give an explicit upper bound for the number of limit cycles of Liénard equation $\dot{x} = y - F(x), \dot{y} = -x$ of even degree in the case its unique singular point $(0, 0)$ is a focus.

M. Caubergh and F. Dumortier give explicit upper estimates for large amplitude limit cycles of such equations [CD]. We estimate the number of mid amplitude limit cycles of Liénard equations using the Growth-and-Zeros theorem proved by Ilyashenko and Yakovenko [IYa].

Our estimate depends on four parameters: n, C, a_1, R. Let $F(x) = x^n + \sum_{i=1}^{n-1} a_i x^i$, where $n = 2l$ is the even degree of the monic polynomial F without constant term, $\forall i |a_i| < C$, so C is the size of a compact subset in the space of parameters, $|a_1|$ stands the distance from the equation linearization to the center case in the space of parameters and R is the size of the neighborhood of the origin, such that there are no bigger than l limit cycles located outside of this neighborhood.

1. Introduction. History of the Hilbert-Smale problem

In 1977 A. Lins Neto, W. de Melo and C. C. Pugh [LMP] examined small perturbations of a linear center for a special class of polynomial vector fields on the plane. This class is called Liénard equations:

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -x \end{cases}, \tag{1}$$

where F is a polynomial of odd degree.

They proved the finiteness of limit cycles for a Liénard equation of odd degree n. Also they conjectured that the number of limit cycles of (1) is not bigger than $\frac{n-1}{2}$.

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In 1998 S. Smale [S] suggested to consider a restriction of the second part of the Hilbert's 16th problem to Liénard equations of odd degree. He conjectured that there exists an integer n and real C such that the number of limit cycles of (1) is not bigger than Cn^q.

In 1999 Yu. Ilyashenko and A. Panov [IP] got an explicit upper bound for the number of limit cycles of Liénard equations through the (odd) power of the monic polynomial F and magnitudes of its coefficients. Their result reclined on the theorem of Ilyashenko and Yakovenko that binds the number of zeros and the growth of a holomorphic function [IYa].

In 2007 F. Dumortier, D. Panazzolo and R. Roussarie [DPR] constructed a counterexample to the conjecture of A. Lins Neto, W. de Melo and C. C. Pugh. They constructed an example of a Liénard equation of odd degree n with at least $\frac{n+1}{2}$ limit cycles.

In 2008 Yu. Ilyashenko [I2] suggested to prove a result analogous to the one of Ilyashenko and Panov for Liénard equations of even degree.

In 2008 M. Caubergh and F. Dumortier in [CD] proved the following theorem for Liénard equations of even degree.

Theorem 1. *Let K be a compact set of polynomials of degree exactly $n = 2l$, then there exists $R > 0$ such that any system having an expression (1) with $F \in K$ has at most l limit cycles having an intersection with $\mathbb{R}^2 \backslash B_R(0)$.*

Here $B_R(0)$ denotes the ball around the origin with radius R.

2. Notations and the Ilyashenko strategy

From now on we will consider a system (1), where F is a monic polynomial of even degree $n = 2l$ without a constant term.

Remark. *The assumption $F(0) = 0$ does not reduce the generality; it may be fulfilled by a shift $y \mapsto y + a$. The assumption that F is monic may be fulfilled by rescalling in x, y and reversing time if necessary.*

Let v be an analytic vector field in the real plane, that may be extended to $\mathbb{C}^2$. For any set D in a metric space denote by $U^\varepsilon(D)$ the ε-neighborhood of D. The metrics in $\mathbb{C}$ and $\mathbb{C}^2$ are given by:

$$\rho(z, w) = |z - w|, \qquad z, w \in \mathbb{C};$$
$$\rho(z, w) = \max(|z_1 - w_1|, |z_2 - w_2|), \qquad z, w \in \mathbb{C}^2.$$

Denote by $|D|$ the length of the segment D. For any larger segment $D' \supset D$, let $\rho(D, \partial D')$ be the Hausdorff distance between D and $\partial D'$.

We want to apply the next theorem proved by Ilyashenko and Panov [IP]. Actually, it is the easy corollary from the Growth-and-Zeros theorem for holomorphic functions proved by Ilyashenko and Yakovenko [IYa].

Consider the system

$$\dot{x} = v(x), \qquad x \in \mathbb{R}^2. \tag{2}$$

Theorem 2. *Let Γ be a cross-section of the vector field v, $D \subset \Gamma$ a segment. Let P be the Poincaré map of (2) defined on D, and $D \subset D' = P(D)$. Suppose that P may be analytically extended to $U = U^{\varepsilon}(D) \in \mathbb{C}$, $\varepsilon < 1$, and $P(U) \subset U^1(D') \subset \mathbb{C}$. Then the number $\#LC(D)$ of limit cycles that cross D admits an upper estimate:*

$$\#LC(D) \leq e^{2|D|\varepsilon^{-1}} \log \frac{|D'| + 2}{\rho(D, \partial D')}. \tag{3}$$

The same is true for P replaced by P^{-1}.

Actually, the *Ilyashenko strategy* is the application of the previous theorem. It requires purely qualitative investigation of a vector field, i.e. a construction of such D for every nest of limit cycles. This strategy was applied before in papers [I1] and [IP].

We take K from the Theorem 1 to be the space of monic polynomials of degree exactly n with coefficients, which moduli are bounded by some positive constant C, i.e.

$$F(x) = x^n + \sum_{i=1}^{n-1} a_i x^i, \qquad \forall i : |a_i| < C.$$

If $|a_1| < 2$ then the unique singular point $(0, 0)$ of the system (1) is a focus. In our work we will consider only this case.

Let us denote by Y the maximal y-coordinate of the point of intersection between the most external limit cycle which lies inside $B_R(0)$ (if it exists, of course) and y-axis.

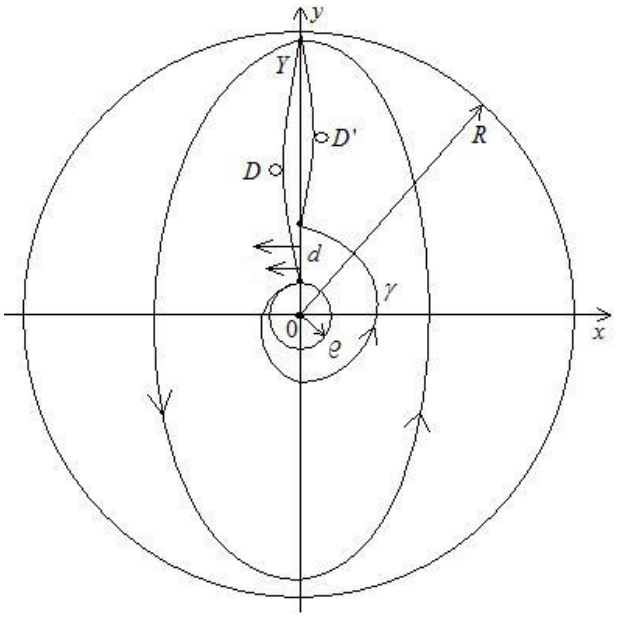

FIGURE 1. This is the reversed time picture.

3. Bendixson trap from within

In this Section we construct an interval D, which lies inside $B_R(0)$ and intersects transversally all limit cycles in $B_R(0)$. Also we find an upper estimate for *the Bernstein index*, $b = \log \dfrac{|D'|+2}{\rho(D,\partial D')}$. To do that we need to estimate $\rho(D,\partial D')$ from bellow, where $D' = P(D) \subset D$ and P is the Poincaré map defined on D.

Let φ, r be polar coordinates on $\mathbb{R}^2$, $\dot\varphi$, $\dot r$ be derivatives with respect to (1).

Lemma 1. *Put* $\varrho = -\dfrac{a_1}{4C}$. *If* a_1 *is negative, then* $\dot r > 0$ *in* $U_\varrho(0)$. *Let* $D = [\varrho, Y] \subset 0y$. *Then* $d = \rho(D,\partial D') \geq \dfrac{\pi a_1^2}{8C}$.

Proof. Let us calculate $\dot r$.

$$\dot r = \frac{x\dot x + y\dot y}{r} = \frac{r\cos\varphi(r\sin\varphi - F(r\cos\varphi)) - r^2\sin\varphi\cos\varphi}{r} =$$

$$= -\cos\varphi F(r\cos\varphi) = -r\cos^2\varphi\sum_{i=1}^{n} a_i(r\cos\varphi)^{i-1}$$

If $r < \varrho$, then $r < \frac{1}{2}$ and $2Cr < \frac{a_1}{2}$. Therefore, $\frac{Cr}{1-r} < \frac{a_1}{2}$. Then

$$\left|\sum_{i=2}^{n} a_i(r\cos\varphi)^{i-1}\right| \leq Cr\frac{1-r^{n-1}}{1-r} < \frac{Cr}{1-r} < \frac{a_1}{2}.$$

So

$$\dot r > r\cos^2\varphi\left(-a_1 + \frac{a_1}{2}\right) = -\frac{a_1}{2}r\cos^2\varphi > 0.$$

This proves the first part of the Lemma.

Consider the orbit γ of the system (1) that passes through the point $(0,\varrho)$. Then the Hausdorff distance d can be estimated as follows:

$$d \geq \left|\int_0^{2\pi} \dot r(\gamma)d\varphi\right| > \int_0^{2\pi} \frac{a_1}{2}r\cos^2\varphi d\varphi > \frac{\pi a_1^2}{8C}.$$

This inequality completes the proof of the Lemma. □

Remark. *For positive* a_1 *we can get the same results just by reversing of the time.*

Now we can estimate b from above:

$$b \leq \log\frac{R+2}{d} \leq \log\frac{8C(R+2)}{\pi a_1^2} < \log\frac{2C(R+2)}{a_1^2} < \frac{2C(R+2)}{a_1^2}. \quad (4)$$

4. Complex domain of the Poincaré map

In [IP] authors proved that the inverse Poincaré map of the Liénard equation (1) may be extended to the domain $U^{\varepsilon}(D) \subset \mathbb{C}$, where

$$\varepsilon = \exp\left(-n^2(X+2)^{2n+3}\right), \quad X = 4(C+1)^3, \quad C \geq 4. \tag{5}$$

This statement is true for our case, but C should be replaced by $\max(R, C)$. One can prove it using absolutely the same arguments as Ilyashenko and Panov.

From now on without loss of generality we can replace C by $\max(R, C)$.

5. Final estimate

Theorem 3. *The number $L(n, C, a_1, R)$ of limit cycles of (1) in the case when n is even and $0 < |a_1| < 2$, admits the following upper bound:*

$$L(n, C, a_1, R) < a_1^{-2} \exp(5R^2 \exp(C^{11n+16})). \tag{6}$$

Proof. By definition, $|D|$ and $|D'|$ are less than R. So estimates (3), (4) and (5) imply:

$$L(n, C, a_1, R) < \exp(2R \exp(n^2(4(C+1)^3+2)^{2n+3})) \frac{2C(R+2)}{a_1^2} <$$
$$< a_1^{-2} \exp(5R^2 \exp(n^2 C^{5(2n+3)+1})) < a_1^{-2} \exp(5R^2 \exp(C^{11n+16})).$$

This calculation completes the proof of the Theorem. □

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Grisha Kolutsky
Department of the Theory of Dynamical Systems
Faculty of Mechanics and Mathematics
Lomonosov Moscow State University
MSU, GSP, Glavnoe Zdanie, Leninskie Gory
119899 Moscow, Russia
e-mail: kolutsky AT mccme DOT ru

Global attractor for the nonlinear Klein-Gordon equation with the mean field interaction

Andrey Komech
Institute for Information Transmission Problems of Russian Academy of Sciences

Abstract. The long time asymptotics for nonlinear wave equations have been the subject of intensive research, starting with the pioneering papers by Segal and Strauss, where the nonlinear scattering and convergence to zero for small initial data were considered. *Global* attraction (for large initial data) to zero may not hold if there are *quasistationary localized solutions* (*solitary waves*) of the form $\psi(x,t) = \phi(x)e^{-i\omega t}$.

The existing results on the existence and stability of solitary waves suggest that the *global attractor* of all finite energy solutions is finite dimensional and coincides with the set of all solitary waves. In the present work, we will prove this statement for the nonlinear Klein-Gordon equation with the mean field interaction. We will prove the convergence to the solitary manifold in the metric which is ε-weaker than the local energy seminorms.

Аннотация. Долговременные асимптотики для нелинейных волновых уравнений являются объектом интенсивных исследований начиная с основополагающих работ Сигала и Штраусса, где было рассмотрено нелинейное рассеяние и сходимость к нулю для малых начальных данных. *Глобальная* сходимость (для произвольных начальных данных) к нулю не имеет места при наличии *квазистационарных локализованных решений* (*уединённых волн*) вида $\psi(x,t) = \phi(x)e^{-i\omega t}$.

Полученные к настоящему времени результаты о существовании и устойчивости уединённых волн наводят на мысль, что *глобальный аттрактор* всех решений конечной энергии конечномерен и совпадает со множеством всех уединённых волн. В данной работе мы докажем это утверждение для нелинейного уравнения Клейна-Гордона с самодействием типа среднего поля. Мы докажем сходимость решений к солитонному многообразию в метрике, которая ε-слабее энергетических полунорм.

Глобальный аттрактор для нелинейного уравнения Клейна-Гордона с самодействием типа среднего поля

Андрей Комеч
Институт Проблем Передачи Информации Российской Академии Наук

1 История солитонных асимптотик для дисперсионных систем

Квантовая теория: Боровские переходы как глобальная сходимость к уединённым волнам

Сосредоточимся на поведении электрона в атоме водорода. Согласно боровским постулатам [Boh13], невозмущённый электрон вечно двигается по некоторой *стационарной орбите*, которую мы обозначим $|E\rangle$ и назовём *квантовым стационарным состоянием*. Находясь в таком состоянии, электрон имеет постоянную энергию E. Старая квантовая теория была основана на условии квантования $\oint \mathbf{p}\cdot d\mathbf{q} = 2\pi\hbar n$, $n\in\mathbb{N}$, и приводило к значениям энергии электрона в атоме водорода в хорошем соответствии с экспериментом. Условие квантования не объясняло вечного кругового движения электрона, так как, согласно классической электродинамике, такое движение сопровождалось бы потерей энергии из-за радиации. Следуя идеям де Бройля, Шрёдингер отождествил боровские *стационарные орбиты*, или квантовые стационарные состояния $|E\rangle$, с волновыми функциями вида

$$\psi(\mathbf{x},t) = \phi_\omega(\mathbf{x})e^{-i\omega t}, \qquad \omega = E/\hbar, \tag{1.1}$$

где $\hbar$ – постоянная Планка. На физическом языке, плотности заряда и тока, соответствующие (квази)стационарным состояниям вида $\psi(\mathbf{x},t) = \phi_\omega(\mathbf{x})e^{-i\omega t}$, не зависят от времени, и таким образом создают постоянное электромагнитное поле. Такое поле не уносит энергию на бесконечность, позволяя электронному облаку бесконечно течь вокруг ядра.

Второй постулат Бора гласит, что под внешним возмущением электроны могут переходить из одного квантового стационарного состояния (боровской *стационарной орбиты*) в другое, излучая или поглощая квант света, энергия которого равна разности энергий E_+ и E_-.

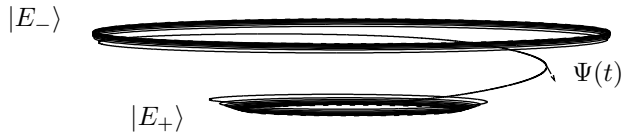

Рис. 1: Под внешним возмущением, волновая функция $\Psi(t)$ переходит из состояния $|E_-\rangle$ в состояние $|E_+\rangle$.

Этот постулат предполагает динамическую интерпретацию боровских переходов как долговременной сходимости

$$\Psi(t) \longrightarrow |E_\pm\rangle, \qquad t\to\pm\infty \tag{1.2}$$

для любой траектории $\Psi(t)$ соответствующей динамической системы. Тогда *квантовые стационарные состояния* следует рассматривать как точки *глобального аттрактора*, который мы обозначим $\mathfrak{A}$. Сходимость (1.2) принимает вид долговременной асимптотики

$$\psi(x,t) \sim \phi_{\omega_\pm}(x)e^{-i\omega_\pm t}, \qquad t\to\pm\infty, \tag{1.3}$$

которая имеет место для каждого решения конечной энергии (фиг. 1). Однако, из-за принципа суперпозиции, асимптотики вида (1.3) в общем случае невозможны для линейного автономного уравнения, будь это уравнение Шрёдингера $i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi - \frac{\mathrm{e}^2}{|\mathbf{x}|}\psi$ или релятивистское уравнение Шрёдингера или уравнение Дирака в кулоновском поле. Адекватное описание такого процесса требует рассмотреть уравнение для волновой функции электрона (уравнение Шрёдингера или Максвелла) вместе с уравнением Максвелла, которое описывает эволюцию четырёх-потенциала $A(x,t) = (\varphi(x,t), \mathbf{A}(x,t))$:

$$\begin{cases} (i\hbar\partial_t - \mathrm{e}\varphi)^2\psi = (c\frac{\hbar}{i}\nabla - \mathrm{e}\mathbf{A})^2\psi + m^2c^4\psi, \\ \Box\varphi = 4\pi\mathrm{e}(\bar\psi\psi - \delta(\mathbf{x})), \qquad \Box\mathbf{A} = 4\pi\mathrm{e}\frac{\bar\psi\cdot\nabla\psi - \nabla\bar\psi\cdot\psi}{2i}. \end{cases} \tag{1.4}$$

Рассмотрение такой системы представляется неизбежным, поскольку, опять же согласно боровским постулатам, переходы $|E_-\rangle \longmapsto |E_+\rangle$ сопровождаются электромагнитным излучением, отвечающим за атомные спектры (которые мы и наблюдаем в эксперименте).

Система взаимодействующих уравнений Максвелла-Шрёдингера была изначально введена в [Sch26]. Она является $\mathbf{U}(1)$-инвариантной нелинейной гамильтоновой системой, для которой можно было бы ожидать такого вот обобщения асимптотики (1.3):

$$(\psi(x,t), A(x,t)) \sim \left(\phi_{\omega_\pm}(x)e^{-i\omega_\pm t}, A_{\omega_\pm}(x)\right), \qquad t \to \pm\infty. \tag{1.5}$$

Асимптотика (1.5) означала бы, что множество всех уединённых волн $\{\left(\phi_\omega e^{-i\omega t}, A_\omega\right) : \omega \in \mathbb{R}\}$ составляет глобальный аттрактор для взаимодействующих уравнений. Таких асимптотик в настоящее время ещё не получено, однако упомянём, что существование уединённых волн для системы Максвелла-Дирака было установлено в [EGS96].

Уединённые волны как глобальные аттракторы в дисперсионных системах

Сходимость к глобальному аттрактору хорошо известна в диссипативных системах, таких, как уравнения Навье-Стокса (смотрите [Hen81, BV92, Tem97]). Глобальный аттрактор в таких системах образован *стационарными состояниями*, и асимптотики (1.3) имеют место только для $t \to +\infty$.

Нас интересует, могут ли гамильтоновы системы также обладать конечномерными глобальными аттракторами, и состоят ли эти аттракторы из уединённых волн. Хотя в таких системах нет диссипации как таковой, мы ожидаем, что сходимость обусловлена определённым трением за счёт дисперсионного механизма (локальное убывание энергии). Из-за трудностей, возникающих в системах взаимодействующих полей, (и, в частности, отсутствием априорных оценок в многих таких системах), мы будем работать с более простыми моделями, которые имеют некоторые общие ключевые свойства с системами взаимодействующих полей Максвелла-Дирака или Максвелла-Шрёдингера. Попробуем выделить эти ключевые свойства:

(1) Система $\mathbf{U}(1)$-инвариантна. Это требуется для существования уединённых волн $\phi_\omega(x)e^{-i\omega t}$.

(2) Линейная часть системы имеет дисперсионный характер. Локальное убывание энергии через излучение придаёт гамильтоновой системе определённые диссипативные свойства.

(3) Система нелинейна. Нелинейность нужна для сходимости к одной уединённой волне. В линейной системе такая сходимость невозможна из-за принципа суперпозиции.

Мы полагаем, что это именно эти свойства отвечают за сходимость (1.3) к "квантовым стационарным состояниям".

Замечание 1.1. Глобальная сходимость (1.3) или (1.5) для $\mathbf{U}(1)$-инвариантных уравнений наводит на мысль соответствующего обобщения на общие $\mathbf{G}$-инвариантные уравнения:

$$\psi(x,t) \sim \psi_\pm(x,t) = e^{\mathbf{\Omega}_\pm t}\phi_\pm(x), \qquad t \to \pm\infty, \tag{1.6}$$

где $\mathbf{\Omega}_\pm$ – элементы соответствующей алгебры Ли. Таким образом, глобальный аттрактор состоял бы из уединённых волн (1.6). В частности, для унитарной группы $\mathbf{G} = \mathbf{SU}(3)$ асимптотики (1.6) связывают "стационарные квантовые состояния" со структурой соответствующей алгебры Ли $\mathbf{su}(3)$. В этой связи, упомянём, что согласно теории Гелл-Мана – Неемана [GMN64], имеет место соответствие между алгебрами Ли и классификацией элементарных частиц которые являются "стационарными квантовыми состояниями". Это соответствие было подтверждено экспериментальным открытием гиперона Омега-Минус.

Существующие результаты по аттракторам в гамильтоновых системах

Локальная и глобальная сходимость к нулю. Асимптотики типа (1.3) были впервые открыты для случая $\psi_\pm = 0$ в теории рассеяния. А именно, Сигал, Моравец и Штраусс изучили (нелинейную) теорию рассеяния для решений нелинейного уравнения Клейна-Гордона в $\mathbb{R}^3$ [Seg66, Str68, MS72]. Эти результаты можно интерпретировать как *локальную* (имея в виду малые начальные данные) сходимость к нулю, $\psi(x,t) \sim \psi_\pm = 0, \qquad t \to \pm\infty$. Такая асимптотика имеет место на произвольном компактном множестве и означает хорошо известное локальное (в пространстве) убывание энергии. Такой *глобальной* (в смысле произвольных начальных данных) сходимости к нулю заведомо не будет при наличии решений вида уединённых волн, $\phi_\omega(x)e^{-i\omega t}$.

Существование уединённых волн. Существование уединённых волн вида $\psi_\omega(x,t)=\phi_\omega(x)e^{-i\omega t}$, $\omega\in\mathbb{R}$, $\phi_\omega\in H^1(\mathbb{R}^n)$, для нелинейного уравнения Клейна-Гордона (как и нелинейного уравнения Шрёдингера) в $\mathbb{R}^n$ в достаточно общей ситуации было установлено в [Str77]. В типичной ситуации, такие решения существуют для ω из интервала или набора интервалов вещественной оси. Упомянём, что существуют многочисленные результаты по существованию решений вида уединённых волн для нелинейных гамильтоновых систем с $\mathbf{U}(1)$ симметрией.

Устойчивость. В то время как все локализованные стационарные решения нелинейных волновых уравнений в размерностях $n\geq 3$ оказываются неустойчивыми ("теорема Деррика", [Der64]), *квазистационарные* уединённые волны могут быть орбитально устойчивыми. Устойчивость уединённых волн ведет своё начало в [VK73] и к настоящему времени хорошо изучена.

Локальная сходимость к уединённым волнам. Первые результаты по асимптотикам вида (1.3) с $\omega_\pm\neq 0$ были получены для нелинейного $\mathbf{U}(1)$-инвариантного уравнения Шрёдингера в контексте асимптотической устойчивости. То есть, выводятся асимптотики типа (1.3), но только для решений, изначально близких к уединённым волнам. Это впервые было сделано Зоффером и Вайнштейном и Буслаевым и Перельман [SW90, BP93].

Глобальная сходимость к уединённым волнам. Глобальная сходимость вида (1.3) с $\psi_\pm\neq 0$ и $\omega_\pm=0$ хорошо известна для некоторого числа нелинейных волновых задач (см. например, [Kom91, Kom99]). В этих задачах аттрактор является множеством всех *статических* стационарных состояний. В [Kom03, KK07a, KK07b] сходимость к множеству уединённых волн доказана для поля Клейна-Гордона взаимодействующего с одним и с несколькими нелинейными осцилляторами.

Мы знаем всего один недавний результат [Tao07] в области нетривиальных (ненулевых) глобальных аттракторов гамильтоновых уравнений в частных производных. В этой работе рассмотрено существование глобального аттрактора для нелинейного уравнения Шрёдингера в размерностях $n\geq 5$. Дисперсионная (уходящая) волна указана явно, благодаря использованию быстрого локального убывания энергии в многомерном случае. Была доказана компактность глобального аттрактора, однако он не был отождествлён с множеством уединённых волн, и его конечномерность также не была доказана [Tao07, Remark 1.18].

2 Модель и основной результат

Рассмотрим комплексное уравнение Клейна-Гордона с самодействием типа среднего поля в N точках:

$$\ddot\psi(x,t)=\Delta\psi(x,t)-m^2\psi(x,t)+\sum_{I=1}^N\rho_I(x)F_I(\langle\rho_I,\psi(\cdot,t)\rangle),\quad x\in\mathbb{R}^n,\quad t\in\mathbb{R},\quad n\geq 3,\tag{2.1}$$

где $m>0$ и $\langle\rho_I,\psi(\cdot,t)\rangle=\int_{\mathbb{R}^n}\bar\rho_I(x)\psi(x,t)\,d^nx$. Мы предполагаем, что $\rho_I(x)=\rho(x-X_I)$, где $X_I\in\mathbb{R}^n$, а ρ – гладкая вещественнозначная функция из пространства Шварца: $\rho\in\mathscr{S}(\mathbb{R}^n)$, $\rho\not\equiv 0$.

Предположение 2.1. Мы предполагаем, что уравнение (2.1) $\mathbf{U}(1)$-инвариантно и что все нелинейности $F_I(z)$, $1\leq I\leq N$, консервативны и полиномиальны:

$$F_I(z)=-\nabla U_I(z),\qquad\text{где}\quad U_I(z)=\sum_{l=1}^p u_{I,l}|z|^{2l},\quad u_{I,l}\in\mathbb{R},\quad u_{I,p_I}>0,\quad\text{и}\quad p_I\geq 2.\tag{2.2}$$

Обозначим через $\|\cdot\|_{L^2}$ норму в $L^2(\mathbb{R}^n)$. Пусть $H^s(\mathbb{R}^n)$, $s\in\mathbb{R}$, – пространство Соболева с нормой $\|\psi\|_{H^s}=\|(m^2-\Delta)^{s/2}\psi\|_{L^2}$. Для $s\in\mathbb{R}$ и $R>0$, обозначим через $H^s_0(\mathbb{B}^n_R)$ пространство распределений из $H^s(\mathbb{R}^n)$ с носителем в $\mathbb{B}^n_R$ (шар радиуса R в $\mathbb{R}^n$). Обозначим через $\|\cdot\|_{H^s,R}$ норму в пространстве $H^s(\mathbb{B}^n_R)$.

Определение 2.2. *(1)* $\mathscr{X}$ обозначает гильбертово пространство состояний $\Psi=(\psi,\pi)\in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ с нормой

$$\|\Psi\|^2_{\mathscr{X}}=\|\pi\|^2_{L^2}+\|\nabla\psi\|^2_{L^2}+m^2\|\psi\|^2_{L^2}=\|\pi\|^2_{L^2}+\|\psi\|^2_{H^1}.$$

(2) Зафиксируем $\varepsilon>0$. Пусть $\|\Psi\|^2_{\mathscr{X}^{-\varepsilon},R}=\|\pi\|^2_{H^{-\varepsilon},R}+\|\psi\|^2_{H^{1-\varepsilon},R}$, $R>0$, и пусть $\mathscr{Y}$ – банахово пространство с нормой

$$\|\Psi\|_{\mathscr{Y}}=\sum_{R=1}^\infty 2^{-R}\|\Psi\|_{\mathscr{X}^{-\varepsilon},R}<\infty.\tag{2.3}$$

Замечание 2.3. Уравнение (2.1) является гамильтоновой системой с фазовым пространством $\mathscr{X}$.

Лемма 2.4. *Вложение $\mathscr{X} \subset \mathscr{Y}$ компактно.*

Определение 2.5. *(1)* Обозначим через $\mathfrak{S}$ множество всех уединённых волн:

$$\mathfrak{S} = \{\psi \in C(\mathbb{R}, H^1(\mathbb{R}^n)): \quad \psi(x,t) = \phi_\omega(x)e^{-i\omega t}, \quad \omega \in \mathbb{R}, \quad \phi_\omega \in H^1(\mathbb{R}^n)\}. \tag{2.4}$$

(2) Солитонное многообразие является множеством соответствующих начальных данных:

$$\mathcal{S} = \left\{(\phi_\omega, -i\omega\phi_\omega)\colon \phi_\omega(x)e^{-i\omega t} \in \mathfrak{S}\right\}. \tag{2.5}$$

Замечание 2.6. Множество $\mathcal{S}$ инвариантно относительно умножения на $e^{i\theta}$, $\theta \in \mathbb{R}$; $\dim \mathcal{S} = 2$.

Определение 2.7 (Глобальный аттрактор). *(1)* $p \in \mathscr{X}$ является омега-предельной точкой решения $\Psi \in C(\mathbb{R}, \mathscr{X})$ если существует такая последовательность $t_j \to \infty$, что $\Psi(t_j) \xrightarrow[j\to\infty]{\mathscr{Y}} p$.

Равносильно, следуя [CV02], множество всех омега-предельных точек решения $\Psi \in C(\mathbb{R}, \mathscr{X})$ можно определить так:

$$\omega(\Psi) = \bigcap_{t\geq 0} \Big[\bigcup_{s\geq t} \Psi(s)\Big]_{\mathscr{Y}},$$

где $\big[\ \cdot\ \big]_{\mathscr{Y}}$ обозначает замыкание множества в топологии пространства $\mathscr{Y}$ из определения 2.2 *(2)*.

(2) Глобальный аттрактор $\mathcal{A} \subset \mathscr{X}$ является множеством всех омега-предельных точек всех решений конечной энергии:

$$\mathcal{A} = \bigcup_{\Psi(0)\in\mathscr{X}} \omega(\Psi).$$

Пусть S_τ, $\tau \in \mathbb{R}$, – оператор сдвига:

$$S_\tau \Psi(t) = \Psi(\tau + t).$$

Определение 2.8 (Траекторный аттрактор). *(1)* $\mathcal{B} \in C(\mathbb{R}, \mathscr{X})$ является *омега-предельной траекторией* решения $\Psi \in C(\mathbb{R}, \mathscr{X})$ если существует такая последовательность $t_j \to +\infty$, что

$$\forall T > 0, \qquad \sup_{-T\leq t\leq T} \|S_{t_j}\Psi(t) - \mathcal{B}(t)\|_{\mathscr{Y}} \xrightarrow[j\to\infty]{\mathscr{Y}} 0.$$

(2) Траекторный аттрактор $\mathfrak{A} \subset C(\mathbb{R}, \mathscr{X})$ является множеством всех омега-предельных траекторий всех решений конечной энергии $\Psi = (\psi, \dot\psi)$ уравнения (2.1).

Определение 2.9. Для $\rho \in \mathscr{S}(\mathbb{R}^n)$, $X_I \in \mathbb{R}^n$, и $\sigma_{IJ}(\omega) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{i(X_I - X_J)\cdot\xi}|\hat\rho(\xi)|^2}{\xi^2 + m^2 - (\omega + i0)^2}\, d^n\xi$, где $1 \leq I, J \leq N$, определим множества

$$Z_\rho = \{\omega \in \mathbb{R}\backslash[-m,m]\colon \hat\rho(\xi) = 0 \text{ для всех } \xi \in \mathbb{R}^n \text{ удовлетворяющих } m^2 + \xi^2 = \omega^2\};$$

$$Z_\sigma^{N'} = \Big\{\omega \in \mathbb{R}\colon \exists\, \mathcal{I}, \mathcal{J} \subset \{1,\dots,N\},\ |\mathcal{I}| = |\mathcal{J}| = N', \det_{I\in\mathcal{I}, J\in\mathcal{J}} \sigma_{IJ}(\omega) = 0\Big\}, \quad 1 \leq N' \leq N.$$

Предположение 2.10. Все $Z_\sigma^{N'}$, $1 \leq N' \leq N$, являются дискретными множествами, и

$$Z_\sigma^{N'} \cap ([-m,m] \cup Z_\rho) = \emptyset, \qquad 1 \leq N' \leq N.$$

Теорема 2.11 (Аттрактор для Клейна-Гордона с самодействием типа среднего поля)**.** *Предположим, что нелинейности $F_I(z)$, $1 \leq I \leq N$, таковы, что предположение 2.1 выполняется. Предположим, что спаривающая функция $\rho(x)$ и точки X_I, $1 \leq I \leq N$, таковы, что предположение 2.10 выполняется. Тогда верны следующие (равносильные) утверждения:*

(1) Траекторный аттрактор уравнения (2.1) *совпадает с множеством уединённых волн:*

$$\mathfrak{A} = \mathfrak{S}.$$

(2) Глобальный аттрактор уравнения (2.1) *совпадает с солитонным многообразием:*

$$\mathcal{A} = \mathcal{S}.$$

(3) Для любых $(\psi_0, \pi_0) \in \mathscr{X}$, решение $\psi(t)$ уравнения (2.1) *с начальными данными $(\psi, \dot\psi)|_{t=0} = (\psi_0, \pi_0)$ сходится к солитонному многообразию $\mathcal{S}$ в метрике пространства $\mathscr{Y}$:*

$$\lim_{t\to\pm\infty} \operatorname{dist}_{\mathscr{Y}}((\psi, \dot\psi)|_t, \mathcal{S}) = 0, \tag{2.6}$$

где $\operatorname{dist}_{\mathscr{Y}}(\Psi, \mathcal{S}) := \inf_{s\in\mathcal{S}} \|\Psi - s\|_{\mathscr{Y}}$.

