Attractors for Equations of Mathematical Physics

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Abstract. The authors study new problems related to the theory of infinite-dimensional dynamical systems that were intensively developed during the last years. They construct the attractors and study their properties for various non-autonomous equations of mathematical physics: the 2D and 3D Navier–Stokes systems, reaction-diffusion systems, dissipative wave equations, the complex Ginzburg–Landau equation, and others. Since, as it is shown, the attractors usually have infinite dimension, the research is focused on the Kolmogorov ε-entropy of attractors. Upper estimates for the ε-entropy of uniform attractors of non-autonomous equations in terms of ε-entropy of time-dependent coefficients of the equation are proved.

Also, the authors construct attractors for those equations of mathematical physics for which the solution of the corresponding Cauchy problem is not unique or the uniqueness is not known (for example, the 3D Navier–Stokes system). The theory of the trajectory attractors for these equations is developed, which is later used to construct global attractors for equations without uniqueness. The method of trajectory attractors is applied to the study of finite-dimensional approximations of attractors. The perturbation theory for trajectory and global attractors is developed and used in the study of the attractors of equations with terms rapidly oscillating with respect to spatial and time variables. It is shown that the attractors of these equations are contained in a thin neighbourhood of the attractor of the averaged equation.

Dedicated to our wives, Katya and Asya

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Introduction

One of the major problems in the study of evolution equations of mathematical physics is the investigation of the behaviour of the solutions of these equations when time is large or tends to infinity. The related important questions concern the stability of solutions as $t \to +\infty$ or the character of the instability if a solution is unstable. In the last decades considerable progress in this area have been achieved in the study of autonomous evolution partial differential equations. For a number of basic evolution equations of mathematical physics it was shown that the long time behaviour of their solutions is characterized by attractors. Attractors were constructed for the following equations and systems: the two-dimensional Navier–Stokes system, various classes of reaction-diffusion systems, nonlinear dissipative wave equations, complex Ginzburg–Landau equations and many other autonomous equations and systems. Mainly the global attractors of these equations were studied.

An autonomous evolution equation can be written in the following abstract form:

$$\partial_t u = A(u),\ u|_{t=0} = u_0(x). \tag{1}$$

Here $u = u(x,t)$ is the solution of equation (1) and x,t denote the spatial and time variables, respectively. Corresponding to this equation is the *semigroup* of nonlinear operators $\{S(t)\} = \{S(t),\ t \geq 0\}$. The operator $S(t)$ maps the initial data $u_0(x)$ to the solution $u(x,t)$ of the Cauchy problem (1) at the time moment $t:\ S(t)u_0(x) = u(x,t),\ t \geq 0$. We assume that the Cauchy problem has a unique solution. The initial data $u_0(x)$ belongs, for example, to a certain Banach (or metric) space E. The space E is chosen in such a way that $u(x,t)$ belongs to E for all $t \geq 0$. Thus, the operator $S(t)$ maps E into E for all $t \geq 0:\ S(t) : E \to E$. The operators $\{S(t)\}$ satisfy the semigroup properties: $S(t_1)S(t_2) = S(t_1 + t_2)$ for all $t_1, t_2 \geq 0$ and $S(0) = \mathrm{Id}$ is the identity operator.

A set $\mathcal{A}$ from E is said to be a *global attractor* of the equation under consideration or, equivalently, of the corresponding semigroup $\{S(t)\}$ if it has the following properties: **(i)** the set $\mathcal{A}$ is compact in E; **(ii)** $\mathcal{A}$ attracts each bounded (in E) set B: $\mathrm{dist}_E(S(t)B, \mathcal{A}) \to 0$ as $t \to +\infty$; **(iii)** the set $\mathcal{A}$ is strictly invariant with respect to $\{S(t)\}$, that is, $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$. Here $\mathrm{dist}_E(\cdot,\cdot)$ denotes the Hausdorff distance in $E : \mathrm{dist}_E(U,V) = \sup_{u\in U} \inf_{v\in V} \|u - v\|_E$. It follows from the definition of the global attractor that the set $\mathcal{A}$ attracts solutions $u(x,t) = S(t)u_0(x)$ as $t \to +\infty$ uniformly with respect to bounded initial data $u_0(x)$. The global attractor $\mathcal{A}$ is unique if it exists. Thus we can say that the global attractor describes all the possible limits of solutions of equation (1).

It was shown that the Hausdorff and fractal dimension of the global attractors are finite for a number of equations and systems of mathematical physics. The estimates from above and from below for the Hausdorff and fractal dimension of attractors were found. For certain types of equations the structure of the global attractor $\mathcal{A}$ was completely described, for example, in the case where the equation has a global Lyapunov function. All these and other problems are treated in great detail in the books by R.Temam [**156**], A.V.Babin and M.I.Vishik [**9**], J.K.Hale [**82**], O.A.Ladyzhenskaya [**114**] and in books of other authors.

In Part 1 of this book we study autonomous evolution equations of the form (1) and their global attractors. We present the theorems on the existence and structure of global attractors of the basic autonomous equations of mathematical physics. We prove optimal (in some sense) estimates from above and from below for the Hausdorff and fractal dimension of global attractors of these equations. Part 1 contains mainly well-known results from the theory of global attractors of autonomous partial differential equations.

