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SHORT COMMUNICATIONS

On Solutions of the Schlesinger Equation in the Neigborhood of the Malgrange Θ-Divisor

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Consider the family

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i(a)}{z - a_i}\right) y \tag{1}$$

of Fuchs systems of p linear differential equations on the Riemann sphere $\overline{\mathbb{C}}$, where the family holomorphically depends on the parameter

$$a = (a_1, \dots, a_n) \in \mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}.$$

The property of being a holomorphic family at infinity is equivalent to the condition

$$\sum_{i=1}^{n} B_i(a) = 0.$$

Denote by $D(a^0)$ a ball of a small radius centered at the point $a^0 = (a_1^0, \dots, a_n^0)$ of the space

$$\mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}.$$

We say that the family (1) is *isomonodromic* if, for all $a \in D(a^0)$, the monodromies

$$\chi_a \colon \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}) \to G = GL(p, \mathbb{C})$$

of the corresponding system are equal to each other. (Under small variations of the parameter a, there exists a canonical isomorphism of the fundamental groups

$$\pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}) \qquad \text{and} \qquad \pi_1(\overline{\mathbb{C}} \setminus \{a_1^0, \dots, a_n^0\})$$

generating the canonical isomorphism

$$\mathrm{Hom}(\pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}), G)/G \cong \mathrm{Hom}(\pi_1(\overline{\mathbb{C}} \setminus \{a_1^0, \dots, a_n^0\}), G)/G$$

of the spaces of classes of the duality representations for these fundamental groups; this allows one to compare χ_a for various $a \in D(a^0)$.)

For example, if the matrix $B_i(a)$ satisfies the *Schlesinger equation*

$$dB_i(a) = -\sum_{j=1,\, j \neq i}^{n} \frac{[B_i(a), B_j(a)]}{a_i - a_j}\, d(a_i - a_j),$$

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then family (1) is isomonodromic (in this case, it is called the *Schlesinger isomonodromic family*). It is well known that, for arbitrary initial conditions $B_i(a^0) = B_i^0$, the Schlesinger equation has a unique solution $\{B_1(a), \dots, B_n(a)\}$ in the ball $D(a^0)$, and the matrices $B_i(a)$ can be extended to the entire universal covering Z of the space

$$\mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}$$

as meromorphic functions (see [1]).

The set $\Theta \subset Z$ of singularities of the extended matrix-valued functions $B_1(a), \dots, B_n(a)$ is called the *Malgrange* Θ*-divisor* (Θ depends on the initial conditions B_i^0). This set is a codimension-one analytical subset of the set Z (otherwise, this set can be empty), i.e., locally in the neighborhood of the point $a^* \in \Theta$, it is defined by the equation $\tau^*(a) = 0$, where $\tau^*(a)$ is a holomorphic function in the neighborhood of the point a^* called a *local* τ *function of the Schlesinger equation.*[1] According to Miwa's theorem [2] (see also [3]), there exists a function $\tau(a)$ holomorphic on the entire space Z whose set of zeros coincides with Θ. In the neighborhood of the point $a^* \in \Theta$, the global τ function differs from the local one by a holomorphic nonzero multiplier and

$$d \ln \tau(a) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1,\, j \neq i}^{n} \frac{\operatorname{tr}(B_i(a) B_j(a))}{a_i - a_j}\, d(a_i - a_j).$$

The differential form

$$\sum_{i=1}^{n} \frac{B_i^0}{z - a_i^0}\, dz$$

of the coefficients of system (1) for $a = a^0$ can be regarded as the connection form ∇^0 in the trivial holomorphic vector bundle F^0 over $\overline{\mathbb{C}}$. According to Malgrange [1], the pair (F^0, ∇^0) can be continued to a bundle F over $\overline{\mathbb{C}} \times Z$ with connection ∇ such that

$$(F, \nabla)|_{\overline{\mathbb{C}} \times \{a^0\}} = (F^0, \nabla^0).$$

In accordance with this interpretation, the *set* Θ *consists of those points* $a^* \in Z$ *for which the bundle* $F|_{\overline{\mathbb{C}} \times \{a^*\}}$ *is holomorphically nontrivial.*

Below we describe the general solution of the Schlesinger equation in the neighborhood of the Θ-divisor, but, first, we briefly describe the method proposed by Bolibrukh for calculating the local τ function (for details, see [3]).