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Phase transition for Ising model on Critical Lorentzian triangulation

Maxim Krikun*†,1 Anatoly Yambartsev‡,2

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[1] Institut Élie Cartan Nancy (IECN), Nancy-Université, CNRS, INRIA, Boulevard des Aiguillettes B.P. 239 F-54506 Vandœuvre-lès-Nancy, France.
E-mail: krikun@iecn.u-nancy.fr
current affiliation: Google Ireland, Dublin, Ireland

[2] Department of Statistics, Institute of Mathematics and Statistics, University of São Paulo, Rua do Matão 1010, CEP 05508–090, São Paulo SP, Brazil.
E-mail: yambar@ime.usp.br

Abstract

Ising model without external field on Lorentzian triangulation sampled from uniform distribution is considered. We prove the coexistence of at least two Gibbs measures in low temperature region. The proof is based on well known contour Peierls method extensively adopted in statistical physics. And we prove the uniqueness of Gibbs measure on the high temperature region.

Keywords: Lorentzian triangulation, Ising model, dynamical triangulation, quantum gravity
AMS 2000 Subject Classifications: 82B20, 82B26, 60J80

1 Introduction

Triangulations, and planar graphs in general, appear in physics in the context of 2-dimensional quantum gravity as a model for the discretized time-space.

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Perhaps the best understood it the model of Euclidean Dynamical Triangulations, which can be viewed as a way of constructing a random graph by gluing together a large number of equilateral triangles in all possible ways, with only topological conditions imposed on such gluing. Putting a spin system on such a random graph can be interpreted as a *coupling of gravity with matter*, and was an object of persistent interest in physics since the successful application of matrix integral methods to the Ising model on random lattice by Kazakov [7].

More recently, a model of Casual Dynamical Triangulations was introduced (see [1] for an overview). The distinguishing feature of this model is its lack of isotropy — the triangulation now has a distinguished time-like direction, giving it a partial order structure similar to Minkowski space, and imposing some non-topological restrictions on the way elementary triangles are glued. This last fact destroys the connection between the model and matrix integrals, in particular the analysis of the Ising model requires completely different methods (see e.g. [3]).

From a mathematical perspective, we deal here with nothing but a spin system on a random graph. Random graphs, arising from the CDT approach, were considered in [8] under the name of Lorentzian models. In the present paper we consider the Ising model on such graphs. When defining the model we pursue the formal Gibbsian approach [4]; namely, given a realisation of an infinite triangulation, we consider probability measures on the set of spin configurations that correspond to a certain formal Hamiltonian.

Our setting is drastically different from e.g. [7] and [3] in that we do not consider "simultaneous randomness", when both the triangulations and spin configurations are included into one Hamiltonian. Instead we first sample an infinite triangulation from some natural "uniform" measure, and then run an Ising model on it, thus the resulting semi-direct product measure is "quenched".

A modest goal of this work is to establish a phase transition for the Ising model in the above described "quenched" setting (the "annealed" version of the problem is surely interesting, but is also more technically challenging, so we don't attempt it for the moment). We use a variant of Peierls method to prove non-uniqueness of the Gibbs measure at low temperature. Quite surprisingly, proving the uniqueness at high temperature is not easy – the difficulty consists in presence of vertices of arbitrarily large degree, which does not allow for immediate application of uniqueness criteria such as e.g. [10]. We resort instead to the method of disagreement percolation [9], and use

the idea of "ungluing", borrowed from the paper [2], to get rid of vertices of very high degree. Finally, we show that the critical temperature is in fact non-random and coincides for a.e. random Lorentzian triangulation.

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2 Definitions and Main Results

Now we define rooted infinite Lorentzian triangulations in a cylinder $\mathsf{C} = S^1 \times [0, \infty)$.

Definition 2.1. *Consider a connected graph G embedded in a cylinder C. A face is a connected component of $\mathsf{C} \setminus G$. The face is a triangle if its boundary meets precisely three edges of the graph. An embedded triangulation T is such a graph G together with a subset of the triangular faces of G. Let the support $S(T) \subset \mathsf{C}$ be the union of G and the triangular faces in T. Two embedded triangulations T and T' are considered equivalent if there is a homeomorphism of $S(T)$ and $S(T')$ that corresponds T and T'.*

For convenience, we usually abbreviate "equivalence class of embedded triangulations" to "triangulation". This should not cause much confusion. We suppose that the number of the vertices of G is finite or countable.

Definition 2.2. *A triangulation T of C is called Lorentzian if the following conditions hold: each triangular face of T belongs to some strip $S^1 \times [j, j+1], j = 0, 1, \ldots,$ and has all vertices and exactly one edge on the boundary $(S^1 \times \{j\}) \cup (S^1 \times \{j+1\})$ of the strip $S^1 \times [j, j+1]$; and the number of edges on $S^1 \times \{j\}$ is positive and finite for any $j = 0, 1, \ldots$.*

In this paper we will consider only the case when the number of edges on the first level $S^1 \times \{0\}$ equal to 1. This is not restriction, only it gives formulas more clean.

Definition 2.3. *A triangulation T is called rooted if it has a root. The root in the triangulation T consists of a triangle t of T, called the root face, with an ordering on its vertices (x, y, z). The vertex x is the root vertex and the directed edge (x, y) is the root edge. The x and (x, y) belong to $S^1 \times \{0\}$.*

Note that this definition also means that the homeomorphism in the definition of the equivalence class respects the root vertex and the root edge.

For convenience, we usually abbreviate "equivalence class of embedded rooted Lorentzian triangulations" to "Lorentzian triangulation" or LT.

In the same way we also can define a Lorentzian triangulation of a cylinder $\mathsf{C}_N = S^1 \times [0, N]$. Let $\mathbb{LT}_N$ and $\mathbb{LT}_\infty$ denote the set of Lorentzian triangulations with support C_N and C correspondingly.

Gibbs and Uniform Lorentzian triangulations. Let $\mathbb{LT}_N$ be the set of all Lorentzian triangulations with only one (rooted) edge on the root boundary and with N slices. The number of edges on the upper boundary $S^1 \times \{N\}$ is not fixed. Introduce a Gibbs measure on the (countable) set $\mathbb{LT}_N$:

$$\mathsf{P}_{N,\mu}(T) = Z^{-1}_{[0,N]} \exp(-\mu F(T)), \tag{2.1}$$

where $F(T)$ denotes the number of triangles in a triangulation T and $Z_{[0,N]}$ is the partition function:

$$Z_{[0,N]} = \sum_{T \in \mathbb{LT}_N} \exp(-\mu F(T)).$$

The measure on the set of infinite triangulations $\mathbb{LT}_\infty$ is then defined as a weak limit

$$\mathsf{P}_\mu := \lim_{N \to \infty} \mathsf{P}_{N,\mu}\,.$$

It was shown in [8] that this limit exists for all $\mu \geq \mu_{cr} := \ln 2$.

Ising model on Uniform Infinite Lorentzian triangulation – quenched case. Let T be some fixed Lorentzian triangulation, $T \in \mathbb{LT}_\infty$. Let T_N be the projection of T on the cylinder C_N. We associate with every vertex v a spin $\sigma_v \in \{-1, 1\}$. Let $\Sigma(T)$ and $\Sigma_N(T)$ denote the set of of all spin configurations on T and T_N, respectively. The Ising model on T is defined by a formal Hamiltonian

$$H(\sigma) = \sum_{\langle v,v' \rangle \in V} \sigma_v \sigma_{v'} \tag{2.2}$$

where $\langle v, v' \rangle$ means that vertices v, v' are neighbours, i.e. are connected by an edge in T. Let ∂T_N be the set of vertices of T that lie on the circle $S^1 \times \{N + 1\}$. Fix some configuration on the boundary ∂T_N and denote it $\partial\sigma$. The Gibbs distribution with boundary condition $\partial\sigma$ is defined by the

following. Let $V(T_N)$ be the set of all vertices in T_N, then the energy of configuration $\sigma \in \Sigma_N(T)$ is

$$H_N(\sigma|\partial\sigma) = \sum_{\langle v,v'\rangle:\ v,v'\in V(T_N)} \sigma_v\sigma_{v'} + \sum_{\langle v,v'\rangle: v\in V(T_N), v'\in\partial T_N} \sigma_v\sigma_{v'} \tag{2.3}$$

which defines the probability

$$P^T_{N,\partial\sigma}(\sigma) = \frac{\exp\{-\beta H_N(\sigma|\partial\sigma)\}}{Z_{N,\partial\sigma}(T)} \tag{2.4}$$

where

$$Z_{N,\partial\sigma}(T) = \sum_{\sigma\in\Sigma_N(T)} \exp\{-\beta H_N(\sigma|\partial\sigma)\}.$$

When $N \to \infty$, for any sequence of boundary conditions $\partial\sigma$, a limit (at least along some subsequence) of measures $P^T_{N,\partial\sigma}$ exists by compactness. Such a limit is a probability measure on $\Sigma(T)$ with a natural σ-algebra, which we refer to as a Gibbs measure.

In general, it is well known that at least one Gibbs measure exists for the Ising model on any locally finite graph and for any value of the parameter β (see, e.g., [5] page 71). It is also known that the existence of more than one Gibbs measure is increasing in β, i.e. there exists a critical value $\beta_c \in [0,\infty]$ such that there is a unique Gibbs measure when $\beta > \beta_c$, and multiple Gibbs measures when $\beta < \beta_c$ (see [6] for an overview of relations between percolation and Ising model on general graphs).

Thus when considering the Ising model on Lorentzian triangulations it is natural to ask whether the critical temperature is finite (different from both 0 and ∞), and whether it depends on the triangulation. In the following theorems we show that the critical temperature is a.s. bounded both from 0 and ∞. And the last theorem proves that the critical temperature obeys a zero-one law and is therefore a.s. constant.

Theorem 1. *There exists a β_0 such that for all $\beta \in (\beta_0,\infty)$ there exist at least two Gibbs measures for $\mathsf{P}_{\mu_{cr}}$-a.e. T.*

Theorem 2. *There exists a small enough β_h such that for every $\beta \in [0,\beta_h)$ for $\mathsf{P}_{\mu_{cr}}$-a.e. Lorentzian triangulation T the Gibbs measure for the Ising model on T is unique.*

Critical temperature is constant a.s. Consider the critical temperature $\beta_c(G)$ of the Ising model on a graph G as a function of G. In the above two theorem we show that when T is a $P_{\mu_{cr}}$-random Lorentzian triangulation, we have $\beta_c(T) \in [\beta_h, \beta_0]$ a.s.

Theorem 3. *$\beta_c(T)$ is constant $\mathsf{P}_{\mu_{cr}}$-a.s.*

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Tail behaviour of multiple random integrals and U-statistics

Péter Major

Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary
e-mail: major@renyi.hu

The following problems will be discussed:
Let $\xi_1, \dots, \xi_n$ be a sequence of independent and identically distributed random variables with some probability distribution μ on a measurable space $(X, \mathcal{X})$ and let μ_n,

$$\mu_n(A) = \frac{1}{n}\#\{j\colon\ \xi_j \in A,\ 1 \le j \le n\}, \quad A \in \mathcal{X},$$

denote its empirical distribution. Let a measurable function $f(x_1, \dots, x_k)$ of k variables be given on the product space $(X^k, \mathcal{X}^k)$. Take the k-fold direct product of the normalized version $\sqrt{n}(\mu_n - \mu)$ of this empirical measure μ_n and define the following integral of the function f with respect to this normalized empirical distribution:

$$J_{n,k}(f) = \frac{n^{k/2}}{k!} \int' f(x_1, \dots, x_k)(\mu_n(\,dx_1) - \mu(\,dx_1)) \dots (\mu_n(\,dx_k) - \mu(\,dx_k)),$$

where the prime in $\int'$ means that the diagonals $x_j = x_l,\ 1 \le j < l \le k$, are omitted from the domain of integration. (1)

Problem A. Give a good estimate on the probabilities $P(J_{n,k}(f) > u)$ under appropriate conditions for the function f for all $u > 0$.

Problem B. Let a nice class $\mathcal{F}$ of functions $f(x_1, \dots, x_k)$ be given on the space $(X^k, \mathcal{X}^k)$. Give a good estimate on the probabilities $P\left(\sup_{f \in \mathcal{F}} J_{n,k}(f) > u\right)$ for all $u > 0$, where $J_{n,k}(f)$ denotes the integral of the function f defined in formula (1).

The solution of Problems A and B is useful in the study of limit theorems for non-parametric maximum likelihood estimates. The classical proof of the central limit theorem for maximum likelihood estimates is based on a linearization argument, where a good asymptotic solution of the maximum likelihood equation is given by means of a Taylor expansion and the omission of the high order terms in it. In the proof of the non-parametric versions of this result a similar method works, but we also need a good estimate in problems A and B in this case. These results guarantee that the linearization procedure we apply in the proof casuses only a negligible error.

It turned out useful to study problems A and B together with their U-statistic analogues. I recall the definition of U-statistics.

The definition of U-statistics. *Let a sequence $\xi_1, \dots, \xi_n$ of independent and identically distributed random variables be given with values on some measurable space $(X, \mathcal{X})$ together with a function $f(x_1, \dots, x_k)$ on the k-fold product space $(X^k, \mathcal{X}^k)$ with some $k \le n$. The expression*

$$I_{n,k}(f) = \frac{1}{k!} \sum_{\substack{1 \le j_s \le n,\ s=1,\dots,k \\ j_s \ne j_{s'} \text{ if } s \ne s'}} f\left(\xi_{j_1}, \dots, \xi_{j_k}\right) \tag{2}$$

is called a U-statistic of order k with kernel function f.

The following versions of problems A and B will be investigated.

Problem A'. Give a good estimate on the probabilities $P(n^{-k/2} I_{n,k}(f) > u)$ under appropriate conditions for the function f for all $u > 0$.

Problem B'. Let a nice class $\mathcal{F}$ of functions $f(x_1, \dots, x_k)$ be given on a (product) space $(X^k, \mathcal{X}^k)$ together with a sequence of independent and identically distributed random variables $\xi_1, \dots, \xi_n$ with values in $(X, \mathcal{X})$. Give a good estimate on the probabilities $P\left(\sup_{f \in \mathcal{F}} n^{-k/2} I_{n,k}(f) > u\right)$ for all $u > 0$ where $I_{n,k}(f)$ denotes the U-statistic of order k with kernel function f defined in formula (2).

It may be useful to remark that a U-statistic of order k with the kernel function f can be rewritten as

$$I_{n,k}(f) = \frac{n^k}{k!} \int' f(x_1, \dots, x_k) \mu_n(\, dx_1) \dots \mu_n(\, dx_k),$$

where μ_n is the empirical distribution of the sequence $\xi_1, \dots, \xi_n$. This shows that the essential difference between the random integrals introduced in formula (1) and the U-statistics is that in the random integrals $J_{n,k}(f)$ integration is taken with respect to the 'normalized' measures $\mu_n - \mu$, while in the integral representation of the U-statistics $I_{n,k}(f)$ with respect to the 'non-normalized' measures μ_n.

First I discuss problems A and A′ in the simplest case $k = 1$. In this case a good estimate on the tail distribution of sum of i.i.d. random variables (with zero expectation) has to be considered. The following classical result, called Bernstein's inequality, gives a useful estimate in this case.

Bernstein's inequality. *Let $\xi_1, \dots, \xi_n$ be independent random variables which satisfy the relations $P(|\xi_j| \le 1) = 1$ and $E\xi_j = 0$, $1 \le j \le n$. Let us introduce the notation $\sigma_j^2 = E\xi_j^2$, $1 \le j \le n$, $S_n = \sum_{j=1}^n \xi_j$ and $V_n^2 = \operatorname{Var} S_n = \sum_{j=1}^n \sigma_j^2$. The inequality*

$$P(S_n > u) \le \exp\left\{ -\frac{u^2}{2V_n^2 \left(1 + \frac{u}{3V_n^2}\right)} \right\} \tag{3}$$

holds for all numbers $u > 0$.

In nice cases Bernstein's inequality yields an estimate on the distribution of sums of independent random variables suggested by the central limit theorem. But in the general case the situation is more complex because of the coefficient $1 + \frac{u}{3V_n^2}$ in the denominator in its upper bound. In more detail, Bernstein's inequality yields the following estimate.

a) If $u \le \varepsilon V_n^2$ with some small number $\varepsilon > 0$, then $P(S_n > u) \le e^{-(1-\varepsilon)u^2/2V_n^2}$. This is almost such a good estimate as the estimate obtained by a formal application of the central limit theorem.

b) If $u \le 3V_n^2$, then $P(S_n > u) \le e^{-\text{const.}\, u^2/2V_n^2}$. This is a bound similar to that suggested by the central limit theorem.

c) If $u \gg V_n^2$, then $P(S_n > u) \le e^{-u}$.

In case c) Bernstein's inequality yields a very weak estimate which strongly differs from the estimate suggested by the central limit theorem. This result can be slightly improved by means of the so-called Bennett's inequality, but some examples can be given which show that no essential improvement of this result is possible. Hence if we are interested in a good estimate in Problem A or A′ for $k = 1$ and such a function f which is bounded by 1, then it is enough to restrict our attention to the case when $0 \le u \le \text{const.}\, nE\xi_i^2$. A similar picture arises for all $k \ge 1$, but to explain it some questions must be clarified.

In the study of Problem A′ in the case $k = 1$ it was natural to assume that $Ef(\xi_1) = 0$. Further investigations show that the natural multivariate counterpart of this condition is that so-called degenerate U-statistics must be estimated. Its definition is given below.

Definition of degenerate U-statistics. *Take a U-statistic $I_{n,k}(f)$ determined by a sequence of independent and identically distributed random variables $\xi_1, \dots, \xi_n$ with distribution μ and a kernel function $f(x_1, \dots, x_k)$. This U-statistic is degenerate if*

$$E(f(\xi_1, \dots, \xi_k) | \xi_1 = x_1, \dots, \xi_{j-1} = x_{j-1}, \xi_{j+1} = x_{j+1}, \dots, \xi_k = x_k) = 0$$
$$\text{for all indices } 1 \le j \le k \text{ and values } x_s \in X,\ s \in \{1, \dots, k\} \setminus \{j\}.$$

The notion of degenerate U-statistics is useful, because such U-statistics behave similarly to sums of independent random variables with *expectation zero*. Beside this, the study of general U-statistics can be reduced to the study of degenerate U-statistics by means of the following Hoeffding-decomposition.

Hoeffding decomposition of general U-statistics. *All U-statistics $I_{n,k}(f)$ of order k can be written in the form of linear combination*

$$I_{n,k}(f) = \sum_{j=0}^{k} n^{k-j} I_{n,j}(f_j) \tag{4}$$

of degenerate U-statistics $I_{n,j}(f_j)$. *The kernel functions* f_j *(of* j *variables) of the degenerate* U*-statistics* $I_{n,j}(f_j)$, $0 \le j \le k$, *can be calculated explicitly.*

The problems about the behaviour of the multiple random integrals $J_{n,k}(f)$ defined in formula (1) can also be reduced to problems about the behaviour of degenerate U-statistics by means of their appropriate decomposition. Such expressions can be written as the linear combination

$$J_{n,k}(f) = \sum_{j=0}^{k} c(n,j) n^{-j/2} I_{n,j}(f_j) \tag{5}$$

of degenerate U-statistics with the same kernel functions f_j which appear in formula (4) and with some appropriate coefficients $c(n,j)$ such that $c(n,j) < K(j)$ with some universal constant $K(j)$.

In the definition of the random integral $J_{n,k}(f)$ integration is taken with respect to the signed measure $\mu_n - \mu$, and this 'normalization' diminishes the value of the integral. This diminishing effect is reflected in the relatively small value of the coefficients $c(n,j)n^{-j/2}$ in formula (5).

In an informal way we can interpret Bernstein's inequality so that sums of independent random variables with expectation zero behave so as the central limit theorem suggests. To find its multivariate version we have to know the appropriate limit theorem for degenerate U-statistics. Such a limit theorem can be formulated by means of multiple Wiener–Itô integrals. To formulate it I recall the definition of white noise and make some comments about the definition of multiple Wiener–Itô integrals.

The notion of white noise. *Let a measure* μ *be given on some measurable space* $(X, \mathcal{X})$. *A system of jointly Gaussian random variables indexed by the measurable sets* $A \subset X$ *such that* $\mu(A) < \infty$ *is a white noise with reference measure* μ *if*

$$E\mu_W(A)\mu_W(B) = \mu(A \cap B) \quad \textit{and} \quad E\mu_W(A) = 0$$

for all measurable sets $A, B \subset X$ *such that* $\mu(A) < \infty$ *and* $\mu(B) < \infty$.

If a white noise μ_W is given with some reference measure μ together with a function $f(x_1, \dots, x_k)$ square integrable with respect to the k-fold product μ^k of the measure μ, then the k-fold Wiener–Itô integral

$$Z_{\mu,k}(f) = \frac{1}{k!} \int f(x_1, \dots, x_k) \mu_W(\,dx_1) \dots \mu_W(\,dx_k) \tag{6}$$

of this function f with respect to the white noise μ_W can be defined in a natural way. (First this integral is defined for simple so-called step functions which take a constant value on finitely many rectangles, and disappear outside them. Then the integral can be extended to general functions by means of an appropriate L_2-isomorphism.)

The following limit theorem holds.

Limit distribution theorem for degenerate U-statistics. *Let us consider such a sequence $I_{n,k}(f)$, $n = k, k+1, \dots$, of degenerate U-statistics which is determined by a sequence of independent and identically distributed random variables $\xi_1, \xi_2, \dots$, on a measurable space $(X, \mathcal{X})$ with distribution μ and a (canonical) function $f(x_1, \dots, x_k)$ square integrable with respect to the measure μ^k. The normalized degenerate U-statistics $n^{-k/2} I_{n,k}(f)$ converge in distribution to the k-fold Wiener–Itô integral*

$$Z_{\mu,k}(f) = \frac{1}{k!} \int f(x_1, \dots, x_k) \mu_W(\, dx_1) \dots \mu_W(\, dx_k)$$

of the function f with respect to a white noise μ_W with reference measure μ if $n \to \infty$.

It is natural to consider that version of problems A and B where Wiener–Itô integrals are considered instead of the integrals $J_{n,k}(f)$. This is is done in the formulation of the following Problems A$''$ and B$''$.

Let us consider the Wiener–Itô integral $Z_{\mu,k}(f)$ of a function $f(x_1, \dots, x_k)$ of k variables with respect to a white noise μ_W with reference measure μ introduced in formula (3.3) and study the following problems.

Problem A$''$. Let us give a good estimate on the probability $P(Z_{\mu,k}(f) > u)$ for all numbers $u > 0$.

Problem B$''$. Let a nice class $\mathcal{F}$ of functions $f(x_1, \dots, x_k)$ of k variables be given. Take the Wiener–Itô integral $Z_{\mu,k}(f)$ of all functions $f \in \mathcal{F}$ with respect to a white noise μ_W. Give a good estimate on the distribution of the supremum of these random integrals, i.e. on the probability $P\left(\sup_{f \in \mathcal{F}} Z_{\mu,k}(f) > u \right)$ for all numbers $u > 0$.

The following result gives the solution of Problem A$''$.

Estimation about the tail distribution of Wiener–Itô integrals. *Let a white noise μ_W be given with reference measure μ together with a function $f(x_1, \dots, x_k)$ of k variables on a measurable space $(X, \mathcal{X})$ such that*

$$\int f^2(x_1, \dots, x_k) \mu(\, dx_1) \dots \mu(\, dx_k) \le \sigma^2$$

with some number $\sigma^2 < \infty$. The Wiener–Itô integral

$$Z_{\mu,k}(f) = \frac{1}{k!} \int f(x_1, \dots, x_k) \mu_W(\, dx_1) \dots \mu_W(\, dx_k)$$

introduced in formula (6) satisfies the inequality

$$P(k!|Z_{\mu,k}(f)| > u) \le C \exp\left\{-\frac{1}{2}\left(\frac{u}{\sigma}\right)^{2/k}\right\}$$

for all numbers $u > 0$ with some constant $C = C(k) > 0$ depending only on the multiplicity k of the integral.

It can be proved that this estimate is sharp.

Similar, but slightly weaker estimates hold for degenerate U-statistics and multiple random integrals with respect to normalized empirical distributions.

Estimate on the tail distribution of degenerate U-statistics. *Let $\xi_1, \dots, \xi_n$ be a sequence of independent and identically distributed random variables on a measurable space $(X, \mathcal{X})$ with distribution μ. Take a function $f(x_1, \dots, x_k)$ on the space $(X^k, \mathcal{X}^k)$ canonical with respect to the measure μ which satisfies the conditions*

$$\|f\|_\infty = \sup_{x_j \in X,\, 1 \le j \le k} |f(x_1, \dots, x_k)| \le 1$$

$$\|f\|_2^2 = \int f^2(x_1, \dots, x_k)\mu(\,dx_1) \dots \mu(\,dx_k) \le \sigma^2$$

with some number $0 < \sigma^2 \le 1$, and consider the (degenerate) U-statistic defined in formula (1.2) with the help of these quantities. Then there exist some constants $A = A(k) > 0$ and $B = B(k) > 0$ depending only on the order k of the U-statistic such that the inequality

$$P(k!n^{-k/2}|I_{n,k}(f)| > u) \le A \exp\left\{-\frac{u^{2/k}}{2\sigma^{2/k}\left(1 + B\left(un^{-k/2}\sigma^{-(k+1)}\right)^{1/k}\right)}\right\}$$

holds for all numbers $0 \le u \le n^{k/2}\sigma^{k+1}$.

Estimate about the tail distribution of random integrals with respect to normalized empirical distributions. *Let a sequence $\xi_1, \dots, \xi_n$ of independent and identically distributed random variables be given with distribution μ which take their values on a measurable space $(X, \mathcal{X})$ together with a function $f(x_1, \dots, x_k)$ on the k-fold product space $(X^k, \mathcal{X}^k)$ which satisfy relations (4.1) and (4.2) with some constant $0 < \sigma \le 1$. Then there exist some constants $C = C_k > 0$ and $\alpha = \alpha_k > 0$ depending only on the multiplicity k of the integral $J_{n,k}(f)$ defined in formula (1.1) such that the following inequality holds:*

$$P\left(|J_{n,k}(f)| > u\right) \le C \exp\left\{-\alpha\left(\frac{u}{\sigma}\right)^{2/k}\right\} \quad \text{for all numbers } 0 < u \le n^{k/2}\sigma^{k+1}.$$

It can be proved that under some not too restrictive conditions similar bound can be given about the tail distribution of the supremum considered in Problems B, B′ and B″ as for the tail distribution of the single terms appearing in these supremums. The introduction of the following definitions proved to be useful.

Definition of L_2-dense classes of functions with respect to some measure. *Let a measurable space be $(Y, \mathcal{Y})$ be given together with a σ-finite measure ν and a class $\mathcal{G}$ of $\mathcal{Y}$-measurable, real valued functions on this space. This class of functions $\mathcal{G}$ is called an L_2-dense class with respect to ν with parameter D and exponent L if for all numbers $1 \ge \varepsilon > 0$ there exists a subclass $\mathcal{G}_\varepsilon = \{g_1, \dots, g_m\} \subset \mathcal{G}$ in the space $L_2(Y, \mathcal{Y}, \nu)$ consisting of $m \le D\varepsilon^{-L}$ elements such that $\inf\limits_{g_j \in \mathcal{G}_\varepsilon} \int |g - g_j|^2 \, d\nu < \varepsilon^2$ for all functions $g \in \mathcal{G}$.*

Definition of L_2-dense classes of functions. *Let us have a measurable space $(Y, \mathcal{Y})$ and a set $\mathcal{G}$ of $\mathcal{Y}$-measurable real valued functions on this space. We call $\mathcal{G}$ an L_2-dense class of functions with parameter D and exponent L if it is L_2-dense with parameter D and exponent L with respect to all probability measures ν on $(Y, \mathcal{Y})$.*

First I formulate a result about the supremum of Wiener–Itô integrals i.e. about Problem B″, and then I present a result on Problems B and B′.

Estimate about the tail distribution of the supremum of Wiener–Itô integrals. *Let us consider a measurable space $(X, \mathcal{X})$ together with a σ-finite non-atomic measure μ on it, and let μ_W be a white noise with reference measure μ on $(X, \mathcal{X})$. Let $\mathcal{F}$ be a countable and L_2-dense class of functions $f(x_1, \dots, x_k)$ on $(X^k, \mathcal{X}^k)$ with some parameter D and exponent L with respect to the product measure μ^k such that*

$$\int f^2(x_1, \dots, x_k)\mu(\,dx_1) \dots \mu(\,dx_k) \le \sigma^2 \quad \text{with some } 0 < \sigma \le 1 \text{ for all } f \in \mathcal{F}.$$

Let us consider the multiple Wiener integrals $Z_{\mu,k}(f)$ introduced in formula (3.3) for all $f \in \mathcal{F}$. The inequality

$$P\left(\sup_{f \in \mathcal{F}} |Z_{\mu,k}(f)| > u\right) \le C(D+1) \exp\left\{-\alpha \left(\frac{u}{\sigma}\right)^{2/k}\right\}$$

holds with some universal constants $C = C(k) > 0$ and $\alpha = \alpha(k) > 0$ if

$$\left(\frac{u}{\sigma}\right)^{2/k} \ge ML \log \frac{2}{\sigma} \quad \textit{with some appropriate constant } M = M(k) > 0.$$

The next result is an estimate on the tail-distribution of the supremum of random integrals $J_{n,k}(f)$ defined in formula (1).

Estimate on the tail distribution of the supremum of multiple integrals with respect to a normalized empirical distribution. *Let us have a probability measure μ on a measurable space $(X, \mathcal{X})$ together with a countable and L_2-dense class $\mathcal{F}$ of functions $f = f(x_1, \dots, x_k)$ of k variables with some parameter D and exponent L, $L \ge 1$, on the product space $(X^k, \mathcal{X}^k)$ such that*

$$\|f\|_\infty = \sup_{x_j \in X,\ 1 \le j \le k} |f(x_1, \dots, x_k)| \le 1,$$

and

$$\|f\|_2^2 = Ef^2(\xi_1, \dots, \xi_k) = \int f^2(x_1, \dots, x_k)\mu(\,dx_1) \dots \mu(\,dx_k) \le \sigma^2$$

for all functions $f \in \mathcal{F}$ with some constant $0 < \sigma \le 1$. Then there exist some constants $C = C(k) > 0$, $\alpha = \alpha(k) > 0$ and $M = M(k) > 0$ depending only on the parameter k such that the supremum of the random integrals $J_{n,k}(f)$, $f \in \mathcal{F}$, defined by formula (1.1) satisfies the inequality

$$P\left(\sup_{f \in \mathcal{F}} |J_{n,k}(f)| \ge u\right) \le CD \exp\left\{-\alpha \left(\frac{u}{\sigma}\right)^{2/k}\right\},$$

provided that

$$n\sigma^2 \ge \left(\frac{u}{\sigma}\right)^{2/k} \ge M(L+\beta)^{3/2} \log \frac{2}{\sigma},$$

where $\beta = \max\left(\frac{\log D}{\log n}, 0\right)$ and the numbers D and L agree with the parameter and exponent of the L_2-dense class $\mathcal{F}$.

A similar estimate holds for the supremum of degenerate U-statistics $I_{n,k}(f)$, $f \in \mathcal{F}$. The only difference in comparison with the above result that in the case of the supremum of U-statistics the additional condition has to be imposed that the U-statistics $I_{n,k}(f)$ must be degenerate.

A more detailed discussion of the results described above can be found in my work [1]. Beside this, I plan to publish a Lecture Note which also contains a complete discussion of the technical details. For the time being this Lecture Note [2] can be found only on my homepage. Both works [1] and [2] contain a more detailed list of references.