The long-time behaviour of solutions of non-autonomous evolution equations of the form

$$\partial_t u = A(u,t),\ u|_{t=\tau} = u_\tau(x) \tag{2}$$

and their attractors were studied in details by many authors for ordinary differential equations ($u \in \mathbb{R}^N$) and for some classes of operator and partial differential equations. The construction of the *skew product flow* of the *process* (the analog of the semigroup in autonomous case) played the main role in this theory; this allowed one to reduce the problem to the study of an attractor of some semigroup acting in an extended function phase space (see, for instance, R.K.Miller [**134**], G.R.Sell [**143**], R.K.Miller and G.R.Sell [**135**], J.K.Hale [**82**]).

Dealing with evolution partial differential equations and especially with systems arising from mathematical physics it is a good idea to extend the phase space by using only the hull of the time-dependent coefficients of the equation under consideration. From this point of view the research was focused in the last decade on attractors for non-autonomous evolution equations of mathematical physics. It was assumed that external forces, interaction functions, and other coefficients in the equations explicitly depend on time. The dependence on time of these parameters can be periodic, quasiperiodic, or almost periodic. The spaces of these functions were studied in great detail in L.Amerio and G.Prouse [**2**], B.Levitan and V.Zhikov [**117**]. In the present book we also study the equations whose time-depending terms are *translation compact* functions in appropriate function spaces. The latter means, that, say, in the case of the external force $g(x,t)$ depending on time $t \in \mathbb{R}$, that all the translations $\{g(x,t+h),\ h \in \mathbb{R}\}$ form a precompact set in the space $L_2([t_1,t_2];H)$ for every interval $[t_1,t_2] \subset \mathbb{R}$. Here H is a Hilbert (or more general) space corresponding to the physical nature of the function $g(x,t)$. Similarly, the translation compactness was defined for other terms of the equation, for example, for interaction functions of the form $f(u,t)$ and so on.

We denote by $\sigma(t)$ the collection of all time-dependent coefficients of a non-autonomous equation. The equation itself can be rewritten in the form

$$\partial_t u = A_{\sigma(t)}(u),\ u|_{t=\tau} = u_\tau(x). \tag{3}$$

The parameter $\sigma(t)$ is said to be the *time symbol* (or just the *symbol*) of the equation. The values of $\sigma(t)$ belong to a metric or Banach space. For example, $\sigma(t) = (f(u,t), g(x,t))$ if the time-dependent terms of the equation are the interaction function $f(u,t)$ and the external force $g(x,t)$. Dealing with non-autonomous equations it is fruitful to study the entire family of equations (3) with time symbols $\sigma(t)$ belonging to a set Σ called the *symbol space*. A typical symbol space is as follows. We are given a fixed initial time symbol $\sigma_0(t)$ of the equation we want to study. Then we consider the set of all time translations of $\sigma_0(t)$, i.e., the set $\{\sigma_0(t+h),\ h \in \mathbb{R}\}$. Moreover, we add to the symbol space Σ all the functions $\sigma(t)$ that are the limits of the sequences of the form $\{\sigma_0(t+h_n)\}$ as $n \to \infty$. The limits are taken in a suitable function space. The resulting family of functions $\{\sigma(t)\}$ is called the *hull* of $\sigma_0(t)$ and is denoted by $\mathcal{H}(\sigma_0)$. For example, if $\sigma_0(t)$ is an almost periodic function with values in a metric space $\mathcal{M}$, then $\mathcal{H}(\sigma_0)$ is the hull of σ_0 in the space $C_b(\mathbb{R}; \mathcal{M})$. We now set $\Sigma = \mathcal{H}(\sigma_0)$ and study the family of equations (3) with symbols $\sigma \in \mathcal{H}(\sigma_0)$.

We start from the fact that the properly defined attractor $\mathcal{A}$ of the initial equation with symbol $\sigma_0(t)$ must simultaneously be the attractor of each equation (3) with symbol $\sigma(t) \in \mathcal{H}(\sigma_0)$ and, moreover, it must be the attractor of the entire family of these equations. This observation leads to the concept of the *uniform (with respect to $\sigma \in \Sigma$) global attractor* $\mathcal{A}_\Sigma$ of the family of equations (3) with symbols $\sigma \in \Sigma$.

The initial data $u_\tau(x)$ for (3) is taken in the Banach space E. We assume that the Cauchy problem (3) is uniquely solvable for every $u_\tau \in E$ and for all $\tau \in \mathbb{R}$. Corresponding to equation (3) is the *process* $\{U_\sigma(t,\tau)\} = \{U_\sigma(t,\tau) \mid t,\tau \in \mathbb{R}, t \geq \tau\}$ acting in the space E. Similarly to the autonomous equation (1) the operator $U_\sigma(t,\tau)$ maps the initial data $u_\tau(x) \in E$ to the solution $u(t,x)$ of the Cauchy problem (3) at the time moment $t:\ U_\sigma(t,\tau)u_\tau(x) = u(x,t),\ t \geq \tau, \tau \in \mathbb{R}$. We assume that $u(x,t)$ belongs to E for all $t \geq \tau$. Thus, the operators $U_\sigma(t,\tau)$ map E into E for all $t \geq \tau, \tau \in \mathbb{R}:\ U_\sigma(t,\tau): E \to E$. The notion of process is a generalization of the notion of semigroup generated by an autonomous evolution equation. The properties $U(t,s)U(s,\tau) = U(t,\tau)$ for all $t \geq s \geq \tau, \tau \in \mathbb{R}$ and $U(\tau,\tau) = \mathrm{Id}$ for all $\tau \in \mathbb{R}$ are the characteristic properties of a process.