Note that the restriction of the bundle F to $\overline{\mathbb{C}} \times \{a^0\}$ is holomorphically trivial; hence its degree is equal to zero. Since the eigenvalues of the matrix-residues of the connection form are invariant under isomonodromic deformations (see [4]), it follows that the bundle F restricted to $\overline{\mathbb{C}} \times \{a\}$ has zero degree at any point $a \in Z$ (the degree of such a bundle is equal to the sum of all eigenvalues of the matrix residues of the connection form over all singular points).

Consider a point $a^* \in \Theta$ and the auxiliary system

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{{}^*B_i}{z - a_i^*} \right) y,$$

which possesses a Fuchs singularity and the unit monodromy matrix at infinity such that the sum $\sum_{i=1}^{n} {}^*B_i$ is equal to

$$K = \operatorname{diag}(k_1, \dots, k_p), \qquad \text{where} \quad k_1 \leq \dots \leq k_p.$$

[1] Formally, a and a^* are points of the universal covering Z, but since we are studying the local properties of the solution of the Schlesinger equation in the neighborhood of the Θ-divisor, which is biholomorphically equivalent to the ball in $\mathbb{C}^n \setminus \cup_{i \neq j} \{a_i = a_j\}$, then, further, we can regard a and a^* as the coordinates $(a_1, \dots, a_n)$ and $(a_1^*, \dots, a_n^*)$ of these points, respectively.

The matrix K defines the splitting type of the bundle $F|_{\overline{\mathbb{C}}\times\{a^*\}}$ and $\operatorname{tr} K = 0$ (because the degree of this bundle is equal to zero).

By using the existence and uniqueness theorem for the solution of the Schlesinger equation, let us include the previously constructed Fuchs system in the isomonodromic Schlesinger family

$$\frac{dy}{dz} = \biggl(\sum_{i=1}^{n} \frac{{}^*B_i(a)}{z-a_i}\biggr) y, \qquad {}^*B_i(a^*) = {}^*B_i, \qquad \sum_{i=1}^{n} {}^*B_i(a) = K. \tag{2}$$

It turns out that if $K \neq 0$, then, for a certain pair of indices k_l and k_m of diagonal elements of the matrix K such that $k_m - k_l > 1$, the corresponding expression

$$b_1(a) := \sum_{i=1}^{n} {}^*B_i^{lm}(a) a_i$$

differs from zero (where ${}^*B_i^{lm}(a)$ is an element of the matrix ${}^*B_i(a)$ with indices (l,m)). In this case, there exists a gauge transformation of the form

$$y' = \Gamma_1(z,a)y, \qquad \Gamma_1(z,a) = C(a)(I + \mathscr{E}_{ml}),$$

such that

i) $C(a)$ is a holomorphic matrix invertible in $D(a^*)$ and $\mathscr{E}_{ml}$ is a matrix whose elements are equal to zero with the exception of the one with indices m, l which is equal to $(k_l - k_m + 1)z/b_1(a)$;

ii) it maps family (2) to the Fuchs family with residue matrices $B_i'(a)$ satisfying the relation

$$\sum_{i=1}^{n} B_i'(a) = K',$$

where

$$K' = \operatorname{diag}(k_1', \dots, k_p'), \quad k_j' = k_j \ \text{ for } j \neq l, m, \quad k_l' = k_l + 1, \quad k_m' = k_m - 1.$$

(This transformation acts only in the exterior of a certain analytical subset Θ_1 of codimension one which is the set of zeros of the function $b_1(a)$.)

By $|K|$, we denote the number $|K| = \sum_{i=1}^{p}(k_i)^2$. Then, according to the above construction, we have $|K'| \le |K| - 2$. Indeed,

$$|K'| - |K| = (k_l+1)^2 + (k_m-1)^2 - (k_l)^2 - (k_m)^2 = 2(1 + k_l - k_m) \le -2.$$

If $K' \neq 0$ (note that $\operatorname{tr} K' = 0$), then the previously defined procedure can be applied once more. After a finite number of iterations s, we obtain the Fuchs family, with residue matrices $B_i^s(a)$, which is holomorphic at infinity. This final family is gauge equivalent to the input family (1) under a constant gauge transformation, i.e., $B_i^s(a) = S^{-1}B_i(a)S$, $i = 1, \dots, n$, where S is a constant nondegenerate matrix.

Thus, *in the neighborhood of the point a^*, the singular set Θ coincides with the set of zeros of the function*

$$b_1(a) \cdots b_s(a), \tag{3}$$

where $b_j(a)$ appears at the jth step of the Bolibrukh procedure as in the above construction of the function $b_1(a)$.