References

1.) Péter Major: Tail behaviour of multiple random integrals and U-statistics. *Probability Reviews.* 448–505, (2005)

2.) Péter Major: On the tail behavior of multiple random integrals and degenerate U-statistics. http://www.renyi.hu/˜major

Dynamics of phase boundary with particle annihilation

V. A. Malyshev, A. D. Manita*

We are happy to dedicate this paper to the memory of R. L. Dobrushin. He was a very curious person and had wide scope in probability. Now obviously he could bring his own vision for statistical physics of economic phenomena which is now at its very beginning.

1 Introduction

On one-dimensional lattice $\mathbf{Z}_\varepsilon = \varepsilon\mathbf{Z} = \{\varepsilon m : m \in \mathbf{Z}\}$, $\varepsilon > 0$, there are particles of two types — plus particles and minus particles. Denote $\nu_m^{\pm}(t)$ the number of plus(minus)-particles at site εm. We define the following continuous time Markov process on $[0, \infty)$ by the following conditions:

1. at any time $t > 0$ the state of the process is the vector $\nu_m^{\pm}(t), m \in \mathbf{Z}$;

2a. any plus particle, independently of other plus particles, performs a simple random walk: that is it jumps from εm to $\varepsilon(m+1)$ with rate μ_+ and from εm to $\varepsilon(m-1)$ with rate λ_+.

2b. any minus particle, independently of other minus particles, performs a simple random walk: that is it jumps from εm to $\varepsilon(m+1)$ with rate λ_- and from εm to $\varepsilon(m-1)$ with rate μ_-;

3. at time 0 all plus particles have positive coordinates, all minus particles have negative coordinates;

4a. if a plus particle jumps to a site where there are minus particles it immediately annihilates with one of the minus particles at this site ;

4b. if a minus particle jumps to a site where there are plus particles it immediately annihilates with one of the plus particles at this site.

There are no problems with the existence of this process. Let $\beta_\varepsilon(t) \in \mathbf{Z}_\varepsilon$ be the point where the last annihilation before the time t happened. We call it the phase boundary.

Besides the interpretation related to annihilation of particles there is another one — the microdynamics of the price formation, where the market contains many players and is formed by their behaviour. Namely, $\beta_\varepsilon(t)$ is the price of some product at time t. Plus particles are bears which want to lower the price of this product (we assume further on that $\alpha_+ = \lambda_+ - \mu_+ > 0$), minus particles are bulls which want to increase the price (we assume $\alpha_- = \lambda_- - \mu_- > 0$). Annihilation is a bargain which is performed when the demand and offer prices coincide. Recent models of price formation [5, 2, 3, 4] have much in common with our model, however they are closer in spirit to queueing models. Our model is closer to statistical physics models. Anyway, all such models cannot pretend on practical implementation, mainly because external influence on the action of players is not taken into account.

*Faculty of Mathematics and Mechanics, Lomonosov Moscow State University, Moscow, Russia.
E-mail: malyshev2@yahoo.com, manita@mech.math.msu.su.
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Our goal is to find asymptotic (for large times) behaviour of the (macro)price $\beta_\varepsilon(t)$ as the result of many micro-bargains.

2 Main result

*Initial distribution of particl*es. We assume for simplicity that at time $t = 0$ the distribution of plus particles is inhomogeneous Poisson with density $\rho_+(\varepsilon m)$, where $\rho_+(x)$ is some strictly positive function on $(0, \infty)$. This means that the random variables $\nu_+(\varepsilon m)$ are independent and have Poisson distribution with rate $\rho_+(\varepsilon m)$. Similarly, the distribution of minus particles is inhomogeneous Poisson with density $\rho_-(\varepsilon m)$, where $\rho_-(x)$ is some positive function on $(0, \infty)$.

Notation. Next we introduce main definitions and give their intuitive interpretation. Our interpretation concerns the situation when, instead of point particles on a lattice, there is a continuous media of infinitesimally small particles of two types, the particles move with fixed velocities $-\alpha_+ < 0, \alpha_- > 0$ and have initial densities $\rho_\pm(x)$ correspondingly. That is there are no fluctuations. Define the functions

$$M_-(r) = \int_{-r}^{0} \rho_-(y)\, dy \qquad \text{and} \qquad M_+(r) = \int_0^r \rho_+(y)\, dy \quad \text{for} \quad r \geq 0. \tag{1}$$

We interprete $M_\pm(r)$ as the cumulative mass of plus (minus) particles on the distance less than r from zero. Under above assumptions on $\rho_\pm$ we see that the functions $M_\pm(r)$ are strictly increasing on $(0, +\infty)$ and, therefore, their inverse functions $r_\pm(M)$, defined by the equation

$$M_\pm(r_\pm(M)) = M$$

exist and are strictly increasing. For example, the functions $r_+(M)$ define the interval $(0, r_+)$ where the mass plus particles equals M. Then the function

$$T(M) := \frac{r_-(M) + r_+(M)}{\alpha_- + \alpha_+} \tag{2}$$

defines the time interval $(0, T(M))$ during which mass M of plus and mass M of minus particles annihilate. The function $T(M)$ is also strictly increasing on $[0, +\infty)$ and is invertible. Denote its inverse function by $M(T)$. The place where the latter of these particles meet

$$r_+(M(T)) - \alpha_+ T = -r_-(M(T)) + \alpha_- T = \beta(T) \tag{3}$$

is the coordinate of the boundary at time T. Excluding from the system (3) the terms that are linear in T, we get

$$\beta(T) = r_+(M(T))\frac{\alpha_-}{\alpha_- + \alpha_+} - r_-(M(T))\frac{\alpha_+}{\alpha_- + \alpha_+}\,.$$

Scaling limit for the stochastic model. Here we return to the stochastic particle model and formulate the main result.

Theorem 1 *For any fixed $\tau \geq 0$ the following convergence in probability holds*

$$\beta_\varepsilon(\varepsilon^{-1}\tau) \to \beta_0(\tau) \qquad\qquad (\varepsilon \to 0),$$

where the function $\beta_0 :\ \mathbf{R}_+ \to \mathbf{R}$ is deterministic and has the following explicit form

$$\beta_0(\tau) = \frac{-\alpha_+ r_-(M(\tau)) + \alpha_- r_+(M(\tau))}{\alpha_- + \alpha_+}\,.$$

Corollary 1 *Consider the homogeneous case* $\rho_-(y) \equiv \rho_-,\ y<0, \quad \rho_+(y) \equiv \rho_+,\ y>0$. *All functions defined above are linear:* $M_-(r) = \rho_- r, \quad M_+(r) = \rho_+ r, \quad r_\pm(M) = M/\rho_\pm$, $T(M) = M\,\frac{\rho_-^{-1}+\rho_+^{-1}}{\alpha_-+\alpha_+}, \quad M(T) = T\frac{\alpha_-+\alpha_+}{\rho_-^{-1}+\rho_+^{-1}}$, *and, hence, the phase boundary* $\beta_\varepsilon(\tau)$ *moves with an asymptotically constant velocity:*

$$\beta_\varepsilon(\tau) \to \beta(\tau) = \tau\,\frac{-\alpha_+\rho_-^{-1}+\alpha_-\rho_+^{-1}}{\rho_-^{-1}+\rho_+^{-1}} = \tau\,\frac{-\alpha_+\rho_+ + \alpha_-\rho_-}{\rho_+ + \rho_-}\,.$$

3 Proof

Our plan is to show that the limiting behavior of $\beta_\varepsilon(t)$ in the stochastic model corresponds to the deterministic evolution described in (1)–(3). To do this we need some control over the random fluctuations in the limit $\varepsilon \to 0$. This control can be achieved by use of exponential bounds for some families of events. The proof uses some ideas from [1].

Definition 1 *We say that a family of events* $\mathcal{A} = \{A_\varepsilon\}_{\varepsilon>0}$ *has a property of exponential asymptotic sureness (e.a.s.), if there exist constants* $\mathcal{K}_\mathcal{A} > 0$, $q_\mathcal{A} > 0$, $\varepsilon_\mathcal{A} > 0$ *such that for all* $\varepsilon < \varepsilon_\mathcal{A}$ *the following inequality holds*

$$\mathsf{P}\,(A_\varepsilon) \geq 1 - \mathcal{K}_\mathcal{A} \exp\left(-q_\mathcal{A}\varepsilon^{-1}\right).$$

In the sequel, for breavity, we say sometimes that the event A_ε has probability exponentially close to one. We will use the following fact: if two sequences $\mathcal{A} = \{A_\varepsilon\}_{\varepsilon>0}$ and $\mathcal{B} = \{B_\varepsilon\}_{\varepsilon>0}$ have the property e.a.s., then this property holds also for the sequence $\mathcal{C} = \{A_\varepsilon \cap B_\varepsilon\}_{\varepsilon>0}$.

It is helpful to enumerate the particles at time 0 somehow with the only condition that

$$\cdots \leq x_3^-(0) \leq x_2^-(0) \leq x_1^-(0) < 0 < x_1^+(0) \leq x_2^+(0) \leq x_3^+(0) \leq \cdots \qquad .$$

Denote by $q_-(1)$ and $q_+(1)$ the indices of plus and minus particles of the first annihilating pair. For concreteness one can assume that if some plus (minus) particle jumps to a site where there are several minus (plus) particles then it annihilates with the minus (plus) particle havine minimal index. Let σ_1 be the time moment when the first annihilation occurs. Since particles move independently, their order can change in time, so, in general, $x^-_{q_-(1)}(0) \neq x_1^-(0)$ and $x^+_{q_+(1)}(0) \neq x_1^+(0)$. Similarly, we define $q_-(m)$ and $q_+(m)$ as indices of particles of the m-th annihilating pair and σ_m as the time moment of the m-th annihilation.

To simplify some expressions below we assume that the densities $\rho_\pm$ are bounded and strictly separated from zero: $0 < d_1 \leq \rho_\pm(\pm y) \leq d_2 < +\infty, \quad y \geq 0$. This assumption implies, in particular, that all functions $M_\pm$ and $r_\pm$ have uniformly bounded derivatives.

Fix some $M > 0$. Let $N_\varepsilon = [M\varepsilon^{-1}]$. Consider the N_ε-th pair of annihilating particles $x^-_{q_-(N_\varepsilon)}$ and $x^+_{q_+(N_\varepsilon)}$. The main idea is to prove that for small ε the random time σ_{N_ε} is close to the value $T(M)\varepsilon^{-1}$ and the random coordinate $x^-_{q_-(N_\varepsilon)}(0) \in \mathbf{Z}_\varepsilon$ is close to $-r_-(M)$. In more precise terms, it is sufficient to prove that for any small fixed positive numbers $\varkappa_0, \varkappa_1, \zeta_-, \zeta_+$ *with probability exponentially close to one* (as $\varepsilon \to 0$)

a) the moment σ_{N_ε} belongs to the time interval $(t_0(M,\varepsilon), t_1(M,\varepsilon))$, where

$$t_0(M,\varepsilon) \;=\; (T(M) - \varkappa_0)\varepsilon^{-1}, \qquad t_1(M,\varepsilon) \;=\; (T(M) + \varkappa_1)\varepsilon^{-1}; \tag{4}$$

b) starting point of the minus particle $x^-_{q_-(N_\varepsilon)}(0)$ belongs to the set $(-r_-(M) - \zeta_-, -r_-(M) + \zeta_-) \cap \mathbf{Z}_\varepsilon$;

c) similarly, starting point of the plus particle $x^+_{q_+(N_\varepsilon)}(0)$ belongs to the set $(r_+(M) - \zeta_+, r_+(M) + \zeta_+) \cap \mathbf{Z}_\varepsilon$.

Assume for a while that the above statements *a)–c)* are proved. Let us explain now how to prove the theorem. Recall that $\beta_\varepsilon(\sigma_{N_\varepsilon} + 0) = x^-_{q_-(N_\varepsilon)}(\sigma_{N_\varepsilon}) = x^+_{q_+(N_\varepsilon)}(\sigma_{N_\varepsilon})$. Individual motion of a minus particle is a simple random walk on $\mathbf{Z}_\varepsilon$ with the mean drift $\alpha_-\varepsilon = (\lambda_- - \mu_-)\,\varepsilon$, so applying the upper bound of the large deviation theory, we get that for any fixed $i \in \mathbf{N}$, $s > 0$ and $\delta_0 > 0$ with probability exponentially close to one

$$x_i^-(s\varepsilon^{-1}) - x_i^-(0) \in ((\alpha_- - \delta_0)\,s, (\alpha_- + \delta_0)\,s)\,.$$

In fact, even more stronger result holds: for fixed $s_2 > s_1 > 0$ and $\delta_0 > 0$ the family of events $\{D_\varepsilon\}$, where $D_\varepsilon = \left\{ x_i^-(s\varepsilon^{-1}) - x_i^-(0) \in ((\alpha_- - \delta_0)\,s, (\alpha_- + \delta_0)\,s)\,, \quad \forall s \in [s_1, s_2] \right\}$, has a property of e.a.s. Together with *a)* this gives that

$$x^-_{q_-(N_\varepsilon)}(\sigma_{N_\varepsilon}) - x^-_{q_-(N_\varepsilon)}(0) \in ((\alpha_- - \delta_0)\,(T(M) - \varkappa_0)\,, (\alpha_- + \delta_0)\,(T(M) + \varkappa_1))$$

with probability exponentially close to one. Combining the latter statement with the statement *b)* we conclude that with probability exponentially close to one

$$x^-_{q_-(N_\varepsilon)}(\sigma_{N_\varepsilon}) \in (\alpha_- T(M) - r_-(M) - \gamma, \alpha_- T(M) - r_-(M) + \gamma)$$

where $\gamma = \gamma(\delta_0, \varkappa_0, \varkappa_1, \zeta_-) > 0$ can be made arbitrary small, i.e., $\gamma(\delta_0, \varkappa_0, \varkappa_1, \zeta_-) \to 0$ as $\max(\delta_0, \varkappa_0, \varkappa_1, \zeta_-) \to 0$. Using (2) we see that

$$\begin{aligned} -r_-(M) + \alpha_- T(M) &= -r_-(M) + \alpha_- \frac{r_-(M) + r_+(M)}{\alpha_- + \alpha_+} \\ &= -r_-(M)\frac{\alpha_+}{\alpha_- + \alpha_+} + r_+(M)\frac{\alpha_-}{\alpha_- + \alpha_+} \end{aligned}$$

and, hence, $\beta_\varepsilon(\sigma_{N_\varepsilon}) \to -r_-(M)\frac{\alpha_+}{\alpha_- + \alpha_+} + r_+(M)\frac{\alpha_-}{\alpha_- + \alpha_+}$ in probability as $\varepsilon \to 0$.

To finish the proof of Theorem we need only to check that $\beta_\varepsilon(\sigma_{N_\varepsilon}) - \beta_\varepsilon(\varepsilon^{-1}T(M)) \to 0$ as $\varepsilon \to 0$. This corresponds to continuity property of the border on the macroscopic time scale $\tau = T$. To establish this fact we should take into account that: 1) due to the drift assumption ($\alpha_\pm > 0$) the random sequence $\{\sigma_{m+1} - \sigma_m,\ m \in \mathbf{N}\}$ admits uniform exponential estimates for distribution tails of $\sigma_{m+1} - \sigma_m$ (we refer reader to [6] for the corresponding technique); 2) in finite microtime t displacements of walking particles have the order $O(\varepsilon)$ while in finite macrotime τ their displacements have the order $O(1)$. We omit details.

To prove the statements *a)* we need the following main lemma. Denote by $\mathcal{N}_-(0, t_m(M, \varepsilon))$ a set of minus particles that collide with plus particles on the time interval $(0, t_m(M, \varepsilon))$.

Lemma 1 *For any sufficiently small $\varkappa_2, \varkappa_3 > 0$ the following events F_ε and G_ε,*

$$F_\varepsilon: \quad |\mathcal{N}_-(0, t_0(M, \varepsilon))| < (M - \varkappa_2)\varepsilon^{-1}, \qquad G_\varepsilon: \quad |\mathcal{N}_-(0, t_1(M, \varepsilon))| > (M + \varkappa_3)\varepsilon^{-1},$$

have probabilities exponentially close to one.

Lemma 1 follows from the next two lemmas. Lemma 2 deals with initial distribution of particles and Lemma 3 controls collective deplacements of minus and plus particles.

Lemma 2 *Let $y_1 < y_2 \le 0$ and $0 \le z_1 < z_2$. Then for any $\delta > 0$ the following families of events*

$$L_\varepsilon = \{\textit{the number of minus particles sitting at time } t=0 \textit{ in the set } (y_1,y_2)\cap \mathbf{Z}_\varepsilon$$

$$\textit{is between } \left(\int_{y_1}^{y_2} \rho_-(y)\,dy - \delta\right)\varepsilon^{-1} \textit{ and } \left(\int_{y_1}^{y_2} \rho_-(y)\,dy + \delta\right)\varepsilon^{-1}\},$$

$$R_\varepsilon = \{\textit{the number of plus particles sitting at time } t=0 \textit{ in the set } (z_1,z_2)\cap \mathbf{Z}_\varepsilon$$

$$\textit{is between } \left(\int_{z_1}^{z_2} \rho_+(y)\,dy - \delta\right)\varepsilon^{-1} \textit{ and } \left(\int_{z_1}^{z_2} \rho_+(y)\,dy + \delta\right)\varepsilon^{-1}\},$$

have probabilities exponentially close to one.

Lemma 3 *For any $\delta_1 > 0$ each family of events*

$$A_\varepsilon = \{\textit{all particles } x_k^{\pm}(0) \in (-r_-(M)+\delta_1\,,\, r_+(M)-\delta_1)\cap \mathbf{Z}_\varepsilon \textit{ collide with particles of opposite sign till the time moment } t(M)\varepsilon^{-1}\},$$

$$B_\varepsilon = \{\textit{on the time interval } t\in(0,s\varepsilon^{-1}) \textit{ none of minus particles, started at } t=0 \textit{ from the set } (-\infty,-r_-(M)-\delta_1)\cap\mathbf{Z}_\varepsilon, \textit{ collides with any plus particle, started at } t=0 \textit{ from the set } (r_+(M)+\delta_1,+\infty)\cap\mathbf{Z}_\varepsilon\},$$

satisfies the e.a.s. property. Moreover, fix any $y,\varkappa>0$ and consider the subsets of $\mathbf{Z}_\varepsilon$

$$S_2^- = (-\infty,-y-\varkappa)\,,\qquad S_1^- = (-y,0)\,,\qquad S_1^+ = (0,y)\,,\qquad S_2^+ = (y+\varkappa,+\infty)\,.$$

The below families $\{V_\varepsilon\}$ and $\{U_\varepsilon\}$ of events have the property of e.a.s.

event V_ε**:** *on the time interval $t\in(0,s\varepsilon^{-1})$ none of minus particles, started at $t=0$ from S_2^-, will meet some minus particle, started from the set S_1^-;*

event U_ε**:** *on the time interval $t\in(0,s\varepsilon^{-1})$ none of plus particles, started at $t=0$ from S_2^+, will meet some minus particle, started from the set S_1^+.*

Proofs of Lemmas 2 and 3 are based on stardand probabilistic methods [6] and are omitted. Let us explain now how using these two lemmas one can get, for example, the upper bound for $|\mathcal{N}_-(0,t_0(M,\varepsilon))|$ in Lemma 1. First we include in this bound all minus particles starting at $t=0$ from the set $(-r_-(\,M(T(M)-\varkappa_0)\,)-\delta_5,0)$ where $\delta_5>0$ is small and will be fixed later. By Lemma 2 there is no more than $(M_-\,(r_-(\,M(T(M)-\varkappa_0)\,)+\delta_5)+\delta_6)\,\varepsilon^{-1}$ of such particles (in the sense of e.a.s.) for small $\delta_6>0$. We should add to this bound all minus particles that started at $t=0$ from the set $(-\infty,-r_-(\,M(T(M)-\varkappa_0)\,)-\delta_5)$ and annihilated on the time interval $(0,t_0(M,\varepsilon))$ with some plus particles. We will show now that with probability exponentially close to one the number $N^\circ(0,t_0(M,\varepsilon))$ of such minus particles can be estimated as $c\varepsilon^{-1}$ where $c>0$ is any prefixed small constant. Indeed, by Lemma 3 (again in the sense of e.a.s.) the mentioned minus particles can annihilate only in collisions with some plus particles, started at $t=0$ from the set $(0,r_+(\,M(T(M)-\varkappa_0)\,)+\delta_5)$. By Lemma 2 the number of the plus particles in this set is bounded by $(M_+\,(r_+(\,M(T(M)-\varkappa_0)\,)+\delta_5)+\delta_6)\,\varepsilon^{-1}$. From this bound we should exclude plus particles which was annihilated in collisions with minus particles started from the set $(-r_-(\,M(T(M)-\varkappa_0)\,)+\delta_5,0)$, since by the part "$V_\varepsilon$" of Lemma 3 during the time interval $(0,t_0(M,\varepsilon))$ the latter minus particles will go ahead of the minus particles started from $(-\infty,-r_-(\,M(T(M)-\varkappa_0)\,)-\delta_5)$. By Lemma 2 initially in the set

$(-r_-\,(\,M(T(M)-\varkappa_0)\,)+\delta_5,0)$ there was no less than $(M_-\,(r_-\,(\,M(T(M)-\varkappa_0)\,)-\delta_5)-\delta_6)\,\varepsilon^{-1}$ particles. So using the mean value theorem from analysis we get

$$\begin{aligned}\varepsilon\cdot N^{\circ}(0,t_0(M,\varepsilon)) &\le (M_+\,(r_+\,(\,M(T(M)-\varkappa_0)\,)+\delta_5)+\delta_6)-\\ &\quad -(M_-\,(r_-\,(\,M(T(M)-\varkappa_0)\,)-\delta_5)-\delta_6)=\\ &= M(T(M)-\varkappa_0)+M'_+(\theta_1)\delta_5+\delta_6-\big((\,M(T(M)-\varkappa_0)\,)-M'_-(\theta_2)\delta_5-\delta_6\big)\le\\ &\le \delta_5(\|M'_-\|_C+\|M'_+\|_C)+2\delta_6.\end{aligned}$$

Hence, in the sense of e.a.s.

$$\begin{aligned}\varepsilon\cdot|\mathcal{N}_-(0,t_0(M,\varepsilon))| &\le (M_-\,(r_-\,(\,M(T(M)-\varkappa_0)\,)+\delta_5)+\delta_6)+\delta_5(\|M'_-\|_C+\|M'_+\|_C)+2\delta_6=\\ &= M(T(M)-\varkappa_0)+M'_-(\theta_3)\delta_5+\delta_5(\|M'_-\|_C+\|M'_+\|_C)+3\delta_6\le\\ &\le M(T(M)-\varkappa_0)+\delta_5(2\|M'_-\|_C+\|M'_+\|_C)+3\delta_6\end{aligned}$$

It follows from (2) and assumptions on $\rho_\pm$ that $M'(t)\ge k$ for some $k>0$. Therefore, $M(T(M)-\varkappa_0)\le M-k\varkappa_0$. Given $\varkappa_0>0$ we are allowed to chose positive constants δ_5 and δ_6 as small as we like. So, finally, we get that with probability exponentially close to one the following estimate holds

$$|\mathcal{N}_-(0,t_0(M,\varepsilon))|\le\left(M-\frac{k\varkappa_0}{2}\right)\varepsilon^{-1}.$$

Lower bound for $|\mathcal{N}_-(0,t_1(M,\varepsilon))|$ can be obtained in a similar way.
To get the proof of statement b) one should combine Lemma 3 with the next lemma.

Lemma 4 *For any* $\varkappa_5>0$

$$\text{event } H_\varepsilon:\qquad x_i^-(0)\in(-(1+\varkappa_5)r_-(M),0)\quad\forall i\in\mathcal{N}_-(0,t_1(M,\varepsilon))$$

has the property of e.a.s.

Proof of this lemma uses arguments similar to the proof of Lemma 1. The statement c) is just a mirror modification of the statement b).

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The MDL-PRINCIPLE in ATTRIBUTING AUTHORSHIP of TEXTS

Mikhail Malyutov
Math. Dept., Northeastern University, Boston, MA 02115

ABSTRACT

We study a new *context-free* computationally simple stylometry-based attributor: the mean sliced *conditional compression complexity (CCC) of literary texts* which is inspired by the incomputable Kolmogorov conditional complexity. Whereas other stylometry tools can occasionally almost coincide for different authors, our CCC-attributor introduced in Malyutov (2005) is asymptotically strictly minimal for the true author, if the query texts are sufficiently large but much less than the training texts, universal compressor is good and sampling bias is avoided. This classifier simplifies the Ryabko and Astola (2006) homogeneity test (partly based on compression) **under insignificant difference of unconditional complexities of training and query texts** which can be verified using its asymptotic normality proved in Szpankowski (2001) and elsewhere for IID and Markov sources and normal plots for real literary texts. It is *consistent* under large text approximation as a *stationary ergodic sequence* which follows from the *lower bound for the minimax compression redundancy of piecewise stationary strings* (Merhav (1993)) and from our elementary combinatorial arguments and simulation for IID sources. The *t-ratio* use measuring how many standard deviations are in the mean difference between slices' mean CCCs enables evaluation of its P-value of statistical significance. It is based on the *asymptotic normality* of slices' CCC verified by their normal plots in all cases studied and expected to be proved soon for simplified statistical models of literary texts.

The *asymptotic CCC study* is complemented by many literary case studies processed by Sufeng Li, Irosha Wickramasinghe, Slava Brodsky, Gabriel Cunningham and Andrew Michaelson: attributing the Federalist papers agreeing with previous results, significant (beyond any doubt) mean CCC-difference between two translations of Shakespeare sonnets into Russian, between the two parts of M. Sholokhov's early short novel and less so between the two Isaiah books from the Bible, intriguing SCCC-relations between certain Elizabethan poems. Two different S. Brodsky's novels *deliberately written in different styles* and various Madison's papers showed insignificant mean CCC-difference as the useless Vitanyi-Cilibrasi test did in ALL cases studied.

Keywords: compression length, authorship attribution, homogeneity testing, asymptotic normality.

1 Discrimination with Universal Compressors

C. Shannon (1948, 1949) created a comprehensive theory of information transmission based on Kolmogorov's statistical theory. In particular, **given** a distribution on an alphabet, the mean length of the Shannon-Fano compression of the IID string with elements from this alphabet attains asymptotically the Shannon's entropy lower bound for the length (complexity) of compression. A.N. Kolmogorov (1965) developed a complexity theory of an **individual string** such that for large strings *belonging to a statistical ensemble* their mean complexity approximates their entropy, and sketched (for IID input) the first so-called **universal compressor** (UC) which adapts to an **unknown** stationary ergodic distribution (SED) of strings attaining asymptotically the Shannon entropy lower bound. $\mathbf{P}$ is the class of SED sources approximated by n-MC's. Compressor family $\mathbf{L} = \{L_n : \mathbf{B}^n \to \mathbf{B}^\infty, n = 1, 2, \ldots\}$ is (weakly) **universal**, if for any $P \in \mathbf{P}$ and $\epsilon > 0, \mathbf{B} = \{0, 1\})$, it holds:

$$\lim_{n\to\infty} P(x \in \mathbf{B}^n : |L_n(x)| + \log P(x) \leq n\epsilon) = 1, \tag{1}$$

where $|L(x)|$ is the length of $L(x)$ and $|L_n(x)| + \log P(x)$ is called *individual redundancy*. Thus for a string generated by a SED, the **UC-compression length is asymptotically its negative loglikelihood** which can be used in **nonparametric** statistical inference, if the *likelihood cannot be evaluated analytically*. First UC used estimating parameters of approximating n-Markov Chains (n-MC) to adapt for good compression. A profoundly smarter method implementing much fuller the above-mentioned Kolmogorov's idea for compressors took more than ten years to emerge in two Lempel-Ziv (LZ) compressor constructions (1977-78). Both LZ-compressors do not use any statistics of strings at all. Instead, LZ-78 constructs the tree of binary patterns unseen before in the string consecutively, starting from the first digit of the string. Wyner and Ziv proved that LZ-78 is an UC implying

$$\lim_{n\to\infty} P(|L_n(x)|/|x| \to h) = 1 \quad as \quad |x| \to \infty \tag{2}$$

for $P \in \mathbf{P}$, where h is the binary entropy rate (per symbol) proved to be the asymptotic lower bound for compressing a SED source in Shannon (1949), where SED strings were first singled out as popular models of natural language. By nineties, versions of LZ became everyday tools in computer practice. Rissanen's pioneering publication on the **Minimum Description Length principle (MDL)** in 1978 (continued in his paper (1984)) and Ziv

(1988) initiated applications of UC to statistical problems for SED sources continued in several recent papers of B. Ryabko with coauthors. Of special interest to us is the **homogeneity test** in Ryabko and Astola (2006).

1.0.1 Ryabko-Astola and U-statistics

Define $|A|$ and $|A_c|$ as the lengths of respectively **binary** string A and its compression A_c.

The *concatenated* string $S = AB$ is the string starting with A and proceeding to text B without stop.

The Ryabko and Astola homogeneity of two strings test statistic T is

$$T = h_n^*(S) - |A_c| - |Q_c|, \tag{3}$$

where the empirical Shannon entropy h_n^* of the concatenated sample S (based on n-MC approximation) is defined in their formula (6). The local context-free structure (microstyle) of long (several Kbytes) literary texts (LT) can be modeled sufficiently accurately only by binary n-MC with n not less than several dozen. Its evaluation for LT is very intensive computationally and unstable for texts of moderate size requiring regularization of small or null estimates for transition probabilities. Therefore, appropriateness of T rather than equally computationally intensive Rosenfeld's (1996) Likelihood methods based on n-MC training is questionable. For shorter LT accuracy of SED model may be insufficient, while for very large LT such as novel affected by long literary form relations ('architecture' features such as 'repeat' variations), the microstyle describes only a local part of the author's style as emphasized in Chomsky (1956).

Consider $U(Q, A) = |S_c| - |A_c| - |Q|$. Quantity $U(Q, A)$ *mimics the Ryabko and Astola statistic* T. In $U(Q, A)$ we *replace their empirical Shannon entropy* h^* of the concatenated sample S (based on n-MC approximation) with $|S_c|$ since both are asymptotically equivalent to $h(|Q|+|A|)$ for identical distribution in Q, A with entropy rate h and exceed this quantity for different A, Q . Test T is asymptotically invariant w.r.t. interchanging A, Q and *strictly positive* for *different* laws of A, Q, if $a < |A|/|Q| < 1/a, a > 0$). The last but not the first property seemed to hold also for $U(Q, A)$ in some range of $|A|/|Q|$ due to the **lower bound for the minimax mean UC-compression redundancy of piecewise-stationary sources (Merhav (1993))** which is logarithmic in $(|Q| + |A|)$.

Claim. The U performance on IID extensive simulations in a large range of $|Q|$ (made recently by NEU PhD student Stefan Savev), was not as predicted above (actually empirical mean of U was negative!) due apparently to the additional subtracting of $|Q_c|$. For small $|Q_c|$ this is due to excessively large 'transition value' of $|Q_c|$, since 'entropy' asymptotics is not yet attained.

For large $|Q_c|$, the small increase of U due to inhomogeneity 'is drowned' in the large 'noise' of variable $|S_c|$. Averaging different slices of identically distributed moderately large $Q_i, i = 1, \ldots$ can make mean U positive, but it is not applicable in our LT studies.

1.0.2 CCC- and CC-statistics

Fortunately, another statistic, CCC defined below, overcomes the shortfalls of statistic U.

In our applications $|A|/|Q|$ is large to statistically assess reliability of attribution and upperbounded by an approximate empirical condition $|Q| \geq$ 2000 bytes (requiring further study) for appropriateness of SED approximation.

The *Conditional Complexity of Compression* of text B given text A are respectively

$$CCC(B|A) = |S_c| - |A_c|. \tag{4}$$

The CCC mimics an abstract conditional Kolmogorov Complexity in our settings and measures how adapting to patterns in the training text helps to compress the query text. It presumably approximates the most powerful Likelihood Ratio Test of Q, A homogeneity under our condition on sample sizes and validity of SED approximation for both Q, A.

The only difference of CCC from U is canceling the $|Q_c|$ removal which prevents the aforementioned inconsistency of U - statistic.

We average sliced CCC of text $Q_i, i = 1, \ldots, m = [|Q|/L]$, given the firmly attributed text A, dividing the *query text* Q into slices of equal length L and used the same UC for all sizes of texts.

$$\overline{CCC(Q|A)} := \sum_{i=1}^{m} \frac{CCC(Q_i|A)}{m} := \sum_{i=1}^{m} \frac{CC(Q_i)}{m}. \tag{5}$$

We call the last two empirical quantities '*Mean* $CCC(Q)$ *and Mean* $CC(Q)$' respectively.

Claim. Both our case studies and statistical simulation in section 3 show that the sliced CCC-attribution has a good homogeneity discrimination power in this range for moderate $|Q|$) in a surprisingly wide range of case studies with **insignificantly varying mean unconditional complexity** CC of compression.