We study the uniform attractor of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ corresponding to the family of equations (3) with symbols $\sigma \in \Sigma$. A set $\mathcal{A}_\Sigma$ from E is said to be a *uniform global attractor* of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if it has the following properties: **(i)** the set $\mathcal{A}_\Sigma$ is compact in E; **(ii)** $\mathcal{A}_\Sigma$ attracts any bounded (in E) set $B = \{u_\tau(x)\}$ of initial data uniformly with respect to $\sigma \in \Sigma$: $\sup_{\sigma\in\Sigma} \mathrm{dist}_E(U_\sigma(t,\tau)B, \mathcal{A}_\Sigma) \to 0$ as $t \to +\infty$ for every $\tau \in \mathbb{R}$; **(iii)** $\mathcal{A}_\Sigma$ is the *minimal* set satisfying **(i)** and **(ii)**, that is, if a set $\mathcal{A}_1$ is compact in E and attracts any bounded set B uniformly with respect to $\sigma \in \Sigma$, then $\mathcal{A}_\Sigma \subseteq \mathcal{A}_1$. The notion of uniform global attractor of a family of processes generalizes the notion of global attractor of a semigroup. The invariance property is replaced by the property of minimality.

In Part 2 of the book we study uniform attractors of basic non-autonomous evolution equations of mathematical physics whose autonomous analogues were treated in Part 1. The analysis of time symbols of these equations and systems is the key element in the theory of non-autonomous partial differential equations. The proposed method is quite simple and allows us to construct uniform attractors, to study their structure, and to estimate some important quantities related to attractors, such as the Hausdorff and fractal dimension and the Kolmogorov ε-entropy.

To describe the structure of uniform attractors we introduce the notion of kernel of an equation or a process. The kernel $\mathcal{K}_\sigma$ of equation (3) is the collection of all bounded (in E) solutions $u(t), t \in \mathbb{R}$ of the equation that are defined on the entire time axis $\mathbb{R}$. The set $\mathcal{K}(t) = \{u(t) \mid u \in \mathcal{K}\} \subseteq E$ is called the kernel section at the time moment $t \in \mathbb{R}$. We prove the following identity for the uniform (w.r.t. $\sigma \in \Sigma$) attractor of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ corresponding to problem (3):

$$\mathcal{A}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0). \tag{4}$$

Clearly, the right-hand side of (4) does not change if we replace 0 by an arbitrary time moment τ.

We construct uniform attractors for the non-autonomous two-dimensional Navier–Stokes system, non-autonomous reaction-diffusion systems, non-autonomous dissipative wave equations, non-autonomous Ginzburg–Landau equations and for other equations and systems. For each equation or system we describe in detail the function space to which the time symbol $\sigma_0(t)$ of this equation belongs. We present the conditions that provide the translation compactness of the symbol $\sigma_0(t)$ or, more precisely, the translation compactness of its components. We prove that the corresponding Cauchy problems have unique solutions in suitable function spaces. Using the property of dissipativity (specific to each problem) we establish the existence of uniformly (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) absorbing or attracting set for the corresponding family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$. We apply and develop various known methods for the investigation of various partial differential equations. We derive the corresponding a priori estimates for solutions $u(x,t)$ of these non-autonomous equations and systems. We also prove the necessary continuity properties of the processes. Then the general theorem implies the existence of a uniform attractor $\mathcal{A}_{\mathcal{H}(\sigma_0)}$ of a non-autonomous equation. In particular, identity (4) holds, that is, the global attractor is the union of all values of all bounded (in E) global solutions of all equations (3) with time symbols $\sigma \in \mathcal{H}(\sigma_0)$.

The notion of time symbol of a non-autonomous equation is also important in the study of the dimension of uniform attractors of the above equations and systems of mathematical physics. Using this approach we prove upper estimates for the Hausdorff and fractal dimension of uniform attractors of these systems. In a number of cases we are able to find lower estimates for the dimension of uniform attractors. For example, we prove such upper and lower estimates for the fractal dimension of the uniform attractor $\mathcal{A}_\Sigma$ of the 2D Navier–Stokes system with quasiperiodic (in time t) external force $g_0(x,t) = G(x, \alpha_1 t, \alpha_2 t, \dots, \alpha_k t)$. Here $G(x, \omega_1, \omega_2, \dots, \omega_k)$ is a function that is 2π-periodic in each variable $\omega_i \in \mathbb{R}$. The symbol space $\Sigma = \mathcal{H}(g_0)$ is diffeomorphic to the k-dimensional torus. We prove that the fractal dimension of the uniform attractor $\mathcal{A}_\Sigma$ of this system does not exceed the sum of two terms: the Grashof number Gr (the known parameter describing the number of "degrees of freedom" of a flow) and the number k of rationally independent frequencies of the external force $g(x,t)$. The number k is the dimension of the symbol space. Thus for $k = 0$ (autonomous case) we obtain the known estimate for the fractal dimension of the global attractor of the autonomous 2D Navier–Stokes system. Examples show that the fractal dimension of $\mathcal{A}_\Sigma$ can be greater than k. These facts reflect the importance of the number k in the estimates of the dimension of uniform attractors for the Navier–Stokes system. Moreover, if $g_0(x,t)$ is a general almost periodic function in time t, then examples show that the fractal dimension of the uniform attractor can be infinite. Similar facts are proved for other non-autonomous equations of mathematical physics having quasiperiodic or almost periodic time symbols.