For definiteness, suppose that $b_1(a), \dots, b_r(a)$ are different functions from the product (3), and $m_1, \dots, m_r$ are their multiplicities, so that $m_1 + \dots + m_r = s$. Then

$$\tau^*(a) = b_1(a) \cdots b_r(a).$$

Consider one step of the Bolibrukh procedure. Since

$$\sum_{i=1}^{n}\frac{B_i'(a)}{z-a_i}=\frac{\partial\Gamma_1}{\partial z}\Gamma_1^{-1}+\Gamma_1\biggl(\sum_{i=1}^{n}\frac{{}^*B_i(a)}{z-a_i}\biggr)\Gamma_1^{-1},$$

the structure of the matrix $\Gamma_1(z,a)$ implies that the matrix $b_1^2(a)B_i'(a)$ is holomorphic in the neighborhood of the point a^*. Hence, by performing all s steps of the procedure, we see that the matrices

$$b_1^{2m_1}(a)\cdots b_r^{2m_r}(a)B_i^s(a)$$

are holomorphic in the neighborhood of the point a^* and, therefore, so are the matrices

$$b_1^{2m_1}(a)\cdots b_r^{2m_r}(a)B_i(a).$$

How can one describe the relationship between the numbers $m_1,\dots,m_r$ and the coefficients $k_1,\dots,k_p$ defining the splitting type of the bundle $F|_{\overline{\mathbb{C}}\times\{a^*\}}$? It is obvious that, in order to transform the matrix K to zero matrix by using the Bolibrukh procedure, it is necessary to decrease sequentially to zero all its positive elements (then all negative elements vanish automatically). Hence the number s of steps of this procedure is equal to the sum of positive elements of the matrix K:

$$2m_1+\dots+2m_r=2s=\sum_{j=1}^{p}|k_j|.$$

Denoting by Δ the maximal value of the differences $k_{j+1}-k_j$, let us estimate the last sum. Suppose that all $k_{l+1},\dots,k_p$ are positive elements of the matrix K, $1\le l\le p-1$. Then

$$\begin{aligned}\sum_{j=1}^{p}|k_j|&\le\sum_{j=1}^{l}(k_{l+1}-k_j)+\sum_{j=l+2}^{p}(k_j-k_l)\\&\le\frac{l(l+1)}{2}\Delta+\frac{(p-l-1)(p-l+2)}{2}\Delta\le\frac{p(p-1)}{2}\Delta.\end{aligned}$$

If the monodromy representation of the isomonodromic Schlesinger family (1) is irreducible, then

$$k_{j+1}-k_j\le n-2,\qquad j=1,\dots,p-1;$$

see [5]. In the general case, these difference can be estimated as follows.

Suppose that the eigenvalues β_i^j of the matrix $B_i(a^0)$ satisfy the condition

$$\mu_i\le\operatorname{Re}\beta_i^j\le M_i,\qquad \mu_i,M_i\in\mathbb{Z},\quad \mu_i<M_i.$$

The proof of the following proposition is analogous to that of Proposition 1 from [6].

Proposition 1. *For the splitting type $(k_1,\dots,k_p)$ of the bundle $F|_{\overline{\mathbb{C}}\times\{a\}}$, $a\in\Theta$, the following inequality holds*:

$$k_{j+1}-k_j\le\sum_{i=1}^{n}(M_i-\mu_i),\qquad j=1,\dots,p-1.$$

Thus, we obtain a specification of the structure of the solution for the Schlesinger equation in the neighborhood of the Θ-divisor.

Theorem 1. *In the neighborhood $D(a^*)$ of the point $a^*\in\Theta$, the matrices $B_i(a)$ satisfying the Schlesinger equation have the following form*:

$$B_i(a)=\frac{H_i(a)}{b_1^{2m_1}(a)\cdots b_r^{2m_r}(a)},$$

where $\Theta\cap D(a^*)=\{b_1(a)\cdots b_r(a)=0\}$, $H_i(a)$ are holomorphic matrices in $D(a^*)$, and

$$2m_1+\dots+2m_r\le\frac{p(p-1)}{2}\sum_{i=1}^{n}(M_i-\mu_i).$$

If the monodromy of the isomonodromic Schlesinger family is irreducible, then

$$2m_1 + \cdots + 2m_r \leq \frac{p(p-1)(n-2)}{2}\,.$$

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