Statistical testing of the latter condition is straightforward due to the **asymptotic normality** results of the compression complexity described in Szpankowski (2001). Its very plausible extension for CCC would theoretically support a quite **unusual sample size relation** for UC-attributing

authorship: **sample size of the training text must dramatically exceed those of slices of a query text**. The training test A being fixed, $VarCCC(Q_i|A)$ of independent copies $Q_i, i = 1, ..., N$ of the query text Q, are of order of $|Q|$, while the mean increase in $CCC(Q|A)$ redundancy for *different distributions* of Q and A *as compared to their identity* seems to be $o(|(A|Q)|^b)$ for any $b > 0$ (accurate upper bound even for LZ78 is absent so far (see some LZ-78 upper bounds in Savari (1997)), the lower bound in Merhav(1993) is only $O(\log(|(A|Q)|))$). Thus, the t-ratio is negligible under the asymptotics $|A| \to \infty, 0 < \epsilon < |Q|/|A|$. Malyutov (2005) explains this informally as follows: if the training A and alternative style query text Q sizes are comparable, then two flaws happen: a UC adapts to both at the extra length cost $o(|(A|Q)|^b)$ for any $b > 0$, this extra amount of $CCC(Q|A)$ is hidden in the noise with $VarCCC((A|Q)|$ of order $|(A|Q)|$. Second, the mean $CCC(Q|A)$ of larger slices of query texts have a **bigger bias** due to **self-adapting of UC to the slices' patterns**.

This makes sample size requirements and symmetry arguments in Cilibrasi and Vitanyi (2005) (CV05) also based on the conditional compression complexity although **ignoring assessment of statistical stability**, unappealing, and explains examples of CV05 misclassification shown in Rocha et al (2006). It can explain also the roots of early heated discussion around simpler development in Benedetto et al (2002), where the *sample size relation and statistical stability* issues were not addressed.

Due to space limitation, we skip sections: **Brief survey of microstylometry tools, Methodology, Simulation study of CCC-attributor, Extended LZ index and many exciting examples of Attribution of literary texts** which are described in detail in my larger paper in Russian under review in 'Problems of Information Transmission', MalBrod09 and in MWL07.

1.1 Follow up Analysis

LZ-78 generates the binary tree of all patterns found in LT: thus for every pattern ν we can evaluate frequency of the cases when ν is a **prefix** of the further text which is the cardinality of the subtree rooted in ν.

G. Cunningham implemented in Perl language my algorithm (MWL07) of economic LZ-tree construction and evaluating cardinalities of interesting subtrees. Subtree rooted in ν is called interesting, if 't-value' for its cardinalities $n(\nu, A)$, is large for competing candidates for authorship.

$$'t' = (n(\nu, A) - n(\nu, A'))/\sqrt{[n(\nu, A)(c_1 - n(\nu, A))/c_1 + n(\nu, A')(c_2 - n(\nu, A)')/c_2]}, \quad (6)$$

Table 1: Most 'interesting patterns for Federalist papers

Binary pattern	t-value	Patterns in English
011100100110010101101001	4,08	rei
001000000110010001100101	3,62	de
011010000110010100100000010100110111010 0	3,43	he St
011000010110100101101110011100 11	3,38	ains
011001010111001001100001011011 00	3,38	eral
011101000110100001101111011100 10	3,28	thor
011010000010000001110111	3,15	h w
011001010110111001100100011001 01	3,15	ende
011011000110010100100000011000 01	3,15	le a
0111010101100100	3,15	ud
001000000110000101101110011001000010000001110010	3,14	and r
011101000110100001100101011011 01	3,12	them
011001100110010101100100	3,12	fed
011001110110111000100000	3,12	gn

where $c_i, i = 1, 2$, are total patterns cardinalities for competing candidates. Finally, the tables of English patterns corresponding to interesting binary patterns are tabulated.

Any solid judgement about corresponding P-values is hard due to vast multiplicities of not independent patterns. Still the tables like the shown one for Federalist papers may be useful source of information for linguists.

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Cramér-von Mises test for the Weibull and Pareto distributions*

Gennadi V. Martynov
Institute for information transmission problems
of the Russian Academy of Sciences (Kharkevich Institute)
martynov@iitp.ru

Abstract

Here we will consider the goodness-of-fit tests for testing a form of the distribution function of the observed random variable. Let a distribution function belongs under hypothesis to a parametric family. Generally, the limit distributions of statistics, based on the empirical process, depend of the unknown parameters. It was stated in 1955 (see [8]) that this dependance is absent for the distribution family $\{G((x-\mu)/\sigma),\ \sigma>0\}$. This class includes the normal distribution. We will present now the second class of the parametric distribution families with such property. This is the family $\{R((x/\beta)^{\alpha})),\ \alpha>0,\ \beta>0,\ x\in\mathcal{X}\subset[0,\infty)\}$, where α and β are unknown parameters. This class includes the Pareto and Weibull distribution families. The exponential distribution family is included in both ones.

1 Introduction

Let $X^n=\{X_1,X_2,...,X_n\}$ be the sample from the r.v. with the distribution function $F(x),\ x\in R_1$. We will test the hypothesis

$$H_0:\ \ F(x)\in\mathcal{G}=\{G(x,\theta),\ \ \theta=(\theta_1,\theta_2,...\theta_k)^{\top}\in\Theta\subset R_k\},$$

where θ is an unknown vector of parameters. We will consider the Cramér-von Mises statistic

$$\omega_n^2=n\int_{-\infty}^{\infty}\psi^2(G(x,\theta_n))(F_n(x)-G(x,\theta_n))^2\,dG(x,\theta_n),$$

θ_n is an estimator of θ, $\psi(t)$ is the weight function, $F_n(x)$ is the empirical distribution function. The results below are applicable also to the Kolmogorov-Smirnov statistic

$$D_n=\sqrt{n}\sup_{-\infty<x<\infty}|\psi(G(x,\theta_n))(F_n(x)-G(x,\theta_n))|.$$

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The exact methods for calculating the limit distribution are developed mostly for the Cramér-von Mises statistic (see [4], [8], [10], [11], [12]).
Let θ_n be the likelihood maximum estimator of θ. Under the certain number of the regularity conditions and under H_0 limit distribution of the statistics ω_n^2 coincide with the distribution of the functional

$$\omega^2 = \int_0^1 \psi^2(t)\xi^2(t,\theta_0)dt$$

of the Gauss process $\psi(t)\xi(t,\theta_0)$ with $E\psi^2(t)\xi(t,\theta_0) = 0$, and with the covariance function

$$K(t,\tau) = E(\psi(t)\xi(t,\theta_0)\psi(\tau)\xi(\tau,\theta_0))$$
$$= \psi(t)\psi(\tau)(K_0(t,\tau) - q^\top(t,\theta_0)I^{-1}(\theta_0)q(\tau,\theta_0)),$$

where $K_0(t,\tau) = \min(t,\tau) - t\tau, \quad t,\tau \in (0,\ 1)$, θ_0 is an unknown value of the parameter θ,

$$q^\top(t,\theta) = (\partial G(x,\theta)/\partial\theta_1, ..., \partial G(x,\theta)/\partial\theta_k)|_{t=G(x,\theta)},$$

$I(\theta)$ is the Fisher information matrix,

$$I(\theta) = (E((\partial/\partial\theta_i)\log g(X,\theta)(\partial/\partial\theta_j)\log g(X,\theta)))_{1\le i,j\le k},$$

$$g(x,\theta) = (\partial G(x,\theta)/\partial x).$$

The follow condition must be fulfilled:

$$\int_0^1 \psi^2(t)K(t,t)dt < \infty.$$

The limit distribution for D_n coincides with the distribution of

$$D = \sup_{0<t<1} |\psi(t)\xi(t,\theta_0)|,$$

but the conditions on $\psi(t)$ and another conditions are different from the conditions for ω^2. They was studied in [3], [13]. The distribution of ω^2 depends generally from θ_0 and the distribution family $\mathcal{G}$. Khmaladze [9] has proposed the method of empirical process transformation for eliminate such dependance. Khmaladze and Haywood [7] has applied this method to exponentiality testing by the Cramér-von Mises statistic.
We will use here the traditional approach consistsing in using of the statistic ω_n^2. It is well known (see for example [8], [10]]) that the empirical process does not depend on unknown parameter θ_0 for the family of the form

$$\mathcal{G} = \{G((x-m)/\sigma),\ \ -\infty < x < \infty,\ \sigma > 0\}.$$

Most known example of such family is the normal distribution family (see [5], [8)].

We will propose here another class of the distribution family with such property:

$$\mathcal{R} = \{R((x/\beta)^\alpha),\ \alpha > 0,\ \beta > 0, x \in \mathcal{X} \subset [0, \infty)\},$$

where $\mathcal{X}$ is the support of the distribution $R((x/\beta)^\alpha)$. Here $R(z)$ is a distribution function with the support $\mathcal{Z} \subset [0, \infty)$. Particular cases of such families are Weibull and Pareto distributions. The limit distributions of Cramér-von Mises and Kolmogorov-Smirnov statistics do not depend on the unknown parameters in both families. Additionally, the limit distribution for Pareto family coincide with analogous distribution for exponential family. The goodness-of-fit tests was discussed for the general Pareto distribution in many articles, particularly, in [1], [2], [6].

The ω^2-distribution can be calculated exactly with using the method of calculation the eigenvalues of the covariance operator. It was presented in [12]. This method is applicable for the power function $\psi(t) = t^\alpha$ $\alpha > -1$. The method for the corresponding quadratic forms calculation was particularly presented in [10].

2 General result

Let $X^n = \{X_1, X_2, ..., X_n\}$ be the sample from the r.v. with a distribution function $F(x)$, $x \in R_1$. We will test the hypothesis

$$H_0 : \ F(x) \in \mathcal{R} = \{R((x/\beta)^\alpha)),\ \alpha > 0,\ \beta > 0,\ x \in \mathcal{X} \subset [0, \infty)\},$$

where α and β are unknown parameters. The set of the alternative distributions contains all another distributions. Here $R(z)$ is the distribution function with the support $\mathcal{Z} \subset [0, \infty)$. We note the corresponding density function by $r(z)$. $\mathcal{R}$ is the family of Pareto distributions with $R(z) = 1 - 1/z,\ z > 1$ and $x > \beta$. The family $\mathcal{R}$ consists of Weibull distributions when $R(z) = 1 - \exp(-z),\ z > 0$, and $x > 0$. We will use the Cramér-von Mises and Kolmogorov-Smirnov tests. Both of them based on the empirical process $\xi_n(x) = \sqrt{n}(F_n(x) - R((x/\hat{\beta})^{\hat{\alpha}})))$, where $\hat{\alpha}$ and $\hat{\beta}$ are the ML estimates of α and β. If the regularity conditions are fulfilled for them we can write the follow covariance function for the transformed to $(0, 1)$ limit Gauss process $\xi(t)$:

$$K(t, \tau) = \min(t, \tau) - t\tau - (1/(B_{11}B_{22} - B_{12}^2))$$
$$\times(B_{22}s_1(t)s_1(\tau) - B_{12}(s_1(t)s_2(\tau) + s_2(t)s_1(\tau)) + B_{11}s_2(t)s_2(\tau)).$$

Here, $t, \tau \in (0,\ 1)$,

$$B_{11} = \int_{\mathcal{Z}} \left(\frac{z \log z\, r'(z)}{r(z)} + \log z + 1\right)^2 r(z)dz,\ B_{22} = \int_{\mathcal{Z}} \left(\frac{z\, r'(z)}{r(z)} + 1\right)^2 r(z)dz,$$

$$B_{12} = \int_{\mathcal{Z}} \left(\frac{z \log z\, r'(z)}{r(z)} + \log z + 1\right)\left(\frac{z\, r'(z)}{r(z)} + 1\right) r(z)dz$$

and

$$s_1(t) = r(R^{-1}(t))R^{-1}(t)\log(R^{-1}(t)), \;\; s_2(t) = r(R^{-1}(t))R^{-1}(t).$$

It follows from these formulae that the limit distributions of the considered statistics do not depend from the parameters α and β. Let β be known. Then the covariance function of the process $\xi(t)$is follow:

$$K(t,\tau) = \min(t,\tau) - t\tau - s_1(t)s_1(\tau)/B_{11}.$$

It does not depend of α in his turn. These results are used in the follow two sections.

3 Pareto distribution

We will consider the Pareto distribution in the form

$$F(x) = 1 - (x/\beta)^{-\alpha}, \; x \geq \beta \geq 0, \; \alpha > 0.$$

For this distribution $R(z) = 1 - 1/z$ and $\mathcal{Z} = [\beta, \infty]$. It exists the supereffective unbiased estimate of β

$$\hat{\beta} = \frac{n\alpha - 1}{n\alpha} \min_{i=1,\ldots,n} X_i.$$

We can transform the sample $X_1, ..., X_n$ to new sample $Y_1, ..., Y_n$, where $Y_i = X_i/\hat{\beta}$. The limit process $\psi(t)\xi(t)$ is equivalent to the process with $\beta = 1$. The MLE of parameter α is

$$\hat{\alpha} = n \Big/ \sum_{i=1}^{n} \log X_i.$$

Hence the covariance function of $\xi(t)$ (without the pound function) is

$$K(t,\tau) = \min(t,\tau) - t\tau - (1-t)\log(1-t))(1-\tau)\log(1-\tau).$$

There

$$s_1(t) = -(1-t)\log(1-t), \quad B_{11} = 1.$$

This covariation function coincides with the corresponding covariance function for the exponential family

$$F(x) = 1 - \exp(-x/\beta), \; \beta \geq 0, \; x \geq 0.$$

It can be concluded that the limit distributions of the considered statistics for both families are the same one.

4 Weibull distribution

Consider the two parametric Weibull distribution family

$$F(x) = 1 - e^{-(x/\beta)^{-\alpha}},\ x \geq 0,\ \beta \geq 0,\ \alpha > 0.$$

We can note that $R(z) = 1 - e^{-z}$ and $\mathcal{Z} = [0, \infty]$. Maximum likelihood estimates $\hat{\beta}$ and $\hat{\alpha}$ for β and α can be found by numerical methods from the equation system

$$\hat{\beta} = \left(\frac{1}{n} \sum_{i=1}^{n} X_i^{\hat{\alpha}} \right)^{1/\hat{\alpha}}, \quad \frac{n}{\hat{\alpha}} + \log \left(\frac{X_1 \cdot \ldots \cdot X_n}{\hat{\beta}^n} \right) - \sum_{i=1}^{n} \left(\frac{X_i}{\hat{\beta}} \right)^{\hat{\alpha}} \log \left(\frac{X_i}{\hat{\beta}} \right) = 0.$$

The covariance function of $\xi(t)$ in this example has the follow elements:

$$s_1(t) = -(1-t)\log(1-t)\log(-\log(1-t)),$$

$$s_2(t) = -(1-t)\log(1-t),$$

$$B_{11}(t) = \int_0^\infty ((1-z)\log\ z - 1)^2\ e^{-z} dz = (1-C)^2 + \frac{\pi^2}{6},$$

$$B_{12}(t) = \int_0^\infty ((1-z)\log\ z - 1)(1-z)\ e^{-z} dz = 1 - C,$$

$$B_{22}(t) = \int_0^\infty (1-z)^2\ e^{-z} dz = 1,$$

$$B_{11}B_{22} - B_{12} = \pi^2/6,$$

where C is the Euler constant. It was found by simulation that the critical levels corresponding to the significance levels 0.1 and 0.05 are approximatively 0.10 and 0.12.

5 Power distribution on $[0, 1]$

We consider now the distribution function

$$F(x) = \left(\frac{x-a}{b-a} \right)^{\alpha},\ x \in [a, b],\ b > a,\ \alpha > 0.$$

A supereffective estimates exist for the parameters a and b. Hence, we can transform the sample to the interval $[0, 1]$ without changing the limit distribution of the statistics. It is sufficient to consider tests for the hypothetical distribution family

$$F(x) = x^{\alpha},\ x \in [0, 1],\ \alpha > 0,$$

with $R(z) = z,\ \mathcal{Z} = [0, 1]$. It's easy to derive the covariance function of the limit empirical process $\xi(t)$:

$$K(t, \tau) = \min(t, \tau) - t\tau - t \log t\, \tau \log \tau.$$

The limit distribution of the statistics ω^{2_n} and D_n for this distribution coincides with the corresponding statistics distributions for the exponential and Pareto distribution and for the Weibull distribution with known parameter α.

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Renormalization group flows: facts and conjectures

M. D. Missarov

Kazan State University

Moukadas.Missarov@ksu.ru

Exact renormalization group (RG) solution of the hierarchical fermionic model reveals several non-trivial dynamical and symmetry properties of RG-transformation. Fermionic model on the hierarchical lattice is defined by the Hamiltonian

$$H(\psi^*;\alpha) = \sum_{i,j\in\Lambda} d_n^{-\alpha}(i,j)[\bar{\psi}_1(i)\psi_1(j) + \bar{\psi}_2(i)\psi_2(j)] +$$

$$\sum_{i\in\Lambda}[r(\bar{\psi}_1(i)\psi_1(i) + \bar{\psi}_2(i)\psi_2(i)) + g\bar{\psi}_1(i)\psi_1(i)\bar{\psi}_2(i)\psi_2(i)],$$

where $d_n(i,j)$ denote hierarchical distance on the hierarchical lattice Λ with elementary cell size n, α is model parameter, r and g are real-valued coupling constants, all components of the fermionic field are generators of a Grassmann algebra.

Action of block-spin RG-transformation in the plane of coupling constants (r, g) is given by the rational map $R(\alpha)$. This map has two trivial and two non-trivial fixed points and cycles of any order. We describe hierarchical structure of the RG-invariant sets and curves and using projective space representation we give global description of RG-flow and critical phenomena in the whole plane of the coupling constants. It is shown that when RG-parameter α goes to 1 all non-trivial fixed points and cycles of RG-map tend to the singular point $r = -1, g = 0$. There is commutative relation between RG and Fourier transformations $R(\alpha)F = FR(2-\alpha)$ which points to the special role of $\alpha = 1$. We show also that formal $(\alpha - 3/2)-$ and $(4 - d)-$ expansions describe the same non-trivial fixed point (where d is formal dimension of the hierarchical lattice) in the fermionic and discuss this problem for the bosonic hierarchical model.

In fact, fermionic hierarchical model can be considered as discretized version of fermionic field model over p-adic space. The relation between coupling constants of p-adic model and its discretized hierarchical version is given by non-trivial functional integral and its convergence follows from Poincare and Siegel theorems. Renormalization procedure can be defined as a normal form to the RG-transformation at the trivial (zero) fixed point. With the use of p-adic quantum field formalism it is possible to construct epsilon-expansion for the critical exponents in the bosonic case and up to the second order of perturbation theory we see interesting analogy with the results of Euclidean models. p-Adic models have strong algebraic simularity with Euclidean ones and we discuss some new non-trivial conjectures for the bosonic and fermionic Euclidean models, generated by exact solution of hierarchical fermionic model.

Stochastic Comparison of Ellipsoidal and Interval Error Estimation in Vector Operations

Alexander Ovseevich*

1 Introduction

Consider the following elementary problem of numerical linear algebra. Suppose we are given a vector $x \in \mathbf{R}^n$, not known exactly but located within a known bounded domain Ω, and a matrix A which is known exactly. We would like to localize the vector Ax as good as possible. Certainly, Ax is contained in $A\Omega$, and that's the best one can say. In practice this answer may be not good enough, since it might be unfit for computer. In particular, the domains Ω of uncertainty should have a simple description, that would allow to check easily (for a computer) whether a given vector is contained in it.

There are at least two classes of suitable domains: boxes $\mathcal{B} = \{x \in \mathbf{R}^n : |x_i - a_i| \leq b_i\}$, and ellipsoids $\mathcal{E} = \{x \in \mathbf{R}^n : \langle Q^{-1}(x-a), x-a\rangle \leq 1\}$. Methods of computations with vectors, localized in boxes, are known as interval analysis, similar methods for vectors, localized in ellipsoids, are known as ellipsoidal analysis.

The present paper is inspired by [4], where some evidences are presented that in the problem of multiplication of a vector by matrix the ellipsoidal analysis is, in certain sense, better than the interval one. More precisely, suppose the vector is localized in a box $\mathcal{B}$, and $\mathcal{E}$ is the minimum volume ellipsoid containing $\mathcal{B}$. Certainly, $\mathcal{E}$ also localizes the vector, and, at this stage, the substitution of $\mathcal{E}$ for $\mathcal{B}$ results in a loss of accuracy. However, upon multiplication by A the domain $A\mathcal{B}$ is not necessarily a box, while the domain $A\mathcal{E}$ is still an ellipsoid. To stay within the interval framework one should substitute the minimal box $\mathrm{Box}(A\mathcal{B})$, containing $A\mathcal{B}$, for $A\mathcal{B}$. Finally we get two localization domains: $\mathrm{Box}(A\mathcal{B})$ and $A\mathcal{E}$. It is suggested in [4] to

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compare the quality of methods by means of volumes of the final localization domains.

1.1 Main inequality

The result of comparison does not depend on the initial box, but only on the matrix A, and is determined by the sign $\begin{matrix}\leq\\ \geq\end{matrix}$ in the inequality

$$\prod_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|\begin{matrix}\leq\\ \geq\end{matrix}\frac{(\pi n)^{\frac{n}{2}}\,|\det A|}{2^n\,\Gamma\left(\dfrac{n}{2}+1\right)}. \tag{1.1}$$

The $\leq$ sign specifies the set of matrices such that the ellipsoidal method turns to be worse than the interval one. The inequality (1.1) comes directly from exact formulas for volumes of ellipsoid and box, while the factor $\pi^{n/2}/\Gamma(\frac{n}{2}+1)$ arises as volume of the circumscribed ball for unit cube. Real problems of numerical linear algebra correspond to a large dimension n. That's why we will compare ellipsoids and boxes as $n\to\infty$.

2 Random matrices

The set Ω_n of $n\times n$-matrices A such that (1.1) holds with $\leq$ sign is quite complicated. In a rather vague way, one can say that Ω_n is relatively poor, i.e. most matrices do not belong to it. Still it is not clear in advance how to measure properly the size of Ω_n, and establish that it is small. We suggest a stochastic approach to this issue. Namely, we assume that the matrix A is random so that its elements are independent Gaussian random variables with zero mean and unit covariance. In particular, the distribution of any element a_{ij} of A takes the form

$$p(x)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right). \tag{2.1}$$

Then, a natural measure for the size of the set Ω_n is its probability $\mathbf{P}(\Omega_n)=(2\pi)^{-n^2/2}\int_{\Omega_n}e^{-\frac{1}{2}\operatorname{Tr}AA^*}dA$. Here, $\operatorname{Tr}AA^*=\sum a_{ij}^2$, and $dA=\prod da_{ij}$.

3 Main result

Theorem 1 *The probability of the event Ω_n that intervals are better than ellipsoids tends to zero as $n\to\infty$. In other words,*

$$\mathbf{P}(\Omega_n)=(2\pi)^{-n^2/2}\int_{\Omega_n}e^{-\frac{1}{2}\operatorname{Tr}AA^*}dA=o(1) \tag{3.1}$$

More precisely, $\mathbf{P}(\Omega_n) = O(1/(n^2 \log n))$.

Denote $\log \sum_{j=1}^{n} \left| \frac{a_{ij}}{\sqrt{n}} \right|$ by χ_i, and define

$$\psi_n = \frac{1}{n} \sum_{i=1}^{n} \chi_i, \; C_n = \log \frac{(\pi n)^{\frac{1}{2}}}{2\,\Gamma\left(\frac{n}{2}+1\right)^{1/n}}, \; \Delta_n = \frac{1}{n} \log \left| \det \left(\frac{A}{\sqrt{n}} \right) \right|.$$

The set Ω_n is defined by inequality

$$\psi_n \le C_n + \Delta_n \tag{3.2}$$

and our main result says that this inequality holds with a very small probability. The reason is that each term in (3.2) has a definite, and even deterministic "limit in probability" as $n \to \infty$: $\psi_n = \frac{1}{2} \log \frac{2n}{\pi} + o(1)$, $C_n = \frac{1}{2} \log \frac{\pi e}{2} + o(1)$, $\Delta_n = -\frac{1}{2} + o(1)$, but the limit inequality $\frac{1}{2} \log \frac{2n}{\pi} \le \frac{1}{2} \log \frac{\pi e}{2} - \frac{1}{2}$ is totally false. In what follows we expound the above arguments.

3.1 A heuristic analysis of inequality (3.2)

The functions χ_i can be regarded as independent random variables on the Gaussian probability space of $n \times n$-matrices, and the left-hand side of (3.2) has a form of a mean value $\psi_n = \frac{1}{n} \sum_{i=1}^{n} \chi_i$. Hence, when $n \to \infty$ one can apply the Law of Large Numbers (LLN) to analyze the left-hand side of (3.2) and conclude that

$$\psi_n = \frac{1}{n} \sum_{i=1}^{n} \chi_i \to \mathbf{E}\chi_1 \text{ in probability.} \tag{3.3}$$

By virtue of the Central Limit Theorem (CLT) the distribution of $f_i = \sum_{j=1}^{n} \left| \frac{a_{ij}}{\sqrt{n}} \right|$ is approximately Gaussian with covariance $1 - \frac{2}{\pi}$ and mathematical expectation $\sqrt{\frac{2n}{\pi}}$. Therefore,

$$\mathbf{E}\chi_1 = \mathbf{E} \log f_1 = \log \sqrt{\frac{2n}{\pi}} + o(1) \tag{3.4}$$

and ψ_n is contained in $o(1)$-neighborhood of $\log \sqrt{\frac{2n}{\pi}}$ with probability $1+o(1)$. Hence, if n is large the inequality (3.2) with an overwhelming probability takes the form

$$\Delta_n \ge \log \sqrt{\frac{2n}{\pi}} + \frac{1}{2} \log \frac{\pi e}{2} + o(1) = \frac{1}{2} \log n + \frac{1}{2} + o(1), \tag{3.5}$$

where ψ_n is absent. Thanks to Siegel [1, 2] we have explicit expression

$$\mathbf{E}|\det A|^k = (2\pi)^{-n^2/2} \int |\det A|^k \, e^{-\frac{1}{2}\operatorname{Tr}(AA^*)} dA = 2^{\frac{kn}{2}} \prod_{i=1}^{n} \frac{\Gamma\left(\frac{k+i}{2}\right)}{\Gamma\left(\frac{i}{2}\right)} \tag{3.6}$$

for $\mathbf{E}|\det A|^k$ with any complex k. In particular, it follows from (3.6) that $\mathbf{E}e^{n\Delta_n} = o(1)$. Therefore, Δ_n can be large only with (exponentially) small probability. In particular, the probability of (3.5) decays as $n \to \infty$.

3.2 Rigorous analysis of the left-hand side of (3.2)

The above arguments tacitly assume that some limit processes commute. We will not justify exactly this, and use subgaussian random variables instead of CLT.

A real random variable is said to be subgaussian if

$$\mathbf{E}e^{\lambda\xi} \le e^{\frac{1}{2}\lambda^2}$$

for any real λ. The fact which is very important for us is this:

Theorem 2 *If x is a standard Gaussian random variable, then, the random variable $\xi = |x| - \mathbf{E}|x|$ is subgaussian.*

A proof is based on the so-called theory of logarithmic concavity [5]. This immediately implies the following corollary.

Corollary 1 *Each random variable $f_i - \sqrt{\frac{2n}{\pi}} = \sum_{j=1}^{n} \frac{|a_{ij}| - \mathbf{E}|a_{ij}|}{\sqrt{n}}$ is subgaussian.*

On the basis of this corollary one can show that

$$\mathbf{E}\log f_i = \log\sqrt{\frac{2n}{\pi}} + o(1) \tag{3.7}$$

$$\mathbf{E}\log^2 f_i = \log^2\sqrt{\frac{2n}{\pi}} + o(1) \tag{3.8}$$

$$\mathbf{E}\left|\log f_i - \mathbf{E}\log f_i\right|^2 = O(\frac{\log n}{n}) \tag{3.9}$$

In particular, the asymptotic equality (3.4) holds, and LLN can be applied in order to justify (3.3). Thus, $\psi_n - \log\sqrt{\frac{2n}{\pi}} \to 0$ in a reasonable sense.

Finally, from (3.9), (3.4) and the Chebyshev inequality we obtain:

$$\mathbf{P}\left(\psi_n \le \frac{1}{4}\log n + C\right) = o(1), \tag{3.10}$$

where C is an arbitrary constant, while $o(1)$ is, in fact, $O(\frac{1}{n^2 \log n})$. Therefore, ψ_n is large with a large probability.

3.3 Analysis of the right-hand side of (3.2)

As to the random variable $\Delta_n = \frac{1}{n}\log\left|\det\left(\frac{A}{\sqrt{n}}\right)\right|$ in the right-hand side of (3.2), it is not large with an overwhelming probability. In fact, one can show that $\Delta_n \to -\frac{1}{2}$ in probability so that the (absolute value of) determinant of a random matrix becomes more and more deterministic as $n \to \infty$.

It follows from the Siegel formula (3.6) that

$$\mathbf{E}e^{n\Delta_n} = \left(\frac{2}{n}\right)^{\frac{n}{2}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(1/2\right)}. \tag{3.11}$$

The logarithm of the right-hand side of (3.11) is

$$-\frac{n}{2} - \frac{1}{2} + o(1)$$

in view of the Stirling formula. Now, by means of the Chebyshev inequality we can estimate the probability of large values of Δ_n as follows:

$$\mathbf{P}(\Delta_n \geq x) \leq e^{-nx}\mathbf{E}e^{n\Delta_n} \sim e^{-nx-\frac{n}{2}-\frac{1}{2}}.$$

In particular,

$$\mathbf{P}\left(\Delta_n \geq \frac{1}{4}\log n + C\right) \leq (1+o(1))e^{-\frac{1}{4}n\log n}, \tag{3.12}$$

where C is an arbitrary constant.

3.4 Summing up

Now we get back to inequality (3.2), where $C_n \to C = \frac{1}{2}\log\frac{\pi e}{2}$ by the Stirling formula. If the inequality (3.2) holds for a large n, then either $\psi_n \leq \frac{1}{4}\log n + C + 1$, or $\Delta_n \geq \frac{1}{4}\log n - C - 1$. But, in view of (3.10), (3.12) these events have small probabilities as $n \to \infty$. This proves the main Theorem 1 to the effect that probability of advantage of intervals over ellipsoids is small as $n \to \infty$. In fact, it is shown that this probability is $O(1/(n^2\log n))$. These considerations can be regarded as an evidence in favor of ellipsoids vs. boxes in linear algebraic computations with a guaranteed accuracy.

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ENERGY TRANSFER AND JOINT DIFFUSION

ZSOLT PAJOR-GYULAI, DOMOKOS SZÁSZ

Dedicated to the memory of R. L. Dobrushin on the occasion of his 80'th anniversary

ABSTRACT. A paradigm model is suggested for describing the diffusive limit of trajectories of two Lorentz disks moving in a finite horizon periodic configuration of smooth, strictly convex scatterers and interacting with each other via elastic collisions. For this model the diffusive limit of the two trajectories is a mixture of joint Gaussian laws (analogous behavior is expected for the mechanical model of two Lorentz disks).

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1. INTRODUCTION

Beside the dynamics itself, the joint motion of two particles interacting with each other and with a dynamical environment also depends on the spatial dimension. In the first model where this question was addressed (cf. [Sz 80]), the asymptotically diffusive motions of the two particles either glue together or are independent depending on the initial distance of the particles. The model was actually that of Harris and Spitzer, (see [S 69]) (equilibrium dynamics of elastically colliding point particles) generalized by Major and Szász, [MSz 80] (non-equilibrium dynamics). On the other hand, Kipnis and Varadhan, [KV 86] have shown that the diffusive limits of two particles in a symmetric exclusion process are independent Brownian motions.

Turning from stochastic dynamics to a deterministic one, let us consider the planar, finite-horizon Lorentz process with a periodic configuration of scatterers. It is known that its limit in the diffusive scaling is a Brownian motion (cf. [BS 81] and [BCS 91]). Of course, two point like Lorentz particles do not interact, but if we take two small disks then the case is quite different.

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The motion of one small disk is itself isomorphic to a Lorentz process, so its diffusive limit is the Wiener process. However, if one considers two small Lorentz disks, then the naïve heuristics would suggest that, since the two particles collide very rarely (i. e. $O(\log n)$ times during the first n collisions), the situation is similar to the locally perturbed Lorentz process where the diffusive limit is the same Brownian motion as it was for the unperturbed Lorentz process (cf. [DSzV 09]). This analogy is, however, misleading and the aim of the present work is exactly to clarify the situation. The difference with the preceding models is that *the interaction: elastic collision of the disks also changes the energies of the two particles*. Moreover, in dimension two, by borrowing heuristics from random walk theory (cf. [S 76]) and estimates from [DSzV 08], one can convince himself/herself that the time intervals between consecutive collisions have a slowly varying tail, and consequently with a probability close to one, for t large, the last collision of the disks preceding time t befell at time $o(t)$. Thus the energies of the disks at time t, determining the limiting variance are the random energies obtained at the aforementioned last collision before t ergo the diffusive limit of each disk is a Brownian motion with a random covariance (and their joint limit can already be calculated based upon the previous line of ideas).