In the cases where the fractal dimension of uniform attractors is equal to infinity it is natural to study other characteristics and quantities of uniform attractors of non-autonomous equations. The famous work A.N.Kolmogorov and V.M.Tikhomirov [**106**] is devoted to the systematic study of the ε-entropy of compact sets in various function spaces. Notice that the uniform attractor $\mathcal{A}_\Sigma$ is a compact set in E. Then it is reasonable to investigate the Kolmogorov ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A}_\Sigma)$ of the uniform attractor. It is well known that the number $\mathbf{H}_\varepsilon(\mathcal{A}_\Sigma)$ is equal to $\log_2 N_\varepsilon(\mathcal{A}_\Sigma)$, where $N_\varepsilon(\mathcal{A}_\Sigma)$ is the minimal number of balls in E with radius ε covering the set $\mathcal{A}_\Sigma$. Since $\mathcal{A}_\Sigma$ is compact, $\mathbf{H}_\varepsilon(\mathcal{A}_\Sigma)$ is finite for every $\varepsilon > 0$. The problem arises to study the rate of growth of the ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A}_\Sigma)$ as $\varepsilon \to 0+$. In the book we find upper estimates for the Kolmogorov ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A}_\Sigma)$ of uniform attractors of non-autonomous evolution equations with translation compact symbols $\sigma_0(t)$ in the corresponding spaces. These estimates are optimal in some sense and generalize the well-known estimates for the fractal dimension of the corresponding autonomous equations and systems considered in Part 1 of the book. In particular the ε-entropy of the uniform (w.r.t. $g \in \Sigma = \mathcal{H}(\sigma_0)$) attractor $\mathcal{A}_\Sigma$ of the 2D Navier–Stokes system does not exceed the sum of two terms: the first term is the Grashof number Gr multiplied by $\log_2\left(\frac{1}{\varepsilon}\right)$ and the second is the ε-entropy $\mathbf{H}_\varepsilon(\mathcal{H}(g_0))$ of the hull of the external force $g_0(x,t)$ measured on the finite time interval $[0,l]$, where $l = O\left(\log_2\left(\frac{1}{\varepsilon}\right)\right)$ (in the quasiperiodic case this term has the form $k \cdot \log_2\left(\frac{1}{\varepsilon}\right)$, where k is the number of rationally independent frequencies of $g_0(x,t)$). In particular, the functional dimension of the uniform attractor does not exceed the functional dimension of the hull $\mathcal{H}(g_0)$. We prove similar results for other non-autonomous equations of mathematical physics. In particular the estimates for the ε-entropy of the uniform attractors imply the estimates for the fractal dimension of the uniform attractors if the symbols of the equations are quasiperiodic functions.

In Part 3 of the book we study attractors of equations of mathematical physics for which the solution of the corresponding Cauchy problem exists on any time interval but, maybe, is not unique or the uniqueness theorem is not proved yet. The classical example is the 3D Navier–Stokes system. It is known from the works of J.Leray and E.Hopf that the Cauchy problem for this system has a weak solution on an arbitrary time interval, but it is not known whether this weak solution is unique. Another famous example is the wave equation with nonlinear interaction term of fast polynomial growth. This hyperbolic equation appears in many branches of modern physics, for example, in relativistic quantum mechanics. The existence of a weak solution (in the sense of distributions) of the Cauchy problem for this equation is known, whereas the uniqueness theorem is proved only for a moderate growth of the interaction function (see J.-L.Lions [**124**]). Even though the complex Ginzburg–Landau equations play a central role in the theory of amplitude equations, the global existence and uniqueness of solutions are not established for all values of the dispersion parameters. For all these equations and systems the theory of global attractors of semigroups descried in Parts 1 and 2 of this book is not applicable. To overcome this difficulty we develop the theory of so-called *trajectory attractors* which enables us to study the limiting behaviour of solutions of equations of mathematical physics without uniqueness. Moreover, it is also possible to construct generalized global attractors for such equations using the trajectory attractors. In particular, this theory covers all the above problems of mathematical physics.

Let us briefly explain the idea of the construction of a trajectory attractor using as an example the 3D Navier–Stokes system

$$\partial_t u + \nu L u + P(u, \nabla)u = Pg(x),\ (\nabla, u) = 0,\ u|_{\partial\Omega} = 0,\ t \geq 0, \tag{5}$$

where $x = (x_1, x_2, x_3) \in \Omega \Subset \mathbb{R}^3,\ u = u(x,t) = (u^1, u^2, u^3)$. Here L is the Stokes operator, $g(x) = (g^1, g^2, g^3)$ is the external force, $\nu > 0$ is the viscosity coefficient, and P denotes the orthogonal projection onto the space H of divergence free vector fields with finite L_2-norm. We study weak solutions $u(x,t), t \geq 0$ of system (5) that satisfy the known energy inequality (see J.-L.Lions [**124**] and Chapter XII of the present book). Notice that all the weak solutions resulting from the Galerkin approximation method always satisfy this energy inequality. Therefore the stock of such weak solution is reasonably large. The collection of all these solutions is denoted by $\mathcal{K}^+$.