The goal of the present work is make the above heuristic argument precise on the level of a stochastic model mimicking the deterministic model of two Lorentz disks.

Remark: For treating the deterministic model the realistic alternative is to rely upon the averaging method of [ChD 07]. Indeed, between two collisions of the disks typically there occur long collision sequences of the particular disks with the periodic configuration of fixed scatterers. During these long intervals their orbits become approximately Brownian and their velocities and the normal of impact incoming into a particular collision of the two disks correspond to an equilibrium distribution and finally their outgoing velocities from the collision can be calculated analogously to the collision operator appearing in the derivation of Boltzmann's equation for a hard disk fluid. To the deterministic model we plan to return in a forthcoming paper.

2. THE STOCHASTIC (PARADIGM) MODEL

2.1. **Continuous Time Random Walks with Continuous Internal States.** Discrete time random walks with a finite number of internal states were introduced by Sinai, [S 81] where the internal states

were meant to represent elements of a Markov partition. Their theory was elaborated in a series of works [KSz 83, KSz 84, KSSz 86]. In our case the internal states will also represent particle velocity therefore we have to generalize Sinai's model and to consider random walks with internal states where the internal states belong to a more general state space. Moreover, for being able to include speed we take continuous time. In [KSz 83], a local limit theorem was established for random walks on $\mathbb{Z}^d$ with a finite number of internal states and we will also use much of the techniques presented there.

Definition 1. *(Sinai, 1980) Let $H, |H| < \infty$ be the set of states. On the set $\mathbb{Z}^d \times H$ the Markov chain $\xi_n = (\eta_n, \varepsilon_n)$ is a Random Walk with Internal States (RWwIS) if for $\forall\ x_n, x_{n+1} \in \mathbb{Z}^d,\ u \in H, A \subset H$*

$$P(\xi_{n+1} = (x_{n+1}, v), v \in A | \xi_n = (x_n, u)) = p_{x_{n+1}-x_n}(u, A)$$

Of course, $\{\varepsilon_n; n \geq 0\}_n$ is also a Markov chain. Our paradigm for the mechanical model will be introduced in two steps. First, the individual motion of each of the two particles will be a random walk with internal states with some state space $\tilde{H}$ and exponential jump rate $\lambda \in \Lambda$ (so far we do not specify $\tilde{H}$).

Definition 2. *Assume we are given a rate $\lambda > 0$ and a family $\{P_x(u, .) | x \in \mathbb{Z}^d\}$ of substochastic kernels over $\tilde{H}$ such that $Q = \sum_{x \in \mathbb{Z}^d} P_x$ is a stochastic kernel over $\tilde{H}$. A continuous time Markov chain with internal states from $\tilde{H}$ and with rate λ is a Markov process $\{\xi_t = (x_t, u_t)\}$ such that*

$$\mathbb{P}(\xi_{t+dt} = \xi_t) = 1 - \lambda dt + o(dt)$$

and for every $(x_t, u) \in \mathbb{Z}^d \times \tilde{H}$ and $\forall u \in \tilde{H}, \forall A \subset \tilde{H}$

$$\mathbb{P}(\xi_{t+dt} = (x_{t+dt}, u'), u' \in A | \xi_t = (x_t, u)) = \lambda P_{x_{t+dt}-x_t}(u, A) dt + o(dt)$$

In other words, the generator for jump $x \in \mathbb{Z}^d$ is described by the operators

$$P_x f(u) = \int_{\tilde{H}} f(u') P_x(u, du') \qquad x \in \mathbb{Z}^d$$

and the generator for the Markov chain $\{\varepsilon_n; n \geq 0\}_n$ of internal states is Q. For simplicity let us assume that $P_0 = 0$.

As said our RWwIS is to mimic a Lorentz disk process in $\mathbb{R}^2$ and from now on we will restrict our discussion to the planar case. Concretely we will have $\tilde{H} = S$ for the directions $\frac{v}{|v|}$ of velocities, whereas λ will be for the speed $|v| = \sqrt{2 \text{ energy}}$. Let $H = S \times \Lambda$ where $\Lambda = [a, b],\ 0 \leq a < b < \infty$. λ is, of course, conserved between collisions of the two disks.

2.2. **Interaction: the collision operator.** Next we define the collision interaction. Let $\xi_t^i = (\eta_t^i, \varepsilon_t^i)$, $i = 1, 2$ be two RWwIS.

Whenever $\eta_t^1 \neq \eta_t^2$, the joint generator of the two Markov processes is the product of the two individual generators. Whenever $\eta_t^1 = \eta_t^2 \ (= x)$,

$$\begin{aligned}\mathbb{P}(\xi_{t+}^1 = (x + z^1, v_+^1), \xi_{t+}^2 =& (x + z^2, v_+^2); v_+^1 \in A^1, v_+^2 \in A^2 \\ & | \xi_{t-}^1 = (x, v_-^1), \xi_{t-}^2 = (x, v_{t-}^2)) \\ & = C_{z^1, z^2}(v_-^1, v_-^2, A^1, A^2)\end{aligned}$$

is the collision kernel. We assume that C satisfies conservation of energy: $(v_-^1)^2 + (v_-^2)^2 = (v_+^1)^2 + (v_+^2)^2$ (momentum is not conserved since the collision kernel contains averaging over normal of impact). Consequently we can and do assume that in our two particle model $(v^1)^2 + (v^2)^2 = 1$.

Warning: the pair (ξ_n^1, ξ_n^2) is not a RWwIS on $\mathbb{Z}^d \times \mathbb{Z}^d$ anymore since translation invariance is hurt on the diagonal.

2.3. **Conditions.**

- Q is ergodic with a positive spectral gap (and with invariant probability measure ρ);
- $\sum_{x \in \mathbb{Z}^2} x \int_{\tilde{H}} P_x(v, \tilde{H}) \rho(dv) = 0$ (i. e. no drift);
- $P_x = 0$ if $|x| > 1$ (for simplicity);
- trivial arithmetic, i. e. the minimal lattice $\mathcal{L}$ the RWwIS is coincides with $\mathbb{Z}^2$;
- nonsingular asymptotic covariance matrix Σ (cf. local CLT of [KSz 83]);
- our definition of the interaction implies that by denoting $\{v_+^1(n), v_+^2(n) |\ n \geq 0\}$ the outgoing velocities at consecutive collisions (of course $(v^1)^2(n) + (v^2)^2(n) = 1$) their process is a Markov chain with transition kernel C. We assume that this Markov chain is ergodic with a gap in its spectrum (as to the proof of existence for the physical model cf. [C 75], Chapter IV, section 6). We denote its invariant measure by $\rho(v_+^1, v_+^2)$ and further by $\rho_s(|v_+|)$ the marginal of $\rho(v_+^1, v_+^2)$ providing the density of $|v_+|$.

3. Main result

Denote by (S_t^1, S_t^2) the joint trajectory of the two interacting RWwIS's.

Theorem 1. *The limiting density of* $\frac{1}{\sqrt{t}}(S_t^1, S_t^2)$ *is*

$$h(x_1, x_2) = \frac{1}{(2\pi)|\Sigma|}\int_0^1 \frac{\rho_s(\lambda)}{\lambda\sqrt{1-\lambda^2}} e^{-\frac{1}{2}\left(\frac{x_1^T(\Sigma)^{-1}x_1}{\lambda}+\frac{x_2^T(\Sigma)^{-1}x_2}{\sqrt{1-\lambda^2}}\right)} d\lambda \qquad (x_1, x_2 \in \mathbb{R}^2)$$

where Σ *is the limiting covariance matrix for any of the interacting RWwIS's with speed* 1*, and* $\rho_s(\lambda)$ *is the stationary probability density of the speed of the first particle in the Markov chain of energy change.*

The proof is based on

- the probability theory of order statistic of random variables with slowly varying distributions (cf. [HM 91])
- a far-reaching generalization of renewal theory for renewal processes
  - with slowly varying renewal distributions;
  - with renewal laws coming from a family of similar distributions;
  - directed by a Markov chain.

To the technical proof we return in a forthcoming paper.

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ZSOLT PAJOR-GYULAI: BUDAPEST UNIVERSITY OF TECHNOLOGY, INSTITUTE OF PHYSICS, DOMOKOS SZÁSZ: BUDAPEST UNIVERSITY OF TECHNOLOGY, MATHEMATICAL INSTITUTE, BUDAPEST, EGRY J. U. 1 HUNGARY H-1111,

E-mail address: moosquibe@yahoo.com, szasz@math.bme.hu

Analytical Approach to the Study of Gibbs Measures

Tetyana Pasurek
Bielefeld University (Germany)
tpasurek@math.uni-bielefeld.de

Developing Dobrushin's ideas [1, 2], we present an analytical method to prove existence and uniform à-priori estimates for Gibbs measures associated with classical particle systems in continuum. The method is based on the choice of appropriate Lyapunov functionals and on the corresponding exponential bounds for the local Gibbs specification. As a further application, we discuss existence and uniqueness problems for Gibbsian fields on graphs with unbounded degree, which extends the result of Bassalygo and Dobrushin [3] to the unbounded spin case.

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Percolation in a lattice model of non-ideal gas.

E. Pechersky[†] and A. Yambartsev[‡]

[†] IITP, 19, Bolshoj Karetny per., Moscow, Russia

pech@iitp.ru

[‡] IME-USP, Rua do Matão, 1010, 05508-090, São Paulo, Brazil

yambar@ime.usp.br

There exist a lot of works about the percolation in as discrete as continuum cases. Overwhelming majority of the works devoted to percolation models with an independence. It is either Bernoulli distribution for discrete cases like, for example, the lattice case, or Poisson distribution for the particle system in $\mathbb{R}^d$ (see, for example, [1] and [2]). There are some works on the percolation where a dependence is presented, where a 'random cluster models' are studied. Those models are connected to Izing and Potts models.

In the present work we study the percolation properties of a lattice model having a finite spin space with interactions between spins. We consider $d = 2$. The similar problem was investigated in [3], where the percolation problem was studied for a field of interacted particles in $\mathbb{R}^2$. There are two parameters λ and β which drive the percolation properties. The parameter λ is connected to a density of the particles and $\beta = \frac{1}{T}$ as usually is the inverse temperature. Our result describes regions in the plane $\{\lambda, \beta\}$, where the percolation does not occur with probability 1, where the percolation occurs with probability 1, and there is a region where we can not answer on the question on the percolation (see Fig.1).

1 Model

We specify three objects to define the model: $\mathbb{Z}^2$, S, $\overline{\varphi}$. The spin space S is finite and we assume that $S = \{0, 1, ..., N\}$. The Gibbs measure on the set of all configurations $\mathcal{X} = \{X : \mathbb{Z}^2 \to S\}$ is defined by two potential functions

$\overline{\varphi} = \{\varphi_0 : S \to \mathbb{R},\ \varphi_1 : S^2 \to \mathbb{R}\}$:

$$\begin{aligned} &\varphi_0(s) = a\binom{s}{2},\ \text{with } a > 0, \\ &\text{and} \\ &\varphi_1(s_1, s_2) = -bs_1s_2,\ \text{with } b > 0. \end{aligned} \tag{1.1}$$

Formal Hamiltonian of the model is

$$H(X) = \sum_{t_1,t_2 \in \mathbb{Z}^d : \|t_1 - t_2\| = 1} \varphi_1(X(t_1), X(t_2)) + \sum_{t \in \mathbb{Z}^d} \varphi_0(X(t)), \tag{1.2}$$

where $\|t\| = \max\{t^1, t^2\}$, when $t = (t^1, t^2) \in \mathbb{Z}^2$. One dimensional marginal of the free measure is $\nu(s) = \frac{1}{\Pi_N} \frac{\lambda^s}{s!}$, where $\Pi_N = \sum_{k=0}^{N} \frac{\lambda^k}{k!}$ and $\lambda > 0$.

The Gibbs distributions are defined in the standard way by specification which is a set of densities with respect to the free measure:

$$p^{\beta}_{V,Y}(X) = \frac{1}{Z^{\beta}_{V,Y}} \exp\{-\beta H(X|Y)\}, \tag{1.3}$$

where $V \subset \mathbb{Z}^2$ is finite volume, X is a configuration on V, Y is a boundary configuration out of V,

$$\begin{aligned} H(X|Y) = &\sum_{t_1,t_2 \in V : \|t_1 - t_2\| = 1} \varphi_1(X(t_1), X(t_2)) + \sum_{t \in V} \varphi_0(X(t)) \\ &+ \sum_{t_1 \in V, t_2 \in V^c : \|t_1 - t_2\| = 1} \varphi_1(X(t_1), X(t_2)) \end{aligned}$$

is conditional Hamiltonian, $Z^{\beta}_{V,Y}$ is a normalized constant, β is a positive constant which is called inverse temperature. The expression for the conditional Hamiltonian $H(X|Y)$ is not formal since all sums in the formula are finite. Existence at least one Gibbs measure is consequence of the finiteness of S. However the number of Gibbs state depends on the parameters.

The model can be interpreted in the following way. Consider a spin value $X(t)$ as the number of particles at a site $t \in \mathbb{Z}^d$. Every pair of the particles at the same site interacts with the repulsion energy equal to a. Thus the repulsion energy of $X(t)$ particles is $a\binom{X(t)}{2}$. Any pair of the particles located at a neighboring sites t_1, t_2 attracts with the energy $-b$. The neighbors in the model are any sites t_1, t_2 such that they are vertices of the same elementary square of $\mathbb{Z}^2$. If $X(t_1)$ and $X(t_2)$ are the numbers of the particles at neighboring sites t_1, t_2 then the attractive energy is $-bX(t_1)X(t_2)$.

2 Problem

We study the percolation over sites where spin values belong to a set $S_k = \{k+1, ..., N\},\ k < N$. The set $U \subseteq \mathbb{Z}^2$ is connected if for any pair $t,\ t^{'} \in U$ there exists a set $E(t, t^{'}) = \{t_1 = t, t_2, ..., t_n = t^{'}\} \subseteq U$ such that $|t_{i+1} - t_i| = 1$ for any $i = 1, ..., n-1$. The norm $|\cdot|$ is Euclidean in contrast $\|\cdot\|$.

A cluster is a pair $(U, X), U \subseteq \mathbb{Z}^d,\ X \in \mathcal{X}$, if U is connected and $X(t) \in S_k$ for any $t \in U$. A cluster (U, X) is infinite if U is an infinite set. We also say in this case that the configuration X has infinite cluster.

The issue we address in this work is: when almost all configurations have infinite clusters. The answer depends on the parameters λ and β. If λ and β are such that almost all configurations have infinite clusters then the models with these parameters calls percolated. If almost all configurations have no infinite clusters then the model does not percolate.

For any k we consider the models with the parameters a and b such that

$$\frac{a}{b} \leq f(k),$$

where $f(\cdot)$ is a function which can be specified.

3 Results

The following two theorems are the main results.

Theorem 3.1. *There exists $\lambda_0 < \infty$ such that for any $\lambda \leq \lambda_0$ there exists $\beta^{-}(\lambda) > 0$ such that for any $\beta \leq \beta^{-}(\lambda)$ all clusters are finite with probability 1. The value $\beta^{-}(\lambda)$ is growing to the infinity when $\lambda \to 0$.*

Theorem 3.2. *There exists a positive value of $\lambda^0 < \infty$ such that for $\lambda > \lambda^0$ there exists an infinite cluster for any β. For $\lambda < \lambda^0$ there exists $\beta^{+}(\lambda)$ such that for any $\beta > \beta^{+}(\lambda)$ there exists an infinite cluster.*

Fig.1 shows the regions of percolation and non-percolation.

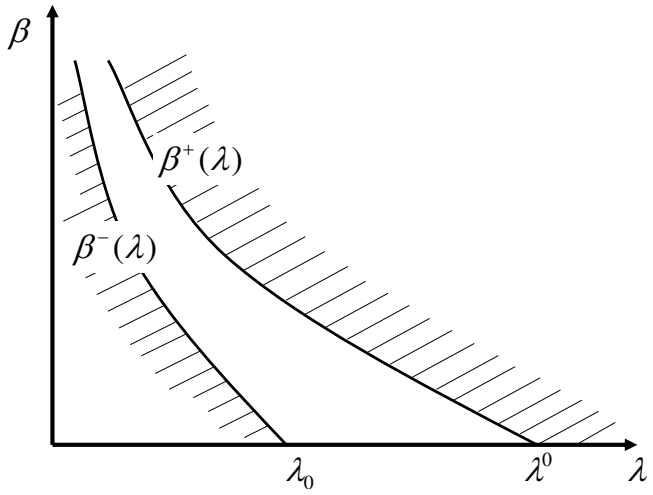

Figure 1: Percolation and non-percolation regions.

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Homogenization of a random singularly perturbed parabolic PDE

Andrey Piatnitski
Lebedev Physical institute, Moscow, Russia
and Narvik University college, Norway

Abstract. The talk will focus on averaging problem for a parabolic equation of the form

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{1}{2}\frac{\partial}{\partial x}\left(a\left(\frac{\cdot}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x}\right)(t,x) + \frac{1}{\sqrt{\varepsilon}}c\left(\frac{x}{\varepsilon}\right)u^\varepsilon(t,x), \quad t \geq 0,\ x \in \mathbf{R};$$

$$u^\varepsilon(0,x) = g(x), \quad x \in \mathbf{R},$$

with random stationary mixing coefficients a and c, in the presence of a large parameter in front of zero order term. We will show that, under proper mixing assumptions on a and c, the family of solutions u^ε converges in law, a $\varepsilon \to 0$, and describe the limits process. It should be noted that the limit dynamics remains random.

RECTANGLING RECTANGLES AND ALTERNATING CURRENT

M. PRASOLOV AND M. SKOPENKOV

This talk deals with the following problem: *for which numbers R and r a rectangle with side ratio R can be dissected into rectangles of side ratio r and $1/r$?* (Here the sides of all rectangles are parallel to coordinate axes, the *side ratio* of a rectangle is the horizontal side divided by the vertical one.) The classical Dehn theorem (1903) asserts that for $r = 1$ such a dissection is possible if and only if R is rational. For arbitrary r and R a result of Freiling, Laszkovich and Rinne (1997) provides a criterion for existence of such a dissection. But this result does not give an algorithm for description of all possible pairs (R, r). A joint result of these authors with Szekeres gives such an algorithm in case when $R = 1$.

We present an analogous "algorithmic" criterion in case when $R = r$:

Theorem. For $R > 0$ the following 4 conditions are equivalent:
(i) A rectangle with ratio R can be dissected into rectangles with ratio R and $1/R$ (in such a way that there is at least one rectangle in the dissection with ratio $1/R$).
(ii) The number R^2 is algebraic and all its algebraic conjugates (distinct from R^2) are negative real numbers.
(iii) For certain positive rational numbers $c_1, \dots, c_n$ we have

$$\cfrac{1}{c_1 R + \cfrac{1}{c_2 R + \dots + \cfrac{1}{c_n R}}} = R.$$

(iv) There is a network of total resistance R ohm consisting of resistors of resistance R ohm and $1/R$ ohm (such that the network contains at least one resistor of resistance $1/R$ with nonzero current).

The proof is based on a physical interpretation of the problem. To a dissection of a rectangle Brooks, Smith, Stone and Tutte (1940) assigned a *direct*-current circuit. Our new approach is based on application of *alternating*-current circuit theory to the problem.

On estimating special type of divergences via Kolmogorov distance *

Viacheslav V. Prelov

Institute for Information Transmission Problems
of the Russian Academy of Sciences
19 Bol'shoi Karetnyi, 127994 Moscow, Russia.

Abstract – *Some upper and lower bounds are presented for the maximum of the divergence $D(P_{X_1\dots X_n}||P_{X_1}\times\cdots\times P_{X_n})$ between the joint distribution $P_{X_1\dots X_n}$ of discrete random variables $X_1,\dots,X_n$ and the product distribution $P_{X_1}\times\cdots\times P_{X_n}$ of these variables via the Kolmogorov distance between $P_{X_1\dots X_n}$ and $P_{X_1}\times\cdots\times P_{X_n}$. In some special cases, our upper and lower bounds coincide or are asymptotically tight.*

Let $P=\{p_i\}$ and $Q=\{q_i\}$, $i\in\mathcal{I}\subseteq\mathbb{N}\doteq\{1,2,\dots\}$, be two discrete probability distribution. Recall that the (information) divergence between P and Q is defined as

$$D(P||Q)=\sum_i p_i\ln\frac{p_i}{q_i},$$

and the variational distance $V(P,Q)$ between these probability distributions is the L_1–distance, i.e.,

$$V(P,Q)=\sum_i|p_i-q_i|.$$

The problem of estimating the information divergence $D(P||Q)$ via $V(P,Q)$ was considered in several papers (see, e.g., [1] and references therein). Among many results in this areas we mention Pinsker's inequality [2,3]

$$D\geq\frac{1}{2}V^2$$

and Vajda's lower bound [4]

$$D\geq\log\left(\frac{2+V}{2-V(}\right)-\frac{2V}{2+V},$$

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where $D = D(P||Q)$ and $V = V(P,Q)$.

Note that in the general case (without additional assumptions on probability distributions P and Q) it is impossible to upper estimate $D(P||Q)$ via $V(P,Q)$ since $D(P||Q)$ can be arbitrary large as $V(P,Q)$ is arbitrary small.

Here, we consider the special case where $P = P_{X_1\dots X_n}$ is the joint distribution of discrete random variables $X_1,\dots,X_n$ and $Q = P_{X_1} \times \cdots \times P_{X_n}$ is the product distribution of these random variables. In what follows we assume that all random variables are discrete such that X_i takes values in $\mathcal{I}_i \subseteq \mathbb{N}$, $i = 1,\dots,n$. In this case the divergence

$$I(X_1;\dots;X_n) \doteq D(P_{X_1\dots X_n}||P_{X_1} \times \cdots \times P_{X_n})$$

is called the mutual information for random variables $X_1,\dots,X_n$ (see, e.g., [3]) which coincides with the usual mutual information $I(X_1;X_2)$ between two random variables X_1 and X_2 for $n = 2$. At the same time, the variational distance

$$\tau(X_1,\dots,X_n) \doteq V(P_{X_1\dots X_n}, P_{X_1} \times \cdots \times P_{X_n})$$

is sometimes called Kolmogorov distance (the latter with the factor $1/2$ and for $n = 2$ was introduced by Kolmogorov [5]).

Consider the quantities

$$I_\tau(X_1;\dots;X_n) \doteq \sup_{Y\,:\,\tau(X_1,\dots,X_n,Y)\le\tau} I(X_1;\dots;X_n;Y), \tag{1}$$

where the supremum is taken over all discrete random variables Y such that $\tau(X_1,\dots,X_n,Y) \le \tau$. Note that $I_\tau(X_1;\dots;X_n)$ is defined only for $\tau \ge \tau(X_1,\dots,X_n)$ since $\tau(X_1,\dots,X_n,Y) \ge \tau(X_1,\dots,X_n)$ for any Y. For given integers $N_1,\dots,N_n$, let

$$I_\tau^{(N_1,\dots,N_n)} \doteq \sup_{X_i\,:\,|X_i|=N_i,\ i=1,\dots,n} I_\tau(X_1;\dots;X_n), \tag{2}$$

where $|X|$ denotes the cardinality of the range of a random variable X.

In the case $n = 1$, the quantities $I_\tau(X)$ and $I_\tau^{(N)}$ were introduced and studied by Pinsker [6]. The further investigations of these quantities were continued in [7,8]. Here we present some upper and lower bounds and asymptotic expressions for quantities $I_\tau(X_1;\dots;X_n)$ and $I_\tau^{(N_1,\dots,N_n)}$ defined above.

The mutual information $I(X_1;\dots;X_n;Y)$ for random variables $X_1,\dots,X_n,Y$ can be represented as

$$\begin{aligned} I(X_1;\dots;X_n;Y) &= \sum_{i=1}^n H(X_i) + H(Y) - H(X_1,\dots,X_n,Y) = \\ &= \sum_{i=1}^n H(X_i) - H(X_1,\dots,X_n) + I(X_1,\dots,X_n;Y), \end{aligned} \tag{3}$$

where, as usual, $H(\cdot)$ denotes the entropy. Therefore, considering the vector $(X_1,\dots,X_n)$ as a single random variable, we can use some results of [6–8] to estimate $\sup\limits_Y I(X_1;\dots;X_n;Y)$ via τ, provided that $\tau((X_1,\dots,X_n),Y) \le \tau$, where

$$\tau((X_1,\dots,X_n),Y) \doteq V(P_{X_1\dots X_nY} \,||\, P_{X_1\dots X_n} \times P_Y).$$

However, our problem is to estimate $\sup_Y I(X_1;\dots;X_n;Y)$ via τ, provided that $\tau(X_1,\dots,X_n,Y) \le \tau$, but there are no direct relations between $\tau((X_1,\dots,X_n),Y)$ and $\tau(X_1,\dots,X_n,Y)$; namely, we can show that $\tau((X_1,\dots,X_n),Y)$ can be both larger and smaller than $\tau(X_1,\dots,X_n,Y)$. Therefore, in general, we cannot apply the results of [6–8], but in the special case, where $X_1,\dots,X_n$ are independent (and therefore, $\tau((X_1,\dots,X_n),Y) = \tau(X_1,\dots,X_n,Y)$), we can.

Let us introduce some definitions which are necessary to formulate our results. For a collection of random variables $X_1,\dots,X_n$, denote the joint and the marginal distributions as

$$p_{i_1\dots i_n} \doteq \Pr\{X_1 = i_1,\dots,X_n = i_n\}, \quad p_{i_k}^{(k)} \doteq \Pr\{X_k = i_k\}, \quad i_k \in \mathcal{I}_k, \quad k = 1,\dots,n.$$

Let

$$\tau^*(X_1,\dots,X_n) \doteq \max_Y \tau(X_1,\dots,X_n,Y),$$

where the maximum is over all random variables Y. It is possible to show that

$$\tau^*(X_1,\dots,X_n) = 2\left(1 - \sum_{i_1,\dots,i_n} p_{i_1\dots i_n} p_{i_1}^{(1)} \dots p_{i_n}^{(n)}\right). \tag{4}$$

Assume that the vectors $(p_{i_1}^{(1)},\dots,p_{i_n}^{(n)})$ are ordered in such a way that $(p_{i_1}^{(1)},\dots,p_{i_n}^{(n)}) \succ (p_{j_1}^{(1)},\dots,p_{j_n}^{(n)})$ if $\prod_{k=1}^n p_{i_k}^{(k)} \le \prod_{k=1}^n p_{j_k}^{(k)}$. For a given vector $\mathbf{s} = (s_1,\dots,s_n)$, $s_k \in \mathcal{I}_k$, $k = 1,\dots,n$, set

$$D_{\mathbf{s}} \doteq \left\{(i_1,\dots,i_n) : (p_{i_1}^{(1)},\dots,p_{i_n}^{(n)}) \succ (p_{s_1}^{(1)},\dots,p_{s_n}^{(n)})\right\}, \tag{5}$$

$$K_{\mathbf{s}} \doteq -\sum_{(i_1,\dots,i_n)\in D_{\mathbf{s}}} \left(\prod_{k=1}^n p_{i_k}^{(k)}\right) \ln\left(\prod_{k=1}^n p_{i_k}^{(k)}\right), \tag{6}$$

and

$$L_{\mathbf{s}} \doteq -\left(\prod_{k=1}^n p_{s_k}^{(k)}\right) \ln\left(\prod_{k=1}^n p_{s_k}^{(k)}\right). \tag{7}$$

Moreover, define

$$M \doteq \sum_{i_1,\dots,i_n} \frac{\left(\prod_{k=1}^n p_{i_k}^{(k)}\right)\left(p_{i_1\dots i_n} - \prod_{k=1}^n p_{i_k}^{(k)}\right)}{1 - \prod_{k=1}^n p_{i_k}^{(k)}} \ln\left(\prod_{k=1}^n p_{i_k}^{(k)}\right) \tag{8}$$

First of all note, that

$$I_\tau(X_1;\dots;X_n) = \sum_{i=1}^n H(X_i)$$

if $\tau \ge \tau^*(X_1,\dots,X_n)$. Indeed, this equality immediately follows from (3) if we put $Y = (X_1,\dots,X_n)$. Therefore, it is sufficient to investigate the behavior of $I_\tau(X_1;\dots;X_n)$

for $\tau(X_1,\dots,X_n) \le \tau < \tau^*(X_1,\dots,X_n)$. Our first proposition gives an upper bound for $I_\tau(X_1,\dots,X_n)$.

Proposition 1. *For any* τ, $\tau(X_1,\dots,X_n) \le \tau < \tau^*(X_1,\dots,X_n)$, *we have*

$$I_\tau(X_1;\dots;X_n) \le K_{\mathbf{s}} + xL_{\mathbf{s}} + M, \tag{9}$$

where the real number x, $0 \le x < 1$, *and the vector* $\mathbf{s} = (s_1,\dots,s_n)$, $s_k \in \mathcal{I}_k$, $k = 1,\dots,n$, *are defined by the equality*

$$\sum_{(i_1,\dots,i_n)\in D_{\mathbf{s}}} (1 - p_{i_1\dots i_n}) p_{i_1}^{(1)} \dots p_{i_n}^{(n)} + x(1 - p_{s_1\dots s_n}) p_{s_1}^{(1)} \dots p_{s_n}^{(n)} = \tau/2, \tag{10}$$

and the quantities $\tau^*(X_1,\dots,X_n)$, $D_{\mathbf{s}}$, $K_{\mathbf{s}}$, $L_{\mathbf{s}}$, *and* M *are defined in* (4)–(8).

An upper bound for $I_\tau^{(N_1\dots N_n)}$ is presented in the next proposition.

Proposition 2. *For any* τ, $0 \le \tau \le 2(1 - 1/N)$, $N \doteq \prod\limits_{k=1}^{n} N_k$, *we have*

$$I_\tau^{(N_1\dots N_n)} \le \frac{\tau}{2} \ln(N-1) + h\left(\frac{\tau}{2}\right), \tag{11}$$

where $h(x) \doteq -x\ln x - (1-x)\ln(1-x)$ *is the binary entropy function and*

$$I_\tau^{(N_1\dots N_n)} = \ln N \quad \text{if} \quad \tau \ge 2(1 - 1/N). \tag{12}$$

Note that the right-hand side of (11) is equal to $\log N$ if $\tau = 2(1 - 1/N)$, and therefore estimate (11) is tight for such τ.

Proposition 3. *For any* τ, $\tau(X_1,\dots,X_n) \le \tau < \tau^*(X_1,\dots,X_n)$, *we have*

$$I_\tau(X_1;\dots;X_n) \ge \sum_{i=1}^{n} H(X_i) - \frac{\tau^*(X_1,\dots,X_n) - \tau}{\tau^*(X_1,\dots,X_n) - \tau(X_1,\dots,X_n)} H(X_1,\dots,X_n). \tag{13}$$

Note that in the special case where random variables $X_1,\dots,X_n$ are independent, it is easy to verify that the upper and lower bounds for $I_\tau(X_1,\dots,X_n)$ and $I_\tau^{(N_1,\dots,N_n)}$ described in propositions 1–3 coincide with corresponding estimates obtained in [6–8] if the vector $(X_1,\dots,X_n)$ is considered as a single random variable whose the set of values is $\mathcal{I} = \mathcal{I}_1 \times \dots \times \mathcal{I}_n$. In particular, this remark allows us to claim that

$$I_\tau(X_1,\dots,X_n) = \frac{\tau N}{2(N-1)} \ln N, \quad 0 \le \tau \le 2(1-1/N), \quad N = \prod_{k=1}^{n} N_k,$$

if $X_1,\dots,X_n$ are independent and each X_i has uniform distribution on the set $\mathcal{I}_i$, $|X_i| = N_i$, $i = 1,\dots,n$.

Note also that for $\tau = \tau(X_1,\dots,X_n)$ (i.e., for the minimum value of τ) the lower bound (13) reduces to the inequality $I_\tau(X_1,\dots,X_n) \ge I(X_1;\dots;X_n)$. It seems that

this estimate is tight, i.e., this inequality should be replaced by the equality, since $\tau(X_1,\dots,X_n,Y)=\tau(X_1,\dots,X_n)$ if Y does not depend on the collection of random variables $X_1,\dots,X_n$, and then we would have $I_\tau(X_1,\dots,X_n)=I(X_1;\dots;X_n)$. However, in the real case, we have the strong inequality $I_\tau(X_1,\dots,X_n)>I(X_1;\dots;X_n)$ if $X_1,\dots,X_n$ are dependent since it is possible to show that there exists a random variable $\overline{Y}$ such that $\tau(X_1,\dots,X_n,\overline{Y})=\tau(X_1,\dots,X_n)$ but, at the same time, $\overline{Y}$ depends on the collections of random variables $X_1,\dots,X_n$, and therefore, we obtain

$$I_\tau(X_1,\dots,X_n)\ge I(X_1;\dots;X_n;\overline{Y})=I(X_1;\dots;X_n)+I(X_1,\dots,X_n;\overline{Y})>I(X_1;\dots;X_n).$$

One can derive the following corollaries from Propositions 1–3.

Corollary 1. *The following relations are valid:*

$$I_\tau^{(N_1,\dots,N_n)}=\left(\frac{\tau}{2}\ln N\right)(1+o(1)),\quad N\doteq\prod_{k=1}^{n}N_k\to\infty$$

and

$$\frac{\tau n}{2(n+1)}\ln\frac{1}{\tau}+O(\tau)\le I_\tau^{(N_1,\dots,N_n)}\le\frac{\tau}{2}\ln\frac{1}{\tau}+O(\tau),\quad \tau\to 0.$$

It seems natural to consider more general quantities $I_\tau^{(m)}(X_1,\dots,X_n)$ defined by the equality

$$I_\tau^{(m)}(X_1,\dots,X_n)\doteq\sup_{Y_1,\dots,Y_m\,:\,\tau(X_1,\dots,X_n,Y_1,\dots,Y_m)\le\tau}I(X_1;\dots;X_n;Y_1;\dots;Y_m).$$

However, this definition is not useful as the following claim shows.

Corollary 2. *For any random variables $X_1,\dots,X_n$, any $\tau>\tau(X_1,\dots,X_n)$, and any integer $m\ge 2$, we have*

$$I_\tau^{(m)}(X_1,\dots,X_n)=\infty.$$

The proof of all statements formulated above can be found in [9]. Here, we only show how to prove equality (4). Let a joint distribution $\{p_{i_1\dots i_n}\}$ of the random variables $X_1,\dots,X_n$ is given. Let us upper estimate $\tau(X_1,\dots,X_{n+1})$ for any random variable X_{n+1}. Let

$$A\doteq\{(i_1,\dots,i_{n+1}):p_{i_1\dots i_{n+1}}>p_{i_1}^{(1)}\dots p_{i_{n+1}}^{(n+1)}\}.$$

Then, using definition of $\tau(X_1,\dots,X_{n+1})$, we obtain

$$\begin{aligned}
\tau(X_1,\dots,X_{n+1})&=2\sum_A\left(p_{i_1\dots i_{n+1}}-p_{i_1}^{(1)}\dots p_{i_{n+1}}^{(n+1)}\right)\le\\
&\le\ 2\sum_A\left(p_{i_1,\dots,i_n}p_{i_{n+1}\,|\,i_1,\dots,i_n}-p_{i_1}^{(1)}\dots p_{i_n}^{(n)}p_{i_1\dots i_{n+1}}\right)=\\
&=\ 2\sum_A p_{i_{n+1}\,|\,i_1\dots i_n}\left(p_{i_1\dots i_n}-p_{i_1\dots i_n}p_{i_1}^{(1)}\dots p_{i_n}^{(n)}\right)\le\\
&\le\ 2\sum_{i_1,\dots,i_{n+1}}\left(p_{i_1\dots i_{n+1}}-p_{i_{n+1}\,|\,i_1\dots i_n}p_{i_1\dots i_n}p_{i_1}^{(1)}\dots p_{i_n}^{(n)}\right)=\\
&=\ 2\left(1-\sum_{i_1,\dots,i_n}p_{i_1\dots i_n}p_{i_1}^{(1)}\dots p_{i_n}^{(n)}\right).
\end{aligned}\tag{14}$$

On the other hand, we have

$$\begin{aligned} \tau^*(X_1,\dots,X_n) &\geq \tau(X_1,\dots,X_n,(X_1,\dots,X_n)) = \\ &= 2\left(1-\sum_{i_1,\dots,i_n} p_{i_1\dots i_n} p_{i_1}^{(1)}\dots p_{i_n}^{(n)}\right). \end{aligned} \tag{15}$$

Equality (4) immediately follows from (14) and (15).

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Spontaneous Resonances and the Coherent States of the Queuing Networks

Alexander Rybko[1], Senya Shlosman[1,2], Alexander Vladimirov[1]
[1] Inst. of the Information Transmission Problems,
Russian Academy of Sciences,
Moscow, Russia
[2] Centre de Physique Theorique, UMR 6207,
CNRS, Luminy, Marseille, France

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Abstract

We present an example of a highly connected closed network of servers, where the time correlations do not go to zero in the infinite volume limit. The limiting interacting particle system behaves in a periodic manner. This phenomenon is similar to the continuous symmetry breaking at low temperatures in statistical mechanics, with the average load playing the role of the inverse temperature.

Keywords: coupled dynamical systems, non-linear Markov processes, stable attractor, phase transition, long-range order.

MSC-class: 82C20 (Primary), 60J25 (Secondary)

1 Introduction

1.1 Interacting particle systems with long range memory

The theory of the phase transition provides, among many results, a positive answer to the question about the possibility of constructing reliable systems

from non-reliable elements. As an example, consider the infinite volume stochastic Ising model at low temperature T in dimension ≥ 2, see [L]. As is well known, if we start this system from the configuration of all pluses, then the evolution under Glauber dynamics has the property that the fraction of plus spins will at any time be exceeding $\frac{1+m^*(T)}{2}$, which is bigger than $\frac{1}{2}$ for $T < T_{cr.}$ (Here $m^*(T)$ is the spontaneous magnetization.) On the other hand, if we consider finite volume Ising model (with empty boundary condition, say), then this property does not hold, and, started from the all plus state, the system at some later (random) times will be found in the state with the majority of the spins to be minuses. Therefore, the infinite system can remember, to some extent, its initial state, while the finite system can not.

There are many other examples of that kind, which belong to the theory of interacting particle systems, such as voter model, contact model, etc. In all these examples we see systems, which are capable of "remembering" their initial state for arbitrary long times.

In the present paper we are constructing a particle system which is "remembering its initial phase". The rough analogy can be described as follows. Imagine a brownian particle $\varphi(t)$, with a unit drift, which lives on a circle. Suppose the initial phase $\varphi(0) = 0$. Then the mean phase $\bar{\varphi}(t) = t \mathrm{mod}(2\pi)$, but with time we know the phase $\varphi(t)$ less and less precisely, since its variance grows, and in the limit $t \to \infty$ the distribution of $\varphi(t)$ tends to uniform. However, one can combine infinitely many such particles, by introducing suitable interaction between them, in such a way that the memory of the initial phase will not vanish and will persist in time.

This is roughly what we will do in the present paper. We will consider a network of simple servers, which are processing messages. Since the service time of every message is random, in the course of time each single server looses the memory of its initial state. So, in particular, the network of non-interacting servers, started in the same state, would become de-synchronized after a finite time. However, if one introduces certain natural interconnection between servers, then it can happen that they are staying synchronized after an arbitrary long time, thus breaking some generally believed properties of large networks. We have to add here that such a phenomenon is possible only if the mean number of particles per server is high enough; otherwise the infinite network becomes de-synchronized, no matter what interaction between servers is taking place. So the parameter of the mean number of particles per server, called hereafter *the load*, plays here the same role as the

temperature in the statistical mechanics.

In other words, the transition we describe happens due to the fact that at low load the behavior of our system is governed by the fixed point of the underlying dynamical system, while at high load the dominant role is played by its periodic attractor. A similar phenomenon was described by Hepp and Lieb in [HL].

Below we present the simplest example of the above behavior. But we believe that the phenomenon we describe is fairly general. Its origin lies in the fact that any large network of the general type possesses in the infinite limit some kind of the continuous symmetry, and it is the breaking of that symmetry at high load which causes the long-range order behavior of the network.

1.2 Information networks and their collective behavior

Now we will describe one pattern of behavior of certain large networks, which was assumed to be universal. It is known under the name of Poisson Hypothesis.

The Poisson Hypothesis is a device to predict the behavior of large queuing networks. It was formulated first by L. Kleinrock in [K], and concerns the following situation. Suppose we have a large network of servers, through which many customers are traveling, being served at different nodes of the network. If the node is busy, the customers wait in the queue. Customers are entering into the network from the outside via some nodes, and these external flows of customers are Poissonian, with constant rates. The service time at each node is random, depending on the node, and the customer. The PH prediction about the (long-time, large-size) behavior of the network is the following:

- consider the total flow $\mathcal{F}$ of customers to a given node $\mathcal{N}$. Then $\mathcal{F}$ is approximately equal to a Poisson flow, $\mathcal{P}$, with a time dependent rate function $\lambda_{\mathcal{N}}(T)$.

- The exit flow from $\mathcal{N}$ – **not** Poissonian in general! – has a rate function $\gamma_{\mathcal{N}}(T)$, which is smoother than $\lambda_{\mathcal{N}}(T)$ (due to averaging, taking place at the node $\mathcal{N}$).

- As a result, the flows $\lambda_{\mathcal{N}}(T)$ at various nodes $\mathcal{N}$ should go to a constant limits $\bar{\lambda}_{\mathcal{N}} \approx \frac{1}{T}\int_0^T \lambda(t)\,dt$, as $T \to \infty$, the flows to different nodes being

almost independent.

- The above convergence is uniform in the size of the network.

Note that the distributions of the service times at the nodes of the network can be arbitrary, so PH deals with quite a general situation. The range of validity of PH is supposed to be the class of networks where the internal flow to every node $\mathcal{N}$ is a union of flows from many other nodes, and each of these flows constitute only a small fraction of the total flow to $\mathcal{N}$. If true, PH provides one with means to make easy computations of quantities of importance in network design.

The rationale behind this conjectured behavior is natural: since the inflow is a sum of many small inputs, it is approximately Poissonian. And due to the randomness of the service time the outflow from each node should be "smoother" than the total inflow to it. (This statement was proven in [RShV] under quite general conditions.) In particular, the variation of the latter should be smaller than that of the former, and so all the flows should go with time to corresponding constant values.

In the paper [RSh] the Poisson Hypothesis is proven for simple networks in the infinite volume limit, under some natural conditions. For systems with constant service times it was proven earlier in [St1].

The purpose of the present paper is to construct a network, satisfying all the above assumptions – namely, that the flow to every given node is an "infinite" sum of "infinitesimally small" flows from other nodes – which network, nevertheless, has coherent states. That means that the states of the servers are evolving in a synchronous manner, and the "phase" of a given server behaves (in the thermodynamic limit – i.e. in the limit of infinite network) as a periodic non-random function, the same for different servers.

We have to stress that our network exhibits these coherent states only in the regime when the average number N of the customers per server – called in what follows *the load* – is large. For low load we expect the convergence to the unique stationary state. This "high temperature" kind of behavior will be the subject of the forthcoming work.

Our network ∇_∞ is constructed from infinitely many elementary "triangular" networks ∇ (described below, in Section **??**). A single triangle network $\nabla = \nabla_1$ with N customers is just a Markov continuous time ergodic jump process with finitely many states. As N becomes large, this Markov process tends (in the appropriate "Euler" limit) to a (5-dimensional) dynamical system Δ, possessing a periodic trajectory $\mathcal{C}$, which turns out to be a stable

(local) attractor. The coordinate φ parameterizing that attractor $\mathcal{C}$ is the "phase", alluded to in the previous subsection. The combined network corresponds in the same sense to the coupled family Δ_∞ of dynamical systems Δ. We establish the synchronization property of that coupled family Δ_∞, and that allows us to construct coherent states of the network ∇_∞.

The networks ∇_M, combined from M triangle networks ∇, are ergodic. Their evolution is given by irreducible finite state Markov processes with continuous time. Let π_M be the invariant measure of the process ∇_M. As $M \to \infty$, the sequence of Markov processes ∇_M converges weakly on finite time intervals to a certain limiting (non-linear Markov) process ∇_∞. By the theorem of Khasminsky – see Theorem 1.2.14 in [L] – any accumulation point of the sequence π_M is a stationary measure of ∇_∞. The special measure χ_∞, describing "the Poisson Hypothesis behavior", is also a stationary measure of ∇_∞. If χ_∞ is a global attractor of ∇_∞, then, of course, the Poisson Hypothesis holds. The proof of the Poisson Hypothesis in [RSh] was based on this argument. The existence of an accumulation point of the sequence π_M that differs from χ_∞ would be the strongest counterexample to the Poisson Hypothesis. This problem will be addressed in forthcoming papers. Here we prove a weaker statement that χ_∞ is not a global attractor for ∇_∞.

In [RSt] Rybko and Stolyar observed that the condition that the workload at every node of a multiclass open queueing network is less than 1 is not sufficient for the network to be ergodic. In connection with this, they introduced a new approach to the analysis of ergodicity of networks, which reduces the problem to the question of stability of the associated fluid models. It was shown by them that considered in [RSt] two-node priority network is ergodic if and only if for every initial state of the corresponding fluid model the total amount of fluid eventually vanishes. This approach was further developed by Dai [D], Stolyar [St2], and Puhalsky and Rybko [PR], who proved that stability of the fluid model is necessary and sufficient for ergodicity of a certain class of general networks. Interesting instances of non-ergodic queueing networks with mean load being smaller than the capacity, where considered by Bramson [B1, B2]. Our construction will be based on the following open network, introduced by Rybko and Stolyar (RS-network) in [RSt].

It is represented by the following 4-dimensional Markov process, which describes the open queuing network with two types of customers. They arrive to the network according to Poisson inflows of constant rate λ. The network consists of two nodes – $\bar{A}$ and $\bar{B}$ – through which the customers go. All the service times are exponential, so to describe the network we need

to specify only the rates, the evolution of types of the customers and the priorities. The customer of type A (respectively, B) arrives to the node $\bar{A}$ (respectively, $\bar{B}$). The customer A is served with the rate γ_A, then is sent to $\bar{B}$, with type AB. There he is served with the rate γ_{AB} and leaves the network. Symmetrically, $\gamma_B = \gamma_A$, and $\gamma_{BA} = \gamma_{AB}$. Each customer AB is served before all the customers B, and vice versa. The nominal workload at nodes $\bar{A}$ and $\bar{B}$ equals to $\rho = \lambda(\gamma_A^{-1} + \gamma_{BA}^{-1})$. The service rates satisfy the conditions $\gamma_{AB} < 2\lambda$ and $\rho < 1$. It is proved in [RSt] that for certain values of the parameters the resulting Markov process is transient. The fluid limit (or Euler limit) of this network evolves in the following non-trivial manner: each node is empty during a positive fraction of time, but at other moments it is non-empty, and moreover the total amount of the fluid in the network grows linearly to infinity.

The rest of the paper is organized as follows. In the next Section 2 we will define our networks ∇_M^N. Here M is the size of the network and N is the load per node. We formulate the preliminary version of our Main Result. In Section 3 we study the limiting network, ∇_∞^N, and we prove the convergence $\nabla_M^N \to \nabla_\infty^N$. In Section 4 we introduce the fluid networks, Δ_M, which are coupled dynamical systems, and their limit, Δ_∞, which turns out to be a non-linear dynamical system, in the sense made precise in this Section. In particular, we show that Δ_∞ is not ergodic. In the next Section 5 we prove the convergence of the Non-Linear Markov Process ∇_∞^N to its Euler fluid limit, Δ_∞, as $N \to \infty$. The last Section 6 contains the formulation and the proof of our Main Result.

To save on notation, we consider throughout this paper the simplest elementary symmetric model, depending on 3 parameters. We stress the fact that this (discrete) symmetry is not essential in our case, and our results are valid for any small 6D-perturbation of our model, since the above mentioned limiting continuous symmetry holds in this more general setting.

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Vladas Sidoravicius

CWI, Amsterdam and IMPA, Rio de Janeiro

Macroscopic sensitivity to localized defects: slow bond and many others...

Abstract: Consider one dimensional Totally Asymmetric Exclusion Process on $\mathbb{Z}$, with the density of particles being $1/2$, and their jump rates being 1 everywhere, but the origin. At the origin the jump rate is different: once particle arrives to the origin its jump rate decreases to $1-\lambda,\ 0<\lambda<1$, and becomes again equal to 1, as soon as particle jumps away from the origin. The central question is to find out for which values of λ the current on the far-right-side of the system is affected by such *local slowdown*? This problem is known as the "slow bond problem", and the search for the critical value λ_c, above which the current is affected and below it is not, was a demanding question.

The model has several equivalent representations such as the Polynuclear Growth Model with "columnar defect" or, as a directed Last Passage Percolation with "reinforced diagonal", and can be treated by Random Matrices techniques as well, using the language of generalized permutations.

Using Polynuclear Growth language (and other ingredients from random matrices and interacting particle systems techniques), I first of all will show that $\lambda_c = 0$. Then, in contrast, I will show that for some "similar" growth systems λ_c could be strictly positive. It will bring to the general discussion about sensitivity of growth systems to localized defects, questions of polymer pinning in presence of bulk disorder and other closely related issues.

The talk based on joint works with V. Beffara, T. Sasamoto and M.E. Vares.

MODULAR VECTOR INVARIANTS OF CYCLIC GROUPS

Sergey A. Stepanov
Institute for Information Transmission Problems,
Russian Academe of Sciences, Moscow, Russia

ABSTRACT. Let F be a field, $V = Fx_1 + \cdots + Fx_n$ a vector space of dimension n over F, and $G \leq GL(n, F)$ a finite group acting on V via F-linear transformations of the basis elements $x_1, \ldots, x_n$. Let $V^{\oplus m} = V \oplus \cdots \oplus V$ be the m-fold direct sum of the space V with diagonal action of the group G. Then the group G naturally acts on the symmetric graded algebra $A_{mn} = F[x_{i1}, \ldots, x_{in} \mid 1 \leq i \leq m]$. Let A_{mn}^G denote the subalgebra of invariants of the polynomial algebra A_{mn} with respect to G. A classical result of Emmy Noether [8], [9] implies that in the non-modular case, that is when the characteristic p of F does not divide $|G|$ (in particular, when char $F = 0$) the ring A_{mn}^G is generated as F-algebra by homogeneous polynomials of degree at most $|G|$, no matter how large m is. On the other hand, it was proved by D. Richman [10]–[12] that this result is no longer hold in the modular case when the characteristic p of F divides $|G|$. Let $p > 2$ be a prime number, $F = F_p$ a finite field with p elements, H a cyclic group of order p acting on a linear F_p-space V of dimension n, and A_{mn}^H the subalgebra of invariants of the polynomial algebra $A_{mn} = F_p[x_{i1}, \ldots, x_{in} \mid 1 \leq i \leq m]$ with respect to H. In this paper we give a further development of the orbit sum method proposed by the author in [16] and determine explicitly a complete system of generators of the algebra A_{mn}^H in the case when $n = 3$. In addition, we find a lower degree bound for the maximal possible degree of homogeneous invariants forming a complete system of generators of the algebra A_{mn}^H. These results extend the corresponding results of D. Richman [10]–[12], a result of Campbell and Hughes [1] concerning the case $n = 2$, and a more general result of the author [16] in the case of cyclic groups H of a special form.

I.INTRODUCTION

Let F be a field, $A = R[x_1, \ldots, x_n]$ a finitely generated commutative R-algebra, G a finite group of its R-algebra automorphisms, and A^G the subalgebra of polynomial invariants of G. If $z_1, \ldots, z_n$ are commuting variables, we set

$$P(z_1, \ldots, z_n) = \prod_{\sigma \in G} (1 + \sigma(x_1)z_1 + \cdots \sigma(x_n)z_n)$$

and denote by $\beta(A^G)$ the smallest positive integer for which A^G can be generated as an R-algebra by polynomials of degree at most β. If each nonzero integer is invertible in R, then Noether's classical result [8], [9] states that A^G is generated over R by the coefficients of $P(z_1, \ldots, z_n)$, so that $\beta(A^G) \leq |G|$. The last inequality is known as *Noether's bound*. The above mentioned result of Richman [12] and standard arguments based on the use of the Noether map [9] show that

Noether's bound $\beta(A^G) \leq |G|$ holds under a much weaker assumption that $|G|!$ is invertible in R. A recent result of Fleishmann [5] demonstrates that in fact the last inequality holds under the assumption that the order $|G|$ of G is invertible in R.

If the characteristic p of F is positive and divide $|G|$, we speak of the *modular case*. Otherwise, we have the *non-modular case*, which includes the classical case of polynomial invariants over algebraically closed fields of characteristic zero. Almost everything that is usually used in the non-modular case (see, for example, [3], [13], [14]) is missing in the modular case: the Cohen–Macaulay property fails in general, we have no Reynolds operator (averaging over $|G|$) and no Molien's formula for the generating Poincaré series. Nevertheless, if F is a field of prime characteristic p, and H a p-Sylow subgroup of G, the modular case admits a possibility of very extensive applications of *generalized orbit Chern classes* related to H, especially, orbit traces (*orbit sums* of monomials) and top orbit classes (*orbit norms* of monomials). Let $F = F_p$ be a prime finite field with p elements, and H a cyclic group of order p acting linearly on the vector space $V = F_p x_1 + \cdots + F_p x_n$. Set $A_{mn} = F_p[V^{\oplus m}]$ and denote by A_{mn}^H the algebra of invariants of A_{mn} with respect to the diagonal action of H on the space $V^{\oplus m} = V \oplus \cdots \oplus V$. It turns out that there exist a F_p-linear space $\tilde{V}$ containing V as a subspace and a cyclic group $\tilde{H}$ (closely related to to the group H and acting linearly on $\tilde{V}$) such that every invariant $u \in A_{mn}^H$ can be written as a special F_p-linear combination of orbit sums $S_{\tilde{H}}(f)$, orbit norms $N_{\tilde{H}}(g)$ (related to the group $\tilde{H}$) and also their products $S_{\tilde{H}}(f)N_{\tilde{H}}(g)$, for various monomials $f, g \in F_p[\tilde{V}^m]$. It should be noted that if H is a cyclic group of prime order p , and $F = F_p$ a prime field with p elements, then $S_{\tilde{H}}(f)$ and $N_{\tilde{H}}(g)$ can be determined exactly that gives a possibility to write out a system of generating elements of A_{mn}^H in an explicit form.

The most significant distinction between the non-modular and the modular case is as follows: Noether's bound no longer holds in the modular case. For the first time this phenomenon was discovered by Richman [10] in course of the study of H-invariant polynomials of the algebra $A_{m2} = F_p[x_i, y_i \mid 1 \leq i \leq m]$, where H is a cyclic group of prime order p. In this paper he proved that $\beta(A_{m2}^H) \geq m(p-1)$. More generally, if G is a finite group of order divisible by a prime number p and A_{nm} a polynomial algebra over an arbitrary field F of characteristic p, Richman [11] proved that

$$\beta(A_{mn}^G) \geq \frac{m(p-1)}{p^{|G|-1}-1} \ .$$

In the case of permutation groups this bound was later improved by Kemper [6] and by the author [15] as follows: if F is a field of prime characteristic p, and $G \subset S_n$ a permutation group which contains an element of order p^α, then

$$\beta(A_{mn}^G) \geq \max\{n, m(p^\alpha - 1)\} \ .$$

This result implies, in particular, that if $R = \mathbb{Z}$ is the ring of integers then $\beta(A_{mn}^{S_n}) \geq \max\{n, m(n-1)/2\}$. It has been recently shown by Fleischmann

[4] that the above low degree bound $\beta(A_{mn}^G) \geq \max\{n, m(p^\alpha - 1)\}$ is sharp: if $n = p^\alpha$ and $m > 1$ then

$$\beta(A_{mn}^{S_n}) \leq \max\{n, m(n-1)\} \ .$$

The last result can be regarded as a refinement of the Campbell-Hughes -Pollack upper bound [2]

$$\beta(A_{mn}^{S_n}) \leq \max\{n, mn(n-1)/2\}, \ ,$$

which holds in the case of an arbitrary polynomial algebra A_{mn} over a commutative ring R.

2. CONSTRUCTION OF A COMPLETE SYSTEM OF GENERATORS

Theorem 1. *Let $H \leq GL(3, F_p)$ be a cyclic group of order p which is generated by the matrix*

$$\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \ .$$

There exists a system of homogeneous polynomials

$$x_{i3}, \quad x_{\tilde{i}_\sigma,3}, \quad \zeta_{\tilde{i}_\sigma,3}, \quad \omega_{\tilde{i}_\sigma,3}, \quad u_{(\tilde{i}_\sigma,\tilde{i}_\tau),3},$$

$$\varphi_{(\tilde{i}_\sigma,\tilde{i}_\tau),3}, \quad \psi_{(\tilde{i}_\sigma,\tilde{i}_\tau),3}, \quad v_{(\tilde{i}_\sigma,\tilde{i}_\tau),3}, \quad w_{(\tilde{i}_\rho,\tilde{i}_\sigma,\tilde{i}_\tau),3}, \quad \varrho_{i,3}, \quad S_H(f') \ ,$$

where

$$\tilde{i}_\sigma = (i_1, \ldots, i_\sigma), \quad 1 \leq i_1 < \ldots < i_\sigma \leq m, \quad 1 \leq \sigma \leq 3,$$

such that every invariant $u \in A_{m3}^H$ is a polynomial over F_p in elements of this system.

Corollary 2. *If $m > 1$, then every system of generating elements of the algebra $A_{m,3}^H$ contains a polynomial of degree at least $m(p-1)$.*

Theorem 1 provides an explicit construction of generating elements of the algebra A_{m3}^H in terms of the orbit sums and the orbit sums of monomials. The algebra of vector invariants A_{m2}^H over F_p, where $H \leq GL(2, F_p)$ is a cyclic subgroup of order p generated by

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

for the first time was studied by Richman in [10]. In particular, he found a minimal generating system of the above algebra in the case when $p = 2$ and conjectured that a similar system would generate A_{m2}^H in the case of odd prime p. This conjecture was proved by Campbell and Hughes in [1]. Another important result of the last paper is that for $m > 2$ the algebra $A_{m,2}^H$ is not Cohen–Macaulay. Independently, an analogous generating system for the algebra A_{mn}^H,

where $n = 2r > 2$ and $H \leq GL(n, F_p)$ is a cyclic subgroup of order p generated by the matrix

$$\gamma = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{pmatrix}$$

with the basic Jordan blocks $J_1, J_2, \ldots, J_r$ of sizes $n_1 = n_2 = \cdots = n_r = 2$, was found by the author [16].

In principle, the result of Theorem 1 can be extended to the general case of the algebra A_{mn}^H of polynomials $f \in A_{mn}$ that are invariant under the action of an arbitrary cyclic subgroup $H \leq GL(n, F_p)$ of order p generated by the $n \times n$ matrix

$$\gamma = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{pmatrix}$$

with the basic Jordan blocks $J_1, J_2, \ldots, J_r$ of sizes $n_1, n_2, \ldots, n_r$, respectively. Moreover, the arguments of the paper give also a possibility to find a low bound for the maximal possible degree of a generator in a complete generating system of the algebra A_{mn}^G in the case when $G \leq GL(n, F_p)$ is an arbitrary group containing the cyclic subgroup H as a subgroup.

3. A BRIEF DESCRIPTION OF THE MAIN IDEAS

We now explain briefly the main ideas underlying the proof of Theorem 1. Let

$$A_{mn} = F[x_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n]$$

be a polynomial algebra over F_p, $G \leq GL(n, F_p)$ a finite group, and f a monomial from A_{mn}. The use of orbit sums

$$S_G(f) = \sum_{u \in \{\sigma(f} \mid \sigma \in G\}u$$

is most efficient in the case when the group G acts on elements of the algebra A_{mn} by permutations of the vector variables

$$x_1 = (x_{11}, \ldots, x_{m1}), \ldots, x_n = (x_{1n}, \ldots, x_{mn}) \ .$$

In that case each invariant $u \in A_{mn}^G$ is an F_p-linear combination of the above orbit sums $S_G(f)$ for various monomials f. This important result is an easy consequence of the following fact: if a monomial f appears in an invariant u with a nonzero coefficient a, then for each $\sigma \in G$ the corresponding monomial $\sigma(f)$ also appears in u with the same coefficient a. Unfortunately, the above property

is no longer hold for finite groups G of a more general form, in particular, for cyclic groups H generated by matrices

$$\gamma = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{pmatrix}$$

with basic Jordan blocks $J_1, J_2, \ldots, J_r$ of sizes $n_1, n_2, \ldots, n_r$, respectively, such that $1 < n_i < p$, for at least one $i = 1, 2, \ldots, r$. On the other hand, if $n_1 = n_2 = \cdots = n_r = p$, then after a non-singular linear transformation we can proceed to a new system of vector variables

$$\tilde{x}_1 = (\tilde{x}_{11}, \ldots \tilde{x}_{m1}), \ldots, \tilde{x}_n = (\tilde{x}_{1n}, \ldots, \tilde{x}_{mn})$$

on which the group H acts by cyclic permutations.

Let H be the cyclic group of prime order $p > 2$ generated by a non-singular square matrix γ with the basic Jordan blocks $J_1, J_2, \ldots, J_r$ of sizes $n_1, n_2, \ldots, n_r$, respectively, and recall that H acts linearly on the vector space V^m of dimension $m(n_1 + \cdots + n_r)$. The proof of Theorem 1 falls into three steps.

(i) At the first step we 'blow up' each Jordan block J_i, $1 \leq i \leq r$, of the matrix γ to a Jordan block of the largest possible size p. As a result, the generating matrix γ of the cyclic group H is transformed into the corresponding square matrix $\tilde{\gamma}$ of size $\nu = rp$ and the group H into the corresponding cyclic group generating by $\tilde{\gamma}$ and acting on the vector space $\tilde{V}^m$ of dimension $m\nu$. It follows from the above that then one can find new vector variables

$$\tilde{x}_1 = (\tilde{x}_{11}, \ldots, \tilde{x}_{m1}), \ldots, \tilde{x}_n = (\tilde{x}_{1n}, \ldots, \tilde{x}_{mn}) \ ,$$

obtained from the original variables

$$\tilde{x}_1 = (\tilde{x}_{11}, \ldots, \tilde{x}_{m1}), \ldots, x_n = (\tilde{x}_{1n}, \ldots, \tilde{x}_{mn})$$

by a non-degenerate linear transformation, on which the group $\tilde{H}$ by cyclic permutations. This property of the group $\tilde{H}$ allows us to show that every invariant v of the algebra $A_{mn}^{\tilde{H}}$ is an F_p-linear combination of the orbit sums $S_{\tilde{H}}(f)$, orbit norms $N_{\tilde{H}}(g)$ and also their products $S_{\tilde{H}}(f)N_{\tilde{H}}(g)$ for various monomials $f, g \in A_{m\nu}$.

(ii) At the second step we we demonstrate that an appropriate embedding of the algebra $A_{m,3}$ into $A_{m,p}$ results a fairly simple test distinguishing among the $\tilde{H}$-invariants $v \in A_{mp}^{\tilde{H}}$ the polynomial invariants with respect to the action of H. The use of this test makes possible an explicit construction of invariants $u \in A_{m,3}$ as F_p-linear combinations of orbit sums $S_{\tilde{H}}(f)$ and orbit norms $N_{\tilde{H}}(g)$ a special form, and also their products products $S_{\tilde{H}}(f)N_{\tilde{H}}(g)$.

(iii) At the third step we form a complete system of generators of the algebra $A_{m,3}^{H}$ by selecting certain families of homogeneous polynomials $u \in A_{m,3}$ of bounded degree.

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Tarasov A.G.

Spectrum of resonances in one dimension

Consider Schrodinger operator on real line. The potential $V(x)$ is supposed to be super-exponentially decaying, i.e such that for every $M > 0$, there exists a constant $C = C(M) > 0$ such that

$$|\,V(x)\,| \;\leqslant\; C\,e^{-M|x|} \qquad (*)$$

for any $x \in \mathbb{R}$. The scattering matrix for operator $L_V = -d^2/dx^2 + V(x)$ is defined as

$$S(k) \;=\; \begin{pmatrix} A(k) & B(k) \\ C(k) & D(k) \end{pmatrix},$$

where the reflection and transitions coefficients (since V satisfies to condition $(*)$) for $k \in \mathbb{C}_+$ can be represented in the following form

$$\begin{aligned} B(k) &= \frac{1}{2ik}\int_{\mathbb{R}} e^{-ikx}\,V(x)\,y^{(-)}(x,k)\,dx\,, \\ C(k) &= \frac{1}{2ik}\int_{\mathbb{R}} e^{ikx}\,V(x)\,y^{(+)}(x,k)\,dx\,, \end{aligned}$$

and

$$\begin{aligned} A(k) &= 1 + \frac{1}{2ik}\int_{\mathbb{R}} e^{-ikx}\,V(x)\,y^{(+)}(x,k)\,dx\,, \\ D(k) &= 1 + \frac{1}{2ik}\int_{\mathbb{R}} e^{ikx}\,V(x)\,y^{(-)}(x,k)\,dx \end{aligned}$$

respectively. Here $y^{(\pm)}(x,k) \;=\; e^{\pm ikx}\big(1 - e^{\mp ikx}R_V(k)e^{\pm ikx}V\big)$, and $R_V(k)$ is the resolvent for L_V.