The traditional theory of global attractors uses the set of initial data $\{u_0(x)\}$ of the Cauchy problem (1) as the phase space E on which the corresponding semigroup $\{S(t)\}$ acts. Now the phase space corresponding to system (5) is the set $\mathcal{K}^+ = \{u(x,t), t \geq 0\}$ of weak solutions defined on the entire time semiaxis $\mathbb{R}_+$. The elements of the phase space are functions depending on time. The set $\mathcal{K}^+$ is called the *trajectory space* of the 3D Navier–Stokes system and the elements of $\mathcal{K}^+$ are called *trajectories*. We consider the translation operators $\{T(h), h \geq 0\}$ acting on $\mathcal{K}^+$ by the formula $T(h)u(x,t) = u(x, t+h)$. The translation $T(h), h \geq 0$ maps any function $u(x,t), t \geq 0$ onto the shifted function $u(x, t+h), t \geq 0$. It follows from the definition of $\mathcal{K}^+$ that $u(x, t+h) \in \mathcal{K}^+$ if $u(x,t) \in \mathcal{K}^+$. It is clear that the translations $\{T(h)\} = \{T(h), h \geq 0\}$ form a semigroup acting on $\mathcal{K}^+$: $T(h) : \mathcal{K}^+ \to \mathcal{K}^+$ for $h \geq 0$. We study the global attractor of the translation semigroup $\{T(h)\}$ on $\mathcal{K}^+$.

In the trajectory space $\mathcal{K}^+$ we consider a weak convergence topology (see Chapter XII). It follows easily that the space $\mathcal{K}^+$ is closed in this topology and the translation semigroup $\{T(h)\}$ is continuous in $\mathcal{K}^+$. We define bounded sets in $\mathcal{K}^+$ and prove the existence of a bounded absorbing set B_0 of the semigroup $\{T(h)\}$ in $\mathcal{K}^+$, that is, for any bounded set $B \subset \mathcal{K}^+$ there exists $h_1 = h_1(B) > 0$ such that $T(h)B \subset B_0$ for all $h \geq h_1$. Since the set B_0 is bounded, it is compact in the weak topology of the space $\mathcal{K}^+$. From this facts it follows that the semigroup $\{T(h)\}$ has a global attractor $\mathfrak{A} \subset \mathcal{K}^+$, that is, $\mathfrak{A}$ is compact in the weak topology, strictly invariant with respect to $\{T(h)\} : T(h)\mathfrak{A} = \mathfrak{A}$ for all $h \geq 0$, and for every bounded set B of trajectories from $\mathcal{K}^+$ the set $T(h)B$ tends to $\mathfrak{A}$ in the weak topology as $h \to +\infty$. The set $\mathfrak{A}$ is called the *trajectory attractor* of the 3D Navier–Stokes system (5).

Notice that the weak topology in $\mathcal{K}^+$ is stronger than the local strong convergence topology of the spaces $L_2^{loc}(\mathbb{R}_+; H^{1-\delta})$ and $C^{loc}(\mathbb{R}_+; H^{-\delta})$, where $0 < \delta \leq 1$. Therefore for any bounded set B from $\mathcal{K}^+$ and for every $M > 0$

$$\operatorname{dist}_{L_2(0,M;H^{1-\delta})}(T(h)B, \mathfrak{A}) \to 0, \tag{6}$$

$$\operatorname{dist}_{C([0,M];H^{-\delta})}(T(h)B, \mathfrak{A}) \to 0 \text{ as } h \to \infty. \tag{7}$$

From (7) we deduce that the set $\mathcal{A} = \mathfrak{A}|_{t=0} \subset H$ is the *global attractor* of the 3D Navier–Stokes system (5). More precisely, $\mathcal{A}$ is bounded and closed in H and satisfies the following attracting property: the restriction $B|_t$ at time t of any bounded set of solutions $B \subset \mathcal{K}^+$ tends to $\mathcal{A}$ as $t \to \infty$ in the space $H^{-\delta}$:

$$\operatorname{dist}_{H^{-\delta}}(B|_t, \mathcal{A}) \to 0\ (t \to \infty),\ 0 < \delta \leq 1. \tag{8}$$

Moreover, $\mathcal{A}$ is the minimal closed (in H) set that satisfies (8). Thus, the set $\mathcal{A}$ has all the properties known for the global attractor of the semigroup corresponding to the Cauchy problem for which the uniqueness theorem holds (for example, the 2D Navier–Stokes system).

Using this scheme we construct trajectory attractors and global attractors for other autonomous equations and systems of mathematical physics of the form (1) for which the uniqueness theorem of the Cauchy problem is not proved or does not hold. For example, we construct trajectory attractors and global attractors and study their properties for the dissipative wave equation with arbitrary polynomial growth of the interaction function.

Notice that in a number of cases it is also reasonable to study trajectory attractors for the equations for which the uniqueness theorem holds. In this case the trajectory attractor $\mathfrak{A}$ consists of all trajectories $u(t), t \geq 0$, that lie on the usual global attractor $\mathcal{A}$:

$$\mathfrak{A} = \{u(t) = S(t)u_0, t \geq 0 \mid u_0 \in \mathcal{A}\}$$

and $\mathfrak{A}$ attracts bounded sets of trajectories in a stronger topology.

The methods of trajectory attractors is also fruitful in the theory of perturbation of attractors and in the study of attractors of equations containing rapidly oscillating terms. For example, we prove that the trajectory attractor $\mathfrak{A}_\varepsilon$ of the wave equation

$$\varepsilon\partial_t^2 u + \gamma\partial_t u = \Delta u - f(u) + g(x)$$

depending on a positive small parameter ε converges as $\varepsilon \to 0+$ in the corresponding space to the trajectory attractor $\mathfrak{A}_0$ of the limiting parabolic equation

$$\gamma\partial_t u = \Delta u - f(u) + g(x).$$

Here $f(u)$ is a function with arbitrary polynomial growth with respect to u. Since for the limiting parabolic equation the uniqueness theorem holds, it has the usual global attractor $\mathcal{A}_0$ and the trajectory attractor $\mathfrak{A}_0$ consists of all solutions $u(t)$ of this equation lying on $\mathcal{A}_0$ for all $t \geq 0$. Besides this case of a singular perturbation we consider other problems of the theory of perturbation of partial differential equations. These results reflect the following general property of the trajectory attractors of equations of mathematical physics: the trajectory attractors of perturbed equations depend upper semicontinuously on the perturbation parameters.

Similarly to autonomous equations we study *uniform trajectory attractors* and *global attractors* for non-autonomous equations of mathematical physics of the form (3) with terms depending on time. We assume that the time symbols $\sigma(t)$ are translation compact in the corresponding spaces. To begin with we consider the equations for which the existence of the Cauchy problem is not proved or does not hold. We construct the uniform trajectory attractor for the non-autonomous 3D Navier–Stokes system with translation compact (in time t) external force $g = g(x, t)$. We also study the dissipative hyperbolic equation containing the interaction function $f(u, t)$ with arbitrary polynomial growth with respect to u. We also consider other non-autonomous equations of mathematical physics. Separately we study the trajectory attractors for non-autonomous equations with uniqueness. This leads to stronger attraction of trajectories to the uniform trajectory attractor.

The trajectory attractors also satisfy the following important property known in the theory of global attractors. For example, we study the 3D Navier–Stokes system (5). We consider the corresponding Galerkin approximation system of order m, that is, the system of ordinary differential equations in m-dimensional Euclidean space. Using the above scheme we construct the trajectory attractor $\mathfrak{A}^{(m)}$ of this system. Recall that $\mathfrak{A}^{(m)}$ consists of all solutions $u_m(x,t), t \geq 0$ of the Galerkin system that lie on the global attractor (in $\mathbb{R}^m$) $\mathcal{A}^{(m)}$ of this system. We prove that $\mathfrak{A}^{(m)}$ converges to $\mathfrak{A}$ in the weak topology as $m \to +\infty$. In particular, $\text{dist}_{H^{-\delta}}\left(\mathcal{A}^{(m)}, \mathcal{A}\right) \to 0 \ (m \to \infty),\ 0 < \delta \leq 1$. Here $\mathfrak{A}$ and $\mathcal{A}$ are the trajectory attractor and the global attractor of the Navier–Stokes system (5), respectively. This property of upper semicontinuity of attractors holds for all equation and systems considered in this book. No matter whether the corresponding uniqueness theorem holds or not.

We investigate the attractors of evolution equations with terms that oscillate rapidly with respect to the spatial or time variable. The parameter $\varepsilon^{-1}, \varepsilon > 0$ characterizes the oscillation frequency. We assume that rapidly oscillating terms and coefficients have averages in a weak sense as $\varepsilon \to 0+$ in the corresponding function spaces. The equation with averaged terms and coefficients is called the *averaged equation.* We prove that the trajectory attractor $\mathfrak{A}_\varepsilon$ of the equation with rapidly oscillating terms converges as $\varepsilon \to 0+$ to the trajectory attractor $\overline{\mathfrak{A}}$ of the averaged equation in a suitable weak sense. Moreover, the global attractors $\mathcal{A}_\varepsilon$ of the original equations with rapidly oscillating terms converge as $\varepsilon \to 0+$ to the global attractor $\overline{\mathcal{A}}$ of the averaged equation as $\varepsilon \to 0+$ in the corresponding function space. We apply these results to the 3D and 2D Navier–Stokes systems with external force of the form $g\left(x, \frac{x}{\varepsilon}\right)$ (or $g\left(x, t, \frac{t}{\varepsilon}\right)$). We assume that the function $g\left(x, \frac{x}{\varepsilon}\right)$ has the average $\bar{g}\,(x)$ as $\varepsilon \to 0+$, for example, in the space H_w. (The space H_w is the space H endowed with the weak topology). Then the trajectory attractors $\mathfrak{A}_\varepsilon$ converge to $\overline{\mathfrak{A}}$ in the following sense: for every $M > 0$

$$\text{dist}_{L_2(0,M;H^{1-\delta})}\left(\mathfrak{A}_\varepsilon, \overline{\mathfrak{A}}\right) \to 0, \tag{9}$$

$$\text{dist}_{C([0,M];H^{-\delta})}(\mathfrak{A}_\varepsilon, \overline{\mathfrak{A}}) \to 0 \ (\varepsilon \to 0+),\ 0 < \delta \leq 1. \tag{10}$$