Definition 1 *A resonance for operator L_V is a pole in $\mathbb{C}_-$ in the meromorphic continuation of $S(k)$.*

Since matrix $S(k)$ is unitary, it follows from the Schwartz symmetry principle that poles of the scattering matrix in $\mathbb{C}_-$ coincide with zeroes of the determinant $E(k) = \det S(-k)$. The transition coefficients are bounden in $\mathbb{C}_+$, so for $k \in \mathbb{C}_-$ one can write

$$E(k) \;=\; 1 + \frac{1}{4k^2}\,B(-k)C(-k) + O(k^{-1})\,.$$

Thus the problem under consideration is reduced to analysis of behavior of the transition coefficients. By means of rude bounds on these coefficients on can determine asymptotically free poles regions.

Statement 1 *Let $V(x)$ satisfies to condition $(*)$. Suppose that Fourier transform $\widehat{V}(k)$ is bounded in the sectors $\big\{|\arg(\pm k)| \leqslant \psi\big\}$. Then there are only finitely many scattering poles in these sectors.*

Let potential be of the form $V(x) = e^{-P(x)}$, where $P(x) = x^{2m}/2m$ and $m \in \mathbb{N}$. In this case the asymptotic formulas for distribution of resonances are inferred.

Theorema 1 *Suppose* $L_V = -d^2/dx^2 + e^{-x^{2m}/2m}$. *Then spectrum of resonances of* L_V *consists of two series*

$$k_{\pm n} = \pm \frac{e^{\mp i\frac{\pi}{4m}}}{2}\left(\frac{2\pi m}{2m-1}\,n\right)^{\frac{2m-1}{2m}}\left[\,1 \mp i\,\frac{(3m-2)(2m-1)}{4\pi m^2}\,\frac{\ln n}{n} \mp \right.$$
$$\left. \mp\; i\,\frac{(2m-1)C_\pm}{4\pi m^2}\,\frac{1}{n} + O\big(n^{\frac{1-3m}{2m}}\big)\right],$$

where $C_\pm = m\,\ln\dfrac{2m-1}{2\pi} + (3m-2)\left(\ln\dfrac{2\pi m}{2m-1} \pm i\,\dfrac{\pi}{2}\right)$.

Function $\widehat{V}(k)$ is super-exponentially decaying in the sectors $|\arg(\pm k)| \leqslant \pi/4m$. Hence, it follows from statement 1 that these sectors are asymptotically free of resonances. By means of saddle point method (see [1]) one can obtain that in region $K := \big\{\arg k \in (-\pi+\pi/4m, -\pi/4m)\big\}$ the following formula holds

$$\widehat{V}(2k) = \frac{1}{\sqrt{P''(z_0(k))}}\,\exp\big(S(z_0(k),k)\big)\Big(1 + O\big(k^{\frac{-m}{2m-1}}\big)\Big).$$

Here $z_0(k) = (2ik)^{\frac{1}{2m-1}}$ is a saddle point for phase function $S(z,k) = 2ikz - P(z)$.

The main part of the proof of the theorem 1 is

Proposition 1 *Let* $k \in K = \big\{k \in \mathbb{C}_-,\ \operatorname{Re} S(z_0(k),k) > 0\big\}$, *then*

$$B(-k) = \sqrt{2\pi}\,\widehat{V}(-2k)\Big(1 + O\big(k^{\frac{1-m}{2m-1}}\big)\Big)$$

and

$$C(-k) = \sqrt{2\pi}\,\widehat{V}(2k)\Big(1 + O\big(k^{\frac{1-m}{2m-1}}\big)\Big).$$

To obtain these formulas one should estimate integrals

$$\int_{\mathbb{R}} e^{\pm 2ikx}\,V(x)\,y^{(\mp)}(x,-k)\,dx.$$

That can be done via some modification of the saddle point method. One of the most important step of this modification is contraction and bound of meromorphic continuation for function $y^{(\mp)}(x,-k)$ to the domain containing the saddle carve $\mathbb{R}_- \cup [0,\, 2z_0(k)] \cup \big(z_0(k) + \mathbb{R}_+\big)$ for $S(x,k)$.

Using asymptotics for $A(-k)$ and $D(-k)$ the equation $E(k) = 0$ can be rewritten in the form

$$S(z_0(k),k) - \frac{3m-2}{2m}\,\ln S(z_0(k),k) + O(k^{\frac{1-m}{2m}}) = \pi i n - \frac{1}{4}\ln\big(\pi/2\big), \quad n \in \mathbb{N},$$

which allows to calculate resonances with qualified error.

The described above scheme of calculating resonances can be extended to rather wide class of potentials.

Theorema 2 *Suppose that function* $V(x)$ *is even and*

$$V(x) = e^{-P(x)}, \quad P(x) = a_{2m}x^{2m} + a_{2n}x^{2n} + \ldots + a_0,$$

where $a_{2m}, a_{2n} > 0$. *Then the spectrum of resonances for* $L_V = -d^2/dx^2 + V(x)$ *consists of two series* $k_\nu^\pm$, *such that sufficiently large* $\nu \in \mathbb{N}$ *the following holds*

$$S\big(z_0(k_\nu^\pm), k_\nu^\pm\big) \;=\; \omega_{\pm\nu} \;+\; \frac{3m-2}{2m}\,\ln\omega_{\pm\nu} \;+\; O\Big(\nu^{-\min\left\{1-\frac{n}{m},\,\frac{1}{2}-\frac{1}{m}\right\}}\Big),$$

where $z_0(k) \sim \Big(ik/ma_{2m}\Big)^{1/(2m-1)}$ *is the saddle point for* $S(z,k) = 2ikz + P(x)$ *and*

$$\omega_{\pm\nu} \;=\; \pm\pi i\nu \;-\; \frac{3m-2}{2m}\,\ln\big((2m-1)a_{2m}\big) \;+\; \frac{1}{2}\,\ln\big(4m^3(2m-1)a_{2m}^3/\pi\big).$$

In this case contraction of saddle carve for $S(z,k)$ is rather difficult problem. For such curve can be chosen the line L consists of two rays $\mathbb{R}_-$ and $\xi(k) + \mathbb{R}_+ \subset z_0(k) + \mathbb{R}_+$ and crosspiece $\gamma(k)$, connecting $z = 0$ with $\xi(k)$. Shape of the γ considerable depends on positional relationship between saddle point $z_0(k)$ and set $\Omega_C = \big\{\mathrm{Re}\; P(z) \geqslant -C\big\}$.

In [2] is determined the leading asymptotics of the resonances counting function for Schrodinger operator in one dimension with super-exponentially decaying potential. Results described above confirm and refinement this information.

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Dynamic Routing; Configuration of Overloaded Interacting Servers

N.D. Vvedenskaya*

We consider a symmetrical network with k servers and l Poisson input flows. The protocol uses dynamic routing: each flow is assigned to a subgroup of m servers, upon its arrival a message is directed to the least busy of these servers. Under the condition that at least m servers are overloaded the number of overloaded servers depends on the rate of input flows. The work prolongs [1], where a similar effect is described for a circle of interacting servers.

1 Description of problem, main theorem

We consider networks that use dynamic routing FIFO protocol where each flow is assigned to a subgroup of m servers. Upon its arrival a message selects the least busy server among the assigned ones. We are interested in probability of large fluctuations where the massages of a flow have large delays.

There are many works investigating large fluctuations in networks with dynamic routing (see for example the bibliography in [1]). This work is a continuation of [1], where a circle network with k servers and k Poisson input flows is considered and the distribution of message lengths has light tails. Each flow is assigned to 2 servers; upon its arrival a message is directed to the server with smallest workload. It is shown that under the condition that at least 2 servers are overloaded the number of overloaded servers depends on the rate of input flows.

Consider a symmetrical system $\mathcal{S} = \mathcal{S}(k, m)$ formed by k identical servers $S = (s_1, ..., s_k)$ and $l = \binom{k}{m}$ independent Poisson flows $F = (f_{A_1}, ..., f_{A_l})$, each of rate λ. Here $A_j = (j_1, ..., j_m)$ are the numbers of servers $S_{A_j} = (s_{j_1}, ..., s_{j_m})$ assigned to f_{A_j}. The servers have infinite buffers and operate with equal rate 1. Upon its arrival with f_{A_j} a message is directed to a server from S_{A_j} that at the time of its arrival has the smallest workload. The flows are described by the sequences of independent pairs

$$(\xi_n^{(A_j)}, \tau_n^{(A_j)}), \quad n = ..., -1, 0, 1, ..., \quad j = 1, ..., l,$$

* Kharkevich Institute for Information Transmission Problems, RAS, Moscow, ndv@iitp.ru

where $\tau_n^{(A_j)}$ are the intervals between arrivals of two messages and $\xi_n^{(A_j)}$ – the message lengths. All variables are iid, $\tau_n^{(A_j)}$ are exponentially distributed, $\Pr(\tau_n^{(A_j)} > t) = e^{-\lambda t}$. The time $t_n^{(A_j)}$ of message $(\xi_n^{(A_j)}, \tau_n^{(A_j)})$ arrival is negative if $n < 0$ and positive if $n \geq 0$. The distributions of $\xi_n^{(A_j)}$ are identical. We require that there exists such θ_+, $0 < \theta_+ \leq \infty$, that

$$\varphi(\theta) = \mathbb{E}e^{\theta\xi_n^{(A_j)}} < \infty \quad \text{as} \quad \theta < \theta_+, \quad \lim_{\theta\uparrow\theta_+} \varphi(\theta) = \infty. \tag{1}$$

The mean intensity of the sum of Poisson flows upon one server is $\Lambda = \frac{\lambda\binom{k}{m}}{k}$. The system is in stationary state,

$$\Lambda\varphi'(0) < 1, \quad \lambda < \widehat{\lambda} = k\left(l\varphi'(0)\right)^{-1}.$$

If during some time period the flow intensity is large the flow is said to be *overheated*, if there is a lot of unserved messages in a buffer of a server the server is said to be *overloaded*.

We shown that if a message of some flow waits for a long time to be served then the performance of network depends on mean flow rate: there exists value of input flow rate such that if the arrival rates are above it a long waiting time for a messages most probably coincides with overload of all servers. On the other hand in case of low rate the overload of m servers most probably is not connected with overload of others. The proof of this facts is based on comparison of $\mathcal{S}$ with a network where each message selects one of m assigned servers with a fixed probability. The probability of large workload decreases exponentially with increase of workload. Remark that in a limit system with *infinite* number of servers in case of dynamic routing the stationary distribution of queue lengths decreases superexponentially (see [2]).

Introduce a notion of virtual messages. These are messages that arrive upon $\mathcal{S}$ at time moment $t = 0$ with flows f_{A_j}, have zero lengths and are directed to the servers according to the dynamic routing protocol. The delay (waiting time) of virtual message that arrived with f_{A_j} is denoted by ω_j. The delay of virtual message that arrived with flow f_{A_1}, $A_1 = (1, 2, ..., m)$, is denoted by ω_1. We are interested in probability of large ω_1 and are looking for the asymptotics of probability

$$J = \lim_{n\to\infty} \frac{-1}{n} \ln \Pr(\omega_1 \geq n)$$

Consider three random processes:

1. The l dimensional compound Poisson process $\boldsymbol{\zeta}(t) = (\zeta^{(A_1}(t), ..., \zeta^{(A_l)}(t))$, $\zeta^{(A_j)}(t) = \sum_{n:\ 0\leq t_n^{(A_j)}<t} \xi_n^{(A_j)}$.

2. The k dimensional process $\widehat{\boldsymbol{w}}(\boldsymbol{t}) = (\widehat{w}^1(t), ..., \widehat{w}^k(t))$ that describes the amount of work delivered to the servers during time $[0, t)$.

3. The k dimensional Markov process $\boldsymbol{w}(t) = (w^1(t), ..., w^k(t))$ that describes the workload (the amount of unserved work) at the servers at moment t.

Consider also the scaled versions of these processes: $\zeta_n^{(A_j)}(t) = \frac{1}{n}\zeta^{(A_j)}(nt)$, $\widehat{w}_n^i(t) = \frac{1}{n}\widehat{w}^i(nt)$, $w_n^i(t) = \frac{1}{n}w^i(nt)$.

It is clear that $\Pr(\omega_1 > nd) = \Pr[\ \min(w_n^1, ..., w_n^m) > d]$.

A topological space of not-decreasing functions defined on $[0, \infty)$ and equal to 0 at $x = 0$ is introduced; the space is equipped with *uniformly week* topology (see [3]); a sequence of measures is defined by the processes $\boldsymbol{\zeta}_n$. On this space a rate function I is considered. By conditions (1) we can consider only the subspace of absolutely continuous functions where I is defined by

$$I(x) = \int_0^\infty \sup_{\theta<\theta_+} \{\theta\dot{x}(t) - \lambda[\varphi(\theta) - 1]\}\mathrm{dt}, \quad \mathrm{x} \in \mathfrak{X}_\mathrm{a},$$

here $\dot{x}$ is a derivative on t. For $x(t)$ that is a trajectory of f_{A_j} we call $\dot{x}$ the *flow speed*.

In our problem as $n \to \infty$ the components of optimal trajectories $\boldsymbol{\zeta}$, where rate function reaches its conditional minimum, converge to the functions of form

$$\dot{x}_{T,a}(t) = \begin{cases} a & \text{as } t \in [0, T), \\ \lambda\varphi'(0) = a^* & \text{as } t > T, \end{cases} \qquad x_{T,a}(0) = 0 \tag{2}$$

Below we always presume that as $n \to \infty$ the trajectories converge to such functions. The asymptotics of large delay probability is estimated for the limit trajectories of form (2). For a set of trajectories $Z(t)$, that converge to $x_{T,a}$ $I(Z) = \inf\limits_{z\in Z} I(z) = T \sup\limits_{\theta} \{\theta a - \lambda[\varphi(\theta) - 1]\}$.

Introduce an event

$$\Gamma_j(n) = (\omega_j \geq nd),$$

it indicates that a virtual message that arrived with f_{A_j}, has a delay $\geq nd$.

Theorem 1 . *For any k, $k \geq 3$, and any $d > 0$ there exist $\lambda_{(k)}$, $\lambda^{(k)}$, $0 < \lambda_{(k)} \leq \lambda^{(k)} < \widehat{\lambda}$, that depend on ξ^{A_j} distribution such that*

- *If $\lambda < \lambda_{(k)}$, then $J = \lim_{n\to\infty} \frac{-1}{n} \ln \Pr(\Gamma_1(n)) = m\theta_m d$, where θ_m is a positive root to equation $m\theta = \lambda[\varphi(\theta) - 1]$.*

  The limit dynamic of conditional process $\boldsymbol{\zeta}_n$ under the condition $\Gamma_1(n)$ is

$$\boldsymbol{\zeta}_n = (x_{T,a_1}, x_{T,a^*}, ..., x_{T,a^*}),\ a_1 = \lambda\varphi'(\theta_m). \tag{3}$$

- *If $\lambda > \lambda^{(k)}$, then $J = \lim_{n\to\infty} \frac{-1}{n} \ln \Pr(\Gamma_1(n)) = k\theta_k d$, where θ_k is a positive root to equation $k\theta = \lambda\binom{k}{m}[\varphi(\theta) - 1]$.*

  The limit dynamic of conditional process $\boldsymbol{\zeta}_n$ under the condition $\Gamma_1(n)$ is

$$\boldsymbol{\zeta}_n = (x_{T,a_k}, , ..., x_{T,a_k}),\ a_k = \lambda\varphi'(\theta_k). \tag{4}$$

Remark that the event Γ_1 in case $\lambda < \lambda_{(k)}$ is mainly defined by f_{A_1}, only this flow is overheated, and in case $\lambda > \lambda^{(k)}$ all flows are overheated.

Below we consider several scenarios that bring the events $\Gamma_1(n)$ and estimate there probability.

2 Comparison of two protocols of routing

In addition to system $\mathcal{S}$ with dynamic routing introduce an auxiliary system $\mathcal{S}^{(0)}$ with similar k servers and l flows. The realization of flows in both systems are identical. At $\mathcal{S}^{(0)}$ the routing is random: a message of flow f_{A_j} with given probability $\alpha(j,r)$ is directed to the server $s_r,\ s_r \in S_{A_j}$. Naturally, $0 \le \alpha(j,r) \le 1,\ \ \sum_{r=1}^{m} \alpha(j,r) = 1$. Remark that in auxiliary system the flows upon the servers are independent and Poisson. We say that f_{A_j} *is served* by servers $s_{r_1}, ...s_{r_q},\ q \le m$ if $\alpha(j,r_1) > 0, ..., \alpha(j,r_q) > 0$.

Suppose that during some time interval T the flows F have the speeds $a_{A_1}, ..., a_{A_l}$. We call these flows *balanced* with respect to servers S if for any j, r there exist such $\alpha(j,r)$ and such $b > 0$ that

$$\sum_j \alpha(j,r) a_{A_j} = \frac{\sum_{j=1}^{l} a_{A_j}}{k} = b.$$

Note that $\alpha(j,r)$ may be defined not uniquely.

Now compare the performance of systems $\mathcal{S}$ and balanced system $\mathcal{S}^{(0)}$.

In case of system $\mathcal{S}^{(0)}$ consider the random processes $\widehat{\boldsymbol{v}}$ and $\boldsymbol{v}$ that are similar to processes $\widehat{\boldsymbol{w}}$ and $\boldsymbol{w}$ for system $\mathcal{S}$.

In space $\mathbb{R}^k$ denote by $\widehat{W}_b(t)$ and $W_b(t)$ the projections of $\widehat{\boldsymbol{w}}(t)$ and $\boldsymbol{w}(t)$ upon the bisectrix $w^1 = ... = w^k$ and denote by $\widehat{W}_p(t)$ and $W_p(t)$ the projections of the same vectors upon a hyperplane orthogonal to the bisectrix. Similarly denote by $\widehat{V}_b(t)$, $V_b(t)$ and $\widehat{V}_p(t)$, $V_p(t)$ the projections of $\widehat{\boldsymbol{v}}(t)$ and $\boldsymbol{v}(t)$ upon the bisectrix $v^1 = ... = v^k$ and upon a hyperplane orthogonal to the bisectrix.

Remark that $|\widehat{W}_b(t)| = |\widehat{V}_b(t)|$. (Here and below $|a|$ is the length of vector a.)

We call both systems *overloaded* if $\sum_{j=1}^{l} a_j > k$ (and $b > 1$).

Below we present without proofs several statements.

Lemma 1 . *If $\mathcal{S}$ and $\mathcal{S}^{(0)}$ are overloaded and flows F are balanced then*

$$\mathbf{E}\big||\widehat{v}^i(t)| - |\widehat{v}^j(t)|\big| < \mathrm{O}\left(t^{\beta}\right), \quad \mathbf{E}\big||v^i(t)| - |v^j(t)|\big| < \mathrm{O}\left(t^{\beta}\right), \quad \frac{1}{2} < \beta < 1, \quad (5)$$

$$\mathbf{E}\big||\widehat{w}^i(t)| - |\widehat{w}^j(t)|\big| < \mathrm{O}\left(t^{\beta}\right), \quad \mathbf{E}\big||w^i(t)| - |w^j(t)|\big| < \mathrm{O}\left(t^{\beta}\right), \quad \frac{1}{2} < \beta < 1. \quad (6)$$

$\big|\mathbf{E}|W_b(t)| - \mathbf{E}|V_b(t)|\big| <$*const uniformly in t.*

Lemma 1 states that both in $\mathcal{S}$ and $\mathcal{S}^{(0)}$ the load vectors are concentrated in the neighborhood of bisectrix. The following example shows that in fact in a system with dynamic routing the load vectors are more concentrated.

Example. Consider a system with 2 servers and 1 flow ($k = 2, l = 1, m = 2$). In auxiliary system set $\alpha(1,1) = \alpha(1,2) = \frac{1}{2}$. At the moment t_n of message arrival vector $V_p(t_n + 0) - V_p(t_n - 0)$ is directed with equal probability towards the bisectrix or towards the opposite direction, $\mathbf{E}|V_p(t)| = \mathrm{O}\left(n^{1/2}\right)$ for $\forall$ t. If

$|W_p(t+0)| \neq |W_p(t-0)|$ then vector $W_p(t+0) - W_p(t-0)$ is directed towards the bisectrix and $\mathbf{E}|W_p(t)| = \mathrm{O}(1)$.

If the flows are not balanced then F and S can be divided into r groups of flows $F(p)$ and servers $S(p)$, $F = \bigcup_{p=1}^{r} F(p)$, $S = \bigcup_{p=1}^{r} S(p)$, so that the flows of $F(p)$ are served by the servers $S(p)$ and $F(p)$ are balanced with respect to $S(p)$. Denote by $b(p)$ the mean sum of speeds of flows upon a server $s_i \in S(p)$, let $b(p) < b(p-1)$.

The dividing can be performed in correspondence with dynamic routing:
C o n d i t i o n 1. *If some flow is assigned to servers of different groups it is served by the servers of the group with the smallest possible $b(p)$.*

The dividing for which Condition 1 is fulfilled is called a *proper dividing.* It can be shown that a proper dividing is unique.

Example of proper dividing: $k = l = 3$, $m = 2$, $A_1 = (1,2), a_{A_1} = 5$, $A_2 = (2,3), a_{A_2} = 1$, $A_3 = (1,3), a_{A_3} = 1$, $\alpha(1,1) = \alpha(1,2) = 1/2$, $\alpha(2,2) = 0, \alpha(2,3) = 1$, $\alpha(3,3) = 1, \alpha(3,1) = 0$, $b(1) = 2.5, b(2) = 2$.

Below we use the same notation $S(p)$, $F(p)$ for groups of $\mathcal{S}^{(0)}$ and $\mathcal{S}$.

After a proper dividing is chosen the space $\mathbb{R}^k$ of $\boldsymbol{v}$ is divided into r subspaces $\mathbb{R}^{k_p}$ where k_p is the number of $s_i \in S(p)$. If $b(p) > 1$ the load vectors $\boldsymbol{v}^p(t) = (v^{p_1}(t), ..., v^{p_{k_p}}(t))$ are "mainly" concentrated in neighborhood of sub-bisectrix $v^{p_1} = ... = v^{p_{k_p}}$ and increase with speed $b(p) - 1$ (compare with (5)). .

L e m m a 2 . *If proper dividing is chosen and $b(p) > 1$ then for load vectors $\boldsymbol{w}^p(t) = (w^{p_1}(t), ..., w^{p_{k_p}}(t))$ the inequalities of type (6) take place, the components of $\boldsymbol{w}^p(t)$ increase with speed $b(p) - 1$.*

Consider an event $\Gamma_j^{(0)}(n) = \left([\min_{s_i \in S_{A_j}} v^i] \geq nd\right)$ that is similar to event $\Gamma_j(n)$.

C o r o l l a r y 1 . *As $n \to \infty$ asymptotics of $\Pr(\Gamma_j(n))$ coincide with asymptotics of $\Pr(\Gamma_j^{(0)}(n))$.*

Let the evens $\Gamma_1(n)$ and $\Gamma_1^{(0)}(n)$ take place under the condition:
C) during time interval T the flows F have the speeds $a_{A_1}, ..., a_{A_l}$ and are divided properly. Let s_r be a server where $r = \mathrm{argmin}_{q=1}^{m} v^q(T)$ and $s_r \in S(p)$. That means that in the limit $n \to \infty$ the speed of increase of v_n^q (and of w_n^q), $1 \leq q \leq m$, is not less then $b(p) - 1$.

Set $F' = \bigcup_{q=1}^{p} F(q)$, $|F'| = l'$ and $S' = \bigcup_{q=1}^{p} S(q)$, $|S'| = k'$, here $l' = \binom{k'}{m}$.

P r o p o s i t i o n 1 . *The rate function of events $\Gamma_1(n)$ and $\Gamma_1^{(0)}(n)$ under the conditions* **C***) is not less then the rate function of similar events for a properly divided flows where speeds of flows $f_{A_j} \in F(q) \subset F'$ are equal to $a'_{A_q} = \frac{a_{A_q} b(p)}{b(q)}$, the speeds of $f_{A_j} \in F \setminus F'$ are equal to a^*, $|v^j(t)|$ and $|w^j(t)|$, $s_j \in S'$, increase with speed $b(p) - 1$.*

From large deviation theory and from above statements follows

Lemma 3 . *The rate function $J(k',m)$ of events $\Gamma_1^{(0)}(n)$ and $\Gamma_1(n)$ under the condition that l' flows assigned to k' servers have equal speeds and are overheated, $l' = \binom{k'}{m}$, while $l - l'$ flows have speed a^*, is equal to*

$$J(k',m) = k'\theta d, \tag{7}$$

where θ is the root to equation

$$k'\theta = \vartheta(\lambda, k') = l'\lambda\left(\varphi(\theta) - 1\right). \tag{8}$$

The speed of overheated flows is $a = \lambda\varphi'(\theta)$.

We see now that the rate function of events $\Gamma_1(n)$ and $\Gamma_1^{(0)}(n)$ caused by some configuration of flows is bounded from below by rate function of similar events caused by configuration of flows where l', $1 \le l' \le l$ flows have equal speeds $a > a^*$ and $l - l'$ flows are of speed a^*.

To prove the theorem me have to investigate the dependance of $J(k',m)$ on λ and k'. Remind that $\lambda = \frac{k\Lambda}{\binom{k}{m}}$, $\Lambda < \left(\varphi'(0)\right)^{-1}$. It follows from (7),(8) that as $\Lambda \to \widehat{\Lambda} = \left(\varphi'(0)\right)^{-1}$ (and $\lambda \to \widehat{\lambda}$) then $\lim_{k'=k,\Lambda\to\widehat{\Lambda}} \vartheta(\lambda,k') = 0$, and $\lim_{k'<k,\Lambda\to\widehat{\Lambda}} \vartheta(\lambda,k') > 0$. Therefore if λ is sufficiently large then $J(k',m)$ takes its minimum at $k' = k$, the event $\omega_1 > nd$ is most probable as all flows are equally overheated, see (4).

Further, as $\lim_{\lambda\to 0}\theta' = \theta^+$ the function $\vartheta(\lambda,k')$ increases the faster the less is k'. Therefore for sufficiently small λ $J(k',m)$ takes its minimum as $k' = m$. The event $\omega_1 > nd$ is most probable as only one flow f_{A_1} is overheated, see (3).

Remark that for not symmetrical system the statement of Theorem 1 may be wrong.

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On games of continuous and discrete randomized forecasting

Vladimir V. V'yugin*

Institute for Information Transmission Problems, Russian Academy of Sciences,
Bol'shoi Karetnyi per. 19, Moscow GSP-4, 127994, Russia
e:mail vyugin@iitp.ru

Abstract. Using the game-theoretic framework for probability, Vovk and Shafer [7] have shown that it is always possible, using randomization, to make sequential probability forecasts that pass any well-behaved statistical test. We show that Vovk and Shafer's result is valid only when the forecasts are computed with unrestrictedly increasing degree of accuracy. We present a test failing any given method of randomized forecasting which uses a fixed level of discreteness.

Using the game-theoretic framework for probability [6], Vovk and Shafer have shown in [7] that it is always possible, using randomization, to make sequential probability forecasts that pass any well-behaved statistical test. This result generalizes work by other authors, among them are Foster and Vohra [2], Kakade and Foster [3], Lehrer [4], Sandrony et al. [5], who consider only tests of calibration.

We complement this result with a lower bound. We show that Vovk and Shafer's result is valid only when the forecasts are computed with unrestrictedly increasing degree of accuracy. We present a test failing any given method of discrete randomized forecasting. To formulate this example, we use the forecasting game presented by Vovk and Shafer [7], namely Binary Forecasting Game II.

Let $\mathcal{P}\{0,1\}$ be the set of all measures on the two-element set $\{0,1\}$. Any measure from $\mathcal{P}\{0,1\}$ is represented by a number $p \in [0,1]$ - the probability of $\{1\}$. Let $\mathcal{P}[0,1]$ be the set of all probability measures on the unit interval $[0,1]$ supplied with the standard Borel σ-field $\mathcal{F}$.

Randomizing forecasting is defined as follows. For each n, a forecaster given a binary sequence of past outcomes $\omega_1 \dots \omega_{n-1}$ (and a sequence of past forecasts $p_1, \dots, p_{n-1}$) outputs a probability distribution $P_n \in \mathcal{P}[0,1]$. The forecasts p_n of the the future event $\omega_n = 1$ are distributed according to this probability distribution.

Vovk and Shafer's [7] *Binary Forecasting Game II* between three players - Forecaster, Skeptic, Reality, Random Number Generator is described by the following *protocol*:

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Let $\mathcal{K}_0 = 1$ and $\mathcal{F}_0 = 1$.
FOR $n = 1, 2, \ldots$
Skeptic announces $S_n : [0,1] \to \mathcal{R}$.
Forecaster announces a probability distribution $P_n \in \mathcal{P}[0,1]$.
Reality announces $\omega_n \in \{0,1\}$.
Forecaster announces $f_n : [0,1] \to \mathcal{R}$ such that $\int f_n(p) P_n(dp) \le 0$.
Random Number Generator announces $p_n \in [0,1]$.
Skeptic updates his capital $\mathcal{K}_n = \mathcal{K}_{n-1} + S_n(p_n)(\omega_n - p_n)$.
Forecaster updates his capital $\mathcal{F}_n = \mathcal{F}_{n-1} + f_n(p_n)$.
ENDFOR

Restriction on Skeptic: Skeptic must choose the S_n so that his capital $\mathcal{K}_n$ is nonnegative for all n no matter how the other players move.

Restriction on Forecaster: Forecaster must choose the P_n and f_n so that his capital $\mathcal{F}_n$ is nonnegative for all n no matter how the other players move.

Vovk and Shafer [7] showed that Forecaster has a winning strategy in the Forecasting Game II, where Forecaster wins if either (i) his capital $\mathcal{F}_n$ is unbounded or (ii) Skeptic's capital $\mathcal{K}_n$ stays bounded; otherwise the other players win.

Using some specific forms of $S_n(p)$, Shafer and Vovk [6] have shown that Forecaster has strategies forcing the strong law of large numbers and the law of iterated logarithm.

Theorem 1. *Forecaster has a winning strategy in Binary Forecasting Game II.*

For completeness of the presentation, we give a sketch of the proof from [7].

At first, at any round n of Binary Forecasting Game II, a simple auxiliary game between Realty and Forecaster is considered: Forecaster chooses $p_n \in [0,1]$, Realty chooses $\omega_n \in \{0,1\}$. Forecaster losses (and Realty gains) $S(p_n)(\omega_n - p_n)$.

For any mixed strategy of Realty $Q_n \in \mathcal{P}\{0,1\}$, let Forecaster's strategy be $p_n = Q\{1\}$. So, the Realty's expected gain is $S(p_n)(1 - Q\{1\})Q\{1\} + S(p_n)(0 - Q\{1\})Q\{0\} = 0$, where $Q\{0\} = 1 - Q\{1\}$.

In order to apply von Neumann's minimax theorem, which requires that move space be finite, we replace Forecaster move space $[0,1]$ with a finite subset of $[0,1]$ dense enough that the value of the game is smaller than some arbitrary small positive number Δ (depending on n). This is possible, since $|S_n(p)| \le \mathcal{K}_{n-1} \le 2^{n-1}$. [2] The minimax theorem asserts that Forecaster has a mixed strategy $P \in \mathcal{P}[0,1]$ such that

$$\int S_n(p)(\omega_n - p) P(dp) \le \Delta \tag{1}$$

for both $\omega_n = 0$ and $\omega_n = 1$.

Let E_Δ be the subset of $\mathcal{P}[0,1]$ consisting all probability measures P satisfying (1) for $\omega_n = 0$ and $\omega_n = 1$. Endowed with the weak topology, $\mathcal{P}[0,1]$ is

[2] Skeptic must choose $S_n(p)$ such that $\mathcal{K}_n \ge 0$ for all n no matter the other players move.

compact. Since each E_Δ is closed, $\cap E_{\Delta_i} \neq \emptyset$, where Δ_i, $i = 1, 2, \ldots$, is some decreasing to 0 sequence of real numbers. So there exists $P_n \in \mathcal{P}[0,1]$ such that

$$\int S_n(p)(\omega_n - p)P_n(dp) \leq 0$$

for both $\omega_n = 0$ and $\omega_n = 1$.

In Binary Forecasting Game II, consider the strategy for Forecaster that uses at any round n the probability distribution P_n just defined and uses as his second move the function f_n defined $f_n(p) = S_n(p)(\omega_n - p)$. Then $\mathcal{F}_n = \mathcal{K}_n$ for all n. So either Skeptic's capital will stay bounded or Forecaster's capital will be unbounded. $\triangle$

In that follows we consider some modification of Binary Forecasting Game II in which Skeptic (but not Forecaster) announces $f_n : [0,1] \to \mathcal{R}$. This means that Skeptic defines the test of randomness he needs.