For the corresponding global attractors $\mathcal{A}_\varepsilon$ and $\overline{\mathcal{A}}$ we have the following relation:

$$\text{dist}_{H^{-\delta}}\left(\mathcal{A}_\varepsilon, \overline{\mathcal{A}}\right) \to 0 \ (\varepsilon \to 0+). \tag{11}$$

We prove similar results for the reaction-diffusion systems and for the dissipative hyperbolic equations with rapidly oscillating terms. If the corresponding Cauchy problem is uniquely solvable, then we prove that relations (9)-(11) hold in more regular space with stronger topology. For example, for the 2D Navier–Stokes system we have that

$$\text{dist}_{H^{1-\delta}}(\mathcal{A}_\varepsilon, \overline{\mathcal{A}}) \to 0 \text{ as } \varepsilon \to 0+\,.$$

For perturbed potential reaction-diffusion systems with rapidly oscillating terms it was recently shown that the distance between the global attractors $\mathcal{A}_\varepsilon$ and $\overline{\mathcal{A}}$ is at most $C\varepsilon^\gamma$, where $\gamma > 0$ (see B.Fiedler and M.I.Vishik [**69**]).

We now describe the content of the book in the order of chapters. The book is divided into three parts and consists of eighteen chapters and two appendices.

Part 1 is devoted to the study of autonomous evolution equations of mathematical physics for which the uniqueness theorems of the corresponding Cauchy problems hold. It has three chapters. For reader's convenience we begin the book with Chapter I that contains main facts from the theory of global attractors of autonomous finite-dimensional dynamical systems. We illustrate the theory with many examples from ordinary differential equations, for example, the well-known Lorenz system.

In Chapter II we present the definitions of the main function spaces we use in the book. We formulate the embedding theorems and prove the necessary differential and integral inequalities that will be used in the next chapters. We present some important theorems concerning the functions with values in Banach spaces. The detailed description of these questions can be found in the books R.Temam [**156**], A.V.Babin and M.I.Vishik [**9**], and many others. Chapter II also deals with the theory of semigroups corresponding to autonomous dissipative evolution equations acting in Banach or metric spaces. We introduce the concept of global attractor of a semigroup. We formulate the theorem on the existence of a global attractor. The chapter also contains the study of the basic autonomous partial differential equations arising from mathematical physics. We consider reaction-diffusion systems of different types, the 2D Navier–Stokes system, and the dissipative hyperbolic equation. For each system we briefly describe the function setting. Then we formulate the theorem on the existence and the uniqueness of the corresponding Cauchy problems. We verify the dissipativity conditions for all these equations which make it possible to apply the general theory and to prove the existence of global attractors for these equations and systems.

Chapter III contains the review of the results concerning the dimension of global attractors of autonomous evolution equations. We prove the upper estimates for the Hausdorff and fractal dimension of attractors. We use the known technique developed in the works of many authors (see A.Douady and J.Oesterle [**51**], P.Constantin, C.Foias, and R.Temam [**43**], R.Temam [**156**], A.V.Babin and M.I.Vishik [**4**]) the key element of which is the investigation of the Lyapunov exponents of the corresponding variational equations. Our main purpose is to prove that under quite general hypotheses the upper estimates for the fractal dimension of global attractors coincides with upper estimates of the Hausdorff dimension of these attractors. We also consider lower bounds for the dimension of global attractors. These lower bounds show that the upper bounds are optimal in some sense. We apply these results to the equations and systems of mathematical physics considered in Chapter II.

Part 2 of the book deals with non-autonomous evolution equations and their attractors. We assume that the corresponding Cauchy problems have unique solutions. Part 2 includes the chapters from IV to IX.

Chapter IV contains a systematic study of the general non-autonomous equations and the corresponding processes. We introduce the concept of time symbol of the equation and we study families of processes with symbols belonging to some symbol space. We consider the main examples of such families of processes when a symbol space is a hull of a given initial symbol which is a translation compact function in an appropriate topological space. We define a uniform global attractor of a family of processes and we prove theorems on the existence and the structure of uniform attractors. We consider the relations between different definitions of uniform and non-uniform attractors.

In Chapter V we study the properties of translation compact functions in various function spaces. We begin the chapter with the important notion of almost periodic function, which is an example of translation compact function in the space $C_b(\mathbb{R};\mathcal{M})$ with uniform (in $\mathbb{R}$) convergence topology. Here $\mathcal{M}$ is a complete metric or Banach space. Then we consider more general translation compact functions in the spaces $C(\mathbb{R};\mathcal{M})$, $L_p^{loc}(\mathbb{R};\mathcal{M})$, and $L_{p,w}^{loc}(\mathbb{R};\mathcal{M})$ with local convergence topologies on any closed interval $[t_1,t_2]\subset\mathbb{R}$. We prove the translation compactness criteria in these spaces. This technical chapter can be omitted in the first reading.

In Chapter VI we apply the results of Chapters IV and V to the basic non-autonomous evolution equations of mathematical physics we want to study. The corresponding autonomous equations were considered in Chapter II. The following PDEs are treated: the 2D Navier–Stokes system with time-dependent external force, the reaction-diffusion systems with interaction function and external forces depending on time, the non-autonomous Ginzburg–Landau equation, and the non-autonomous dissipative hyperbolic equation with time-dependent terms. For each equation we study the following questions: the time symbol of the equation and the translation compactness criterion for it in a physically relevant space; the existence and uniqueness of the solution of the Cauchy problem for this equation; the corresponding family of processes and its properties; the existence of compact uniformly attracting or absorbing sets for the family of processes, the kernel of the equation; and finally, the existence of the uniform global attractor and its structure in terms of kernel sections.

Chapter VII deals with the so-called semiprocesses and their attractors. A semiprocess corresponds to a non-autonomous equation with time symbol defined on the semiaxis $\mathbb{R}_+$. Such equations describe dynamical systems for which the past history is unknown. We study the behaviour of solutions as $t\to+\infty$. Similarly to processes we study the family of semiprocesses and their uniform attractors. We prove the theorem on the existence of uniform attractors. If the symbol of the equation satisfies the backward uniqueness property, then we prove that there exists the unique family of processes whose uniform attractor coincides with the uniform attractor of the original family of semiprocesses. This reduction is important in applications especially if the corresponding family of processes has a simple symbol space, for example, a periodic function with its shifts or a hull of a quasiperiodic function diffeomorphic to a torus. Usually this reduction also leads to the simplification of the structure of the uniform attractor of the original equation. As an application, we study non-autonomous equations with asymptotically almost periodic symbols and so-called cascade systems with symbols generated by an autonomous evolution equation.

In Chapter VIII we study kernels of non-autonomous evolution equations. The notion of a kernel is important because kernels are used in the description of the general structure of the uniform global attractors of autonomous and non-autonomous equations. In this chapter we establish certain weak invariance and attracting properties of the kernel of a given process. Notice that, unlike the previous chapters, we do not assume that the symbol of the equation is a translation compact function in the corresponding space. We also study the fractal dimension of a kernel section. We prove that the fractal dimension of kernel sections is finite and has the uniform upper bound similar to the corresponding autonomous case. These results are applied to non-autonomous partial differential equations studied in Chapter IV but without the hypothesis that their symbols are translation compact functions.

In Chapter IX we prove upper estimates for the Kolmogorov ε-entropy of uniform attractors of non-autonomous evolution equations with translation compact symbols. This theory generalizes the results of Chapter III on the upper estimates for the fractal dimension of global attractors of autonomous evolution equations. We apply these estimates to various non-autonomous partial differential equations with translation compact terms. The estimates so obtained depend explicitly on the parameters of the equations. We also consider the important case where the time symbol of the equation is a quasiperiodic function. In Chapter IX we also consider some classes of symbol spaces with infinite fractal dimension and deduce upper bounds for their ε-entropy.

In Part 3 of the book, consisting of Chapters X–XVIII, we study trajectory attractors of autonomous and non-autonomous evolution equations of mathematical physics. We now do not assume that the corresponding Cauchy problem has a unique solution.

In Chapter X we explain the method of trajectory attractors using autonomous ordinary differential equations. We define trajectory spaces and trajectory attractors for these equations. We prove the theorem on the existence of a trajectory attractor.

In Chapter XI we prove the theorem on the existence of a global attractor of an abstract semigroup acting in a general Hausdorff topological space. Usually in the literature results of this type are proved for complete metric spaces. However, in application we deal with translation semigroups acting in various topological spaces with local weak topology which is not metrizable. That is the reason why we include in the book this abstract topological result. In the beginning of the chapter we recall the main general notions, definitions, and results from the theory of topological spaces. This material is useful for a better understanding of the next chapters.

Chapter XII presents the theory of trajectory attractors for abstract autonomous evolution equations. It is not necessary to assume any longer the unique solvability of the Cauchy problems for these equations. We explain the properties of trajectory spaces of these equations that we need for the construction of a trajectory attractor. We prove the theorem on the existence of a trajectory attractor. We define a kernel for an equation and describe the structure of the trajectory attractors in terms of kernels. Moreover, we define a global attractor for an autonomous equation without the assumption that the solution of the corresponding Cauchy problem is unique. The trajectory attractor is useful in the construction of global attractors of such equations.

In Chapter XIII we study trajectory attractors and global attractors for the following autonomous evolution equations of mathematical physics: the 3D Navier–Stokes system and the dissipative hyperbolic equation with nonlinear term of high polynomial growth. For these equations the uniqueness theorem is not proved yet. Nevertheless, it is possible to study the limiting behaviour of their solutions using trajectory attractors and generalized global attractors. We also study the perturbation of trajectory and global attractors for these equations containing small parameters.

Chapter XIV is devoted to the construction of a uniform trajectory attractor for a non-autonomous evolution equation written in an abstract operator form. Here we assume that the corresponding Cauchy problem has a solution which is not necessarily unique. Similarly to Chapter IV we consider the entire family of non-autonomous equations with symbols from the hull of a given initial symbol. Then we define the united trajectory space and study the global attractor of the translation semigroup acting on the united trajectory space. This attractor is called the uniform trajectory attractor of the initial non-autonomous equation. Using the uniform trajectory attractor we also construct a uniform global attractor of this non-autonomous equation which is a generalization of the notion of a uniform attractor considered in Chapter IV. We also consider uniform trajectory attractors for non-autonomous equations with symbol defined on the semiaxis $\mathbb{R}_+$. This material is an extension and generalization of Chapter VII.

The results of Chapter XIV are applied to non-autonomous partial differential equations in Chapter XV. We study the non-autonomous 2D and 3D Navier