Also, at each step n, Skeptic divide his capital into two accounts: $\mathcal{K}_n = \mathcal{Q}_n + \mathcal{F}_n$; he uses the capital $\mathcal{F}_n$ to force Random Number Generator to generate random numbers which pass the test f_n.

Let $\mathcal{K}_0 = 2$.
FOR $n = 1, 2, \ldots$
Skeptic announces $S_n : [0,1] \to \mathcal{R}$.
Forecaster announces a probability distribution $P_n \in \mathcal{P}[0,1]$.
Reality announces $\omega_n \in \{0,1\}$.
Skeptic announces $f_n : [0,1] \to \mathcal{R}$ such that $\int f_n(p)P_n(dp) \leq 0$.
Random Number Generator announces $p_n \in [0,1]$.
Skeptic updates his capital $\mathcal{K}_n = \mathcal{K}_{n-1} + S_n(p_n)(\omega_n - p_n) + f_n(p_n)$.
ENDFOR

We divide the Skeptic's capital into two parts:
$\mathcal{K}_n = \mathcal{Q}_n + \mathcal{F}_n$ for all n, where
$\mathcal{Q}_0 = 1$ and $\mathcal{F}_0 = 1$.
$\mathcal{Q}_n = \mathcal{Q}_{n-1} + S_n(p_n)(\omega_n - p_n)$ and
$\mathcal{F}_n = \mathcal{F}_{n-1} + f_n(p_n)$ for all $n > 0$.

Restriction on Skeptic: Skeptic must choose the S_n and f_n so that his capital $\mathcal{K}_n$ is nonnegative for all n no matter how the other players move.

Actually, Skeptic will choose the S_n and f_n so that both of his capitals $\mathcal{Q}_n$ and $\mathcal{F}_n$ are nonnegative for all n no matter how the other players move.

Assume for each n, the probability distribution P_n is concentrated on a finite subset D_n of $[0,1]$, say, $D_n = \{p_{n,1}, \ldots, p_{n,m_n}\}$. The number $\Delta = \liminf_{n\to\infty} \Delta_n$, where

$$\Delta_n = \inf\{|p_{n,i} - p_{n,j}| : i \neq j\},$$

is called *the level of discreteness* of the corresponding forecasting scheme on the sequence $\omega_1\omega_2\ldots$. In general case D_n is measurable with respect to the σ-field $\mathcal{F}^{n-1}$, depending on $\omega_1 \ldots \omega_{n-1}$.

A typical example is the uniform rounding: for each n, rational points $p_{n,i}$ divide the unit interval into equal parts of size $0 < \Delta < 1$ and P_n is concentrated

on these points. In this case the level of discreteness equals Δ for an arbitrary sequence $\omega_1\omega_2\ldots$.

We prove that when Forecaster uses finite subsets of $[0,1]$ for randomization Realty and Skeptic can defeat Forecaster (and Random Number Generator) in this forecasting game, where Realty and Skeptic win if Skeptic's capital $\mathcal{K}_n$ is unbounded; otherwise Forecaster and Random Number Generator win.

Theorem 2. *Assume Forecaster's uses a randomized strategy with a positive level of discreteness on each infinite sequence ω. Then Realty and Skeptic win in the modified Binary Forecasting Game II.*

Proof. Define a strategy for Realty: at any step n Realty announces an outcome

$$\omega_n = \begin{cases} 0 \text{ if } P_n((0.5,1]) > 0.5 \\ 1 \text{ otherwise.} \end{cases}$$

We follow Shafer and Vovk's [6] method of defining the defensive strategy for Skeptic.

Let $\epsilon_k = 2^{-k}$, $k = 1, 2, \ldots$. We define recursively by n: $\mathcal{Q}_0^{s,k} = 1$, $S_0^{s,k}(p) = 0$, $s = 1, 2$, and for $n \geq 1$

$$S_n^{1,k}(p) = -\epsilon_k \mathcal{Q}_{n-1}^{1,k} \xi(p > 0.5), \tag{2}$$

$$S_n^{2,k}(p) = \epsilon_k \mathcal{Q}_{n-1}^{2,k} \xi(p \leq 0.5), \tag{3}$$

where $\xi(true) = 1$, $\xi(false) = 0$, and for $n \geq 1$

$$\mathcal{Q}_n^{1,k} = \mathcal{Q}_{n-1}^{1,k} + S_n^{1,k}(p_n)(\omega_n - p_n)), \tag{4}$$

$$\mathcal{Q}_n^{2,k} = \mathcal{Q}_{n-1}^{2,k} + S_n^{2,k}(p_n)(\omega_n - p_n)). \tag{5}$$

We combine $S_n^{1,k}(p)$ and $S_n^{2,k}(p)$ in the Skeptic's strategy $S_n(p) = \frac{1}{2}(S_n^1(p) + S_n^2(p))$, where $S_n^1(p) = \sum\limits_{k=1}^{\infty} \epsilon_k S_n^{1,k}(p)$ and $S_n^2(p) = \sum\limits_{k=1}^{\infty} \epsilon_k S_n^{2,k}(p)$. It can be proved by the mathematical induction on n that $0 \leq Q_n^{i,k} \leq 2^n$ and $|S_n^{i,k}(p)| \leq 2^{n-1}$ for $i = 1, 2$ and for all k, p and n. Then these sums are finite for each n and p.

By (4)-(5) the Skeptic's capital $\mathcal{Q}_n$ at step n, when he follows the strategy $S_n(p)$, equals $\mathcal{Q}_n = \frac{1}{2} \sum\limits_{k=1}^{\infty} \epsilon_k (\mathcal{Q}_n^{1,k} + \mathcal{Q}_n^{2,k})$.

Define for each n the function $g_n(p) = (2\xi(p \leq 0.5) - 1)(\omega_n - p)$. Let $E_{P_n}(g_n) = \int g_n(p) P_n(dp)$.

Let Forecaster be used some randomized strategy P_n, $n = 1, 2, \ldots$.

We define recursively by n: $\mathcal{F}_0^k = 1$, $g_0^k(p) = 0$, and for $n \geq 1$

$$g_n^k(p) = -\epsilon_k \mathcal{F}_{n-1}^k (g_n(p) - E_{P_n}(g_n)), \tag{6}$$

where $\epsilon_k = 2^{-k}$, and $\mathcal{F}_n^k = \mathcal{F}_{n-1}^k + g_n^k(p_n)$ for $n \geq 1$.

By definition for any k and n,

$$\mathcal{F}_n^k = \prod_{j=1}^{n} (1 - \epsilon_k (g_j(p_j) - E_{P_j}(g_j))). \tag{7}$$

By (7) $0 \le \mathcal{F}_n^k \le 2^n$ for all n and k. Finally, Skeptic defines at step n, $f_n(p) = \sum_{k=1}^{\infty} \epsilon_k g_n^k(p)$. By definition $\int f_n(p) P_n(dp) \le 0$.

By (7) the Skeptic's capital $\mathcal{F}_n$ at step n, when he follows the strategy $f_n(p)$, equals $\mathcal{F}_n = \sum_{k=1}^{\infty} \epsilon_k \mathcal{F}_n^k$. Also, $\mathcal{F}_n \ge 0$ for all n.

Suppose that $\sup_n \mathcal{F}_n = C < \infty$, where $C > 0$. Then $\sup_n \mathcal{F}_n^k < \frac{C}{\epsilon_k}$ for each k.

We have for each k,

$$\ln \mathcal{F}_n^k \ge -\epsilon_k \sum_{j=1}^{n} (g_j(p_j) - E_{P_j}(g_j)) - n\epsilon_k^2.$$

Here we use the inequality $\ln(1+r) \ge r - r^2$ for all $|r| \le \frac{1}{2}$.

Since $\mathcal{F}_n$ is bounded by $C > 0$, we have for any k

$$\frac{1}{n} \sum_{j=1}^{n} (g_j(p_j) - E_{P_j}(g_j)) \ge \frac{-\ln C + \ln(\epsilon_k)}{n\epsilon_k} - \epsilon_k \ge -2\epsilon_k \tag{8}$$

for all sufficiently large n.

Define two variables $\vartheta_{n,1} = \sum_{j=1}^{n} \xi(p_j > 0.5)(\omega_j - p_j)$ and $\vartheta_{n,2} = \sum_{j=1}^{n} \xi(p_j \le 0.5)(\omega_j - p_j)$. By definition of g_j, $\vartheta_{n,2} - \vartheta_{n,1} = \sum_{j=1}^{n} g_j(p_j)$. Define $g_{1,j}(p) = \xi(p > 0.5)(\omega_j - p)$ and $g_{2,j}(p) = \xi(p \le 0.5)(\omega_j - p)$. Then $g_j(p) = g_{2,j}(p) - g_{1,j}(p)$.

Assume for any n the probability distribution P_n is concentrated on a finite set $\{p_{n,1}, \dots, p_{n,m_n}\}$.

For technical reason, if necessary, we add 0 and 1 to the support set of P_n and set their probabilities to be 0. Denote $p_n^- = \max\{p_{n,t} : p_{n,t} \le 0.5\}$ and $p_n^+ = \min\{p_{n,t} : p_{n,t} > 0.5\}$.

By definition ω_n, p_n^+ and p_n^- are predictable and $p_n^+ - p_n^- \ge \Delta$ for all n, where $\Delta > 0$. We have

$$\sum_{j=1}^{n} E_{P_j}(g_{1,j}) \le \sum_{\omega_j=0} P_j\{p > 0.5\}(-p_j^+) + \sum_{\omega_j=1} P_j\{p > 0.5\}(1 - p_j^+) \le$$
$$-0.5 \sum_{j=1}^{n} \xi(\omega_j = 0) p_j^+ + 0.5 \sum_{j=1}^{n} \xi(\omega_j = 1)(1 - p_j^+). \tag{9}$$

$$\sum_{j=1}^{n} E_{P_j}(g_{2,j}) \ge \sum_{\omega_j=0} P_j\{p \le 0.5\}(-p_j^-) + \sum_{\omega_j=1} P_j\{p \le 0.5\}(1 - p_j^-) \ge$$
$$-0.5 \sum_{j=1}^{n} \xi(\omega_j = 0) p_j^- + 0.5 \sum_{j=1}^{n} \xi(\omega_j = 1)(1 - p_j^-). \tag{10}$$

Subtracting (9) from (10), we obtain

$$\sum_{j=1}^{n} E_{P_j}(g_j) = \sum_{j=1}^{n} E_{P_j}(g_{2,j}) - \sum_{j=1}^{n} E_{P_j}(g_{1,j}) \geq 0.5\Delta n.$$

Using (8), we obtain for all sufficiently large n

$$\frac{1}{n}(\vartheta_{n,2} - \vartheta_{n,1}) = \frac{1}{n}\sum_{j=1}^{n} g_j(p_j) \geq \frac{1}{n}\sum_{j=1}^{n} E_{P_j}(g_j) - 2\epsilon_k \geq 0.5\Delta - 2\epsilon_k. \quad (11)$$

Now we compute a lower bound of Skeptic's capital. We have from the definition (2)-(3) $\mathcal{Q}_n^{1,k} = \prod_{j=1}^{n}(1 - \epsilon_k \xi(p_j > 0.5)(\omega_j - p_j))$, and $\mathcal{Q}_n^{2.k} = \prod_{j=1}^{n}(1 + \epsilon_k \xi(p_j \leq 0.5)(\omega_j - p_j))$. By these inequalities, $0 \leq \mathcal{Q}_n^{i,k} \leq 2^n$ for all n and for $i = 1, 2$, no matter how the other players move. Also at step n, $\ln \mathcal{Q}_n^{1,k} \geq -\epsilon_k \vartheta_{n,1} - \epsilon_k^2 n$ and $\ln \mathcal{Q}_n^{2,k} \geq \epsilon_k \vartheta_{n,2} - \epsilon_k^2 n$. These inequalities and (11) imply

$$\limsup_{n\to\infty} \frac{\ln \mathcal{Q}_n^1 + \ln \mathcal{Q}_n^2}{n} \geq 0.5\epsilon_k\Delta - 2\epsilon_k^2 \geq 2\epsilon_k^2 \quad (12)$$

for all sufficiently large n, where $\epsilon_k \leq \frac{1}{8}\Delta$. From this, we obtain

$$\limsup_{n\to\infty} \frac{\ln \mathcal{Q}_n^{i,k}}{n} \geq \epsilon_k^2$$

for $i = 1$ or for $i = 2$, and for all sufficiently large n.

Hence, we obtain for the total capital of Skeptic $\mathcal{K}^n = \mathcal{Q}^n + \mathcal{F}^n$

$$\limsup_{n\to\infty} \mathcal{K}_n = \infty$$

no matter how Forecaster moves if Realty uses her strategy defined above.

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The M/D/1—EPS QUEUE REVISITED

S.F.Yashkov*

Abstract

Starting from previous results of studying the M/G/1 queue with egalitarian processor sharing (EPS), we consider a special case: the M/D/1—EPS, and show how to obtain the (conditional) sojourn time distribution for this special case from more general results. Some new properties of such queues are discovered. We also establish some connections between the M/D/1—EPS queue and some known results from geometrical probability and uniform spacings.

Keywords: processor sharing, queues, Laplace transforms, sojourn time distribution, branching processes, covering a circle

1 Introduction

One of the most interesting service disciplines in queueing theory is that of egalitarian processor sharing (EPS): when $n > 0$ jobs are present in the system, then every job is being served with rate $1/n$. In other words, all these jobs simultaneously receive $1/n$ times the rate of service which a solitary job in the processor (server) would receive. Jumps of the service rate occur at the instants of arrivals and departures from the system. Therefore, the rate of service received by a specific job fluctuates with time and, importantly, its sojourn time depends not only on the jobs in the server at its time of arrival there, but also on subsequent arrivals shorter of which can overtake a specific job. This makes the EPS queue intrinsically harder to analyse than, say, the classical First Come—First Served (FCFS) queue or many other classical disciplines. The system works in steady state.

EPS queue was introduced by Kleinrock [1] in 1964 and has been the subject of much research over the past 40+ years. In this model one of the main measures of performance is a (tagged) job's sojourn time distribution, conditioned on that job's service time (job's size). The (stationary) sojourn time is the time the tagged job leaves the system after being served, assuming the job arrives at time zero.

We denote by $V(u)$ the conditional sojourn time, with u being the service time. If the tagged job arrived to an empty system and no further arrivals occurred in the time interval $[0, u]$, then $V(u) = u$. But in general $V(u) > u$ as the tagged job must share the capacity of the server. We denote by $\beta(u)$ the service time density, by $v(x|u)$ the conditional sojourn time density, and by $v(x) = \int_0^x v(x|u)\beta(u)\,du$ the unconditional sojourn time density. In general, $v(x|u)$ has a probability mass along $x = u$, but $v(x)$ is generally continuous function.

The M/M/1—EPS queue assumes Poisson arrivals and i.i.d. service times with density $\beta(u) = \mu e^{-\mu u}$. An expression for $\mathbb{E}[e^{-sV(u)}]$ (that is, for the Laplace transform (LT) of $v(x|u)$) is known since 1970 (see, e.g., Kleinrock's book [2, Eq. (4.19)] (1976)).

*Institute for Information Transmission Problems of RAS (Kharkevich Institute), 19, Bol'shoi Karetny, GSP-4, Moscow, 127994, Russia.

A more difficult model is the M/G/1—EPS queue, where the service density is general. This was independently analysed by the author in [3] (1981), [4] (1983) and by Schassberger [5] (1984) by means of completely different new analytical methods (in particular, the papers [3, 4] use the view of the EPS queue as a branching process.) These authors obtained an explicit, albeit complicated, expression for $\mathbb{E}[\mathrm{e}^{-sV(u)}]$. Inverting the LT leads to an expression for $v(x|u)$ as a contour integral (see Theorem 1), but the integrand is a nonlinear function of another contour integral, which is in turn defined in terms of the LT of the service density.

In this paper we will give a brief derivation of the Laplace–Stieltjes transform (LST) of the conditional sojourn time distribution in the M/D/1—EPS queue with deterministic service density. This was derived by the author (see [6, p. 73]) and more recently in [7, 8]. However, in most cases authors use arguments that are specific to the case $G = D$. But these results also follow easily from the general M/G/1—EPS model. This note shows how such special results can be obtained from the general results. We also give some insight to the properties of the main ingredient of sojourn time in the M/D/1—EPS queue, which are related to well–known problems from geometrical probability and uniform spacings.

2 Preliminaries

In this section we give a short representation about the main results of the determination of the stationary sojourn time distribution (in terms of double Laplace transforms) for the M/G/1—EPS queue.

Let jobs arrive to the single server according to a Poisson process $N = \{N(t) : t \geq 0\}$ with the rate $\lambda > 0$. Their sizes are i.i.d. random variables with a general distribution function $B(u) = \mathbb{P}(B \leq u)$, $(B(0) = 0,\ B(\infty) = 1)$ with the mean $\beta_1 < \infty$ and the LST $\beta(s) \stackrel{\triangle}{=} \int_{0-}^{+\infty} \mathrm{e}^{-su}\, dB(u)$. [1] We assume that $\rho = \lambda\beta_1 < 1$.

We recall that $V(u)$ denotes the conditional sojourn time of a job of the size u upon its arrival. The LST of $V(u)$ is defined by $v(s,u) \stackrel{\triangle}{=} \mathbb{E}[\mathrm{e}^{-sV(u)}]$ for $\mathrm{Re}\, s \geq 0$ and $u \geq 0$.

Let $\pi(s)$ be the LST of the busy period distribution. In other words, it is the positive root of the well–known Kendall–Takács functional equation

$$\pi(s) = \beta(s + \lambda - \lambda\pi(s)) \tag{2.1}$$

with the smallest absolutely value.

To obtain the LST $v(s,u) \stackrel{\triangle}{=} \int_0^{\infty} \mathrm{e}^{-sx}\, d\mathbb{P}(V(u) \leq x)$ the following (non–trivial) decomposition of the random variable $V(u)$ was carried out. We tagged some (virtual) job of the length u and examined the process of the accumulation of its attained service time. We assume that this tagged job enters into the EPS system at time $t = 0$ under the condition that it meets at its arrival time $n \geq 0$ other jobs (the ancestors) in the system with the remaining sizes which lie in infinitesimal neighbourhood of the points $x_1, \ldots, x_n$ (that is, the EPS system is in the state $(n; x_1, \ldots, x_n)$ if $n > 0$ or the system is empty if $n = 0$). Then the sojourn time of the tagged virtual job is decomposed as:

$$V_n(u|(n; x_1, \ldots, x_n)) \stackrel{d}{=} \sum_{i=1}^{n} \Phi(x_i, u) + D(u). \tag{2.2}$$

[1] We assumed that $B(\cdot)$ has no atom in the origin. For otherwise, the pattern of busy and idle periods is essentially the same as in a queueing process for which arrival rate is reduced to $\lambda[1 - \mathbb{P}(B = 0)]$, and service time has the distribution of B given that $B > 0$.

Here $\Phi(x,u)$ is the sum of increments of attained service time (an age) of a job–ancestor of the initial size x and its direct jobs–descendants for the time interval during which the remaining size of other ancestor (say, the tagged job) is reduced by u. This random variable may be also considered as some (Markovian) functional of the corresponding branching process which describes its total lifetime. However, it is more simple to interpret the random variable $\Phi(x,u)$ as a duration of some *terminating* (sub)busy period initiated by ancestor with the size x. It is terminated at time when the increment of the attained service time of the tagged job reaches the level u. The probabilistic structure of the ingredients of such a busy period is reminiscent of the structure of the components of a standard busy period, but with the important difference that each subsequent component depends on the termination time of a branching process and the size of a descendant. Therefore a subsequent component is *stochastically smaller* than a preceding component (in the sense of some stochastic order relation of type $\overset{1}{\leq}$ for the distribution functions.)

As $u \to \infty$, then the random variable $\Phi(x,u)$ is reduced to the standard busy period with the fixed size x of the job which opens it. The random variable $\Phi(x,u)$ does not depend on x as $x \geq u$. For convenience, the special notation for this case was introduced in Eq. (2.2):

$$D(u) \stackrel{d}{=} \Phi(x,u) \quad \text{for} \quad x \geq u. \tag{2.3}$$

The components of the stochastic equality (2.2) (which were called *delay elements* in [4]) are independent of each other. The independence of these random variables is an another non–trivial fact which was elegantly proved by means of two ways: using an equiprobable random selection mechanism for a distinction of jobs–descendants [4, pp.138–139], and using the *random time change* [6, §2.8].

To find the distributions of the components of the decomposition (2.2), we need to derive and solve some system of differential equations (with initial–boundary conditions). Let $\varphi(s,x,u) \stackrel{\Delta}{=} \mathbb{E}[\mathrm{e}^{-s\Phi(x,u)}]$ and $\delta(s,u) \stackrel{\Delta}{=} \mathbb{E}[\mathrm{e}^{-sD(u)}]$. Then

$$\frac{\partial\varphi(s,x,u)}{\partial x} + \frac{\partial\varphi(s,x,u)}{\partial u} + \left[s + \lambda - \lambda\int_0^\infty \varphi(s,y,u)\,dB(y)\right]\varphi(s,x,u) = 0, \tag{2.4}$$

$$\frac{\partial\delta(s,u)}{\partial u} + \left[s + \lambda - \lambda\int_0^\infty \varphi(s,y,u)\,dB(y)\right]\delta(s,u) = 0, \tag{2.5}$$

$$\delta(s,0) = \varphi(s,0,u) = \varphi(s,x,0) = 1. \tag{2.6}$$

Thus

$$\mathbb{E}[\mathrm{e}^{-sV(u)}|(n;x_1,\ldots,x_n)] = \delta(s,u)\prod_{i=1}^{n}\varphi(s,x_i,u), \quad \text{Re } s > 0. \tag{2.7}$$

From here we obtain after removing the condition on $(n;x_1,\ldots,x_n)$ (that is, after averaging on the stationary distribution density of the Markov process of the number of jobs with the remaining sizes which lie in infinitesimal neighbourhood of the points $x_1,\ldots,x_n$) the following statement.

Theorem 1 *When $\rho < 1$, then*

$$v(s,u) \stackrel{\Delta}{=} \mathbb{E}[\mathrm{e}^{-sV(u)}] = (1-\rho)\delta(s,u)\left[1 - \rho\int_0^\infty \varphi(s,x,u)\frac{(1-B(x))}{\beta_1}dx\right]^{-1}, \tag{2.8}$$

where

$$\varphi(s,x,u) = \begin{cases} \delta(s,u) & \text{for } x \geq u, \\ \delta(s,u)/\delta(s,u-x) & \text{for } x < u, \end{cases} \tag{2.9}$$

and

$$\delta(s,u) = \mathrm{e}^{-u(s+\lambda)}/\psi(s,u), \quad u \geq 0 \tag{2.10}$$

are the solutions of the system of equations (2.4) *and* (2.5) *(together with* (2.6)*). Here* $\psi(s,u)$ *is the LST (with respect to* x*) of some function* $\Psi(x,u)$ *of two variables (possessing the probability density on variable* x*), which, in turn, has a LT with respect to* u *(argument* q*)*

$$\tilde{\psi}(s,q) = \frac{q+s+\lambda\beta(q+s+\lambda)}{(q+s+\lambda)(q+\lambda\beta(q+s+\lambda))} \quad (s \geq 0,\, q > -\lambda\pi(s)). \tag{2.11}$$

Eq. (2.8) is a representation of the random variable $V(u)$ in the form of some geometric random sum. Here we do not consider various subtleties of the proof (all this was described in the works cited). It is worth noting that the function $\tilde{\psi}(s,q)$ in (2.11) is given in the form of the two–dimensional transform of a function $\Psi(x,u)$)

$$\tilde{\psi}(s,q) = \int_0^\infty \int_0^\infty \mathrm{e}^{-sx-qu} d_x\Psi(x,u)du. \tag{2.12}$$

In other words, $\psi(s,u)$ is the Laplace transform inversion operator, namely, $\psi(s,u) = \mathcal{L}^{-1}(\tilde{\psi}(s,q))(s,u)$, that is, the contour Bromvich integral

$$\psi(s,u) = \frac{1}{2\pi i}\int_{-i\infty+0}^{+i\infty+0} \tilde{\psi}(s,q)\mathrm{e}^{qu}\, dq.$$

Remark 1 *Briefly, we have derived the expression for* $\mathbb{E}[\mathrm{e}^{-sV(u)}]$ *by writing the sojourn time as some functional on a branching process (like the processes by Crump–Mode–Jagers, see, for example [9]).Using the structure of the branching process, we found and solved a system of partial differential equations (of the first order) determining the components of a decomposition of* $V(u)$*. It leads to* $\mathbb{E}[\mathrm{e}^{-sV(u)}]$*. Many important details can be found in [6, 10] where the stationary solutions are further extended to the time–dependent cases.*

In some cases, it can be useful the equivalent forms of (2.9). For example,

$$\varphi(s,x,u) = \mathrm{e}^{-(x\wedge u)(s+\lambda)+\lambda\int_0^{x\wedge u}\varphi_B(s,u-y)\,dy},\ x \in [0,\infty), \tag{2.13}$$

where

$$\varphi_B(s,t) \stackrel{\triangle}{=} \int_0^\infty \varphi(s,x,t)\,dB(x) = \int_0^t \mathrm{e}^{-\int_{t-x}^t (s+\lambda-\lambda\varphi_B(s,y))\,dy}\,dB(x) + (1-B(t))\mathrm{e}^{-\int_0^t (s+\lambda-\lambda\varphi_B(s,y))\,dy}. \tag{2.14}$$

The equality (2.14) represents the functional equation that must be satisfied by the function $\varphi_B(s,\cdot)$. The function $\varphi_B(s,t)$ is the LST of the distribution of some non–trivial terminating busy period (it terminates at time t) for the M/G/1—EPS queue. The solution of the equation (2.14) was obtained in terms of the function ψ $(\psi(s,t) \stackrel{\triangle}{=} \exp(-\lambda\int_0^t \varphi_B(s,y)\,dy))$ (more precisely, in terms of the LT for this function, see (2.11)). This also shows that the study of the sojourn time in the M/G/1 queue requires deeper analysis in comparison with an analysis that is expected at first sight.

Remark 2 *It is worth mentioning that the random variable* $D(u)$ *in (2.2) constitutes a "main" ingredient of the sojourn time: it has the distribution of the sojourn time of a job with the size* u *that enters into an empty system. When the system is not empty, the* i*th job (among the jobs which are sharing the capacity of the processor), having remaining length* x_i*, "adds" a delay* $\Phi(x_i,u) = \Phi(x_i \wedge u, u)$ *to the new job's sojourn times.*

Next we consider a special case of the M/G/1—EPS queue in equilibrium: the M/D/1 system with egalitarian processor sharing.

3 The M/D/1—EPS queue

Let us begin from the form of deterministic distribution

$$B(x) = \begin{cases} 0, & 0 \le x < u, \\ 1, & x \ge u. \end{cases}$$

Hence the LST of this distribution has the form $\beta(s) = \exp(-su)$ with the moments $\beta_i = u^i, \quad i = 1, 2, \ldots$. The offered load is equal to $\rho = \lambda u < 1$. In this special case, the distributions of conditioned and unconditioned sojourn times coincide, hence we may use $V = V(u)$ to denote the steady–state sojourn time of a job in the queue M/D/1—EPS.

Corollary 1 *The LST of the stationary distribution of $V(u)$ in the special case M/D/1—EPS has simpler form in comparison with* (2.8)*.*

$$v(s) = v(s,u) = \frac{(1-\rho)(s+\lambda)^2 \mathrm{e}^{-u(s+\lambda)}}{s^2 + \lambda[s + (s+\lambda)(1-\rho)]\mathrm{e}^{-u(s+\lambda)}}. \tag{3.1}$$

In the case considered, the formula (2.10) takes the form

$$\delta(s,u) = \frac{s+\lambda}{\lambda + s\mathrm{e}^{u(s+\lambda)}}. \tag{3.2}$$

Proof. The solution for $v(s,u)$ for the case M/D/1—EPS can be found from Theorem 1 in explicit form. In our case, the equation (2.8) is reduced to the form

$$v(s) = v(s,u) = \frac{(1-\rho)\delta(s,u)}{1 - \lambda\delta(s,u)\int_0^u \frac{dx}{\delta(s,u-x)}} \tag{3.3}$$

where $\delta(s,u)$ is given by (3.2). To obtain (3.2), it is easier to use the equation (3.15) from [4] for the unknown function $\delta(s,u)$ (reflected as (2.29) in [6] or (2.20) in [10]) instead of inverting the function $\tilde{\psi}(s,q)$ that is given by (2.11). (However, such inversion is also possible, see [11, pp. 42–43]. Similar inversion for the case M/M/1—EPS was also executed in [6, p. 74]). Then above equation reduces to the form

$$\frac{\partial\delta(s,x)}{\partial x} + (s+\lambda)\delta(s,x) - \lambda\delta(s,x)^2 = 0 \tag{3.4}$$

with the additional condition $\delta(s,0) = 1$. This is a Bernoulli equation. It is reduced to linear one after division of each term by $\delta(s,x)^2$ and the change of variable $1/\delta(s,x) = u$. The solution of (3.4) is given by (3.2). The final result (3.1) follows after the substitution (3.2) into (3.3). □

Remark 3 *The general expression for the variance of $V(u)$ in the M/G/1—EPS queue (see Eq. (3.20) in [4] or (2.33) in [10]) reduces for the M/D/1—EPS system to the form*

$$\mathrm{Var}[V(u)] = \frac{u^2}{(1-\rho)^2} - \frac{2u^2(\mathrm{e}^\rho - 1 - \rho)}{\rho^2(1-\rho)}.$$

Another way of obtaining $\mathrm{Var}[V(u)]$ in the M/D/1—EPS queue is described in [8]. That approach is also based on the results of [4].

Remark 4 *In addition to [4], we can give the two new interpretation of a random variable $D(u)$ whose LST is given by $\delta(s,u)$ (for the case M/D/1—EPS queue). (See [3], [4] concerning previous interpretation via lifetime of some branching process.) First, it is the sojourn time of the first job that arrives to the empty M/D/1—EPS queue. The explanation of this fact is as follows: until the service requirement of the first job is completed, a number of other jobs may arrive but none leave the system before that time, since under EPS discipline with deterministic service time jobs depart from the system without overtaking, that is, in order of their arrival. Second, the random variable $D(u)$ may be interpreted in terms of the maximal length of the pieces of a stick with length u broken randomly (see, for example, the books [12, 13] for detail).*

Let $L(t)$ be the number of jobs at time t. Then $D^*(t) = N(t) - L(t)$ be the number of departures by time t ($N(t)$ was introduced in the begin of §2), and $D_1^* = \inf\{t:\ D^*(t) = 1\}$ be the time until the first departure from the EPS queue. The following theorems comment Remark 4.

Theorem 2 *Let the M/G/1—EPS queue starts from the state $L(0) = 1$. Then the distribution of D_1^* is given by the LST* (3.2) *for $u = t$ in the case $G = D$, and by*

$$\mathbb{E}[\mathrm{e}^{-sD_1^*}] = \frac{(s+\lambda)\beta(s+\lambda)}{s+\lambda\beta(s+\lambda)} \tag{3.5}$$

in general case.

Proof. Omitted. (We means the proof of (3.5) because the assertion for $G = D$ follows from Remark 4 and definition of $D(u)$ in (2.3).) □

Consider a stick of length u that is randomly broken into n pieces with lengths $S_1, S_2, \ldots$. It is known that the distribution of the largest piece (maximal uniform spacing) $\mathbb{P}(\max_{i=1,\ldots,n} S_i \le x)$ is given by Whitworth's formula[2] [12, p. 31], [13, p. 29]:

$$\mathbb{P}(\max_{i=1,\ldots,n} S_i \le x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(1 - k\frac{x}{u}\right)_+^{n-1}, \tag{3.6}$$

where $x_+ = \max(0, x)$.

It can be shown [14] that

$$\max_{i=1,\ldots,n} S_i \stackrel{d}{=} \sum_{i=1}^{n} \frac{S_i}{i}.$$

Then it holds

Theorem 3 *For the M/D/1—EPS queue as $t \ge 0$,*

$$\mathbb{P}(D(u) \le t) = 1 - \mathrm{e}^{-\lambda t}\mathbf{1}_{\{t \le u\}} + \mathrm{e}^{-\lambda t} \sum_{n=1}^{\infty} \sum_{k=0}^{n+1} \frac{(\lambda t)^n}{n!} \binom{n+1}{k} (-1)^k \left(1 - k\frac{x}{t}\right)_+^n. \tag{3.7}$$

Proof. Omitted. □

[2] Another interpretation of Whitworth's formula is connected with the determination of a probability that the circle is completely covered by the arcs of lengths x that are attached to each random point on a circle of length u (these n points are randomly located on a circumference). It is the so-called Steven's formula [12] in such interpretation.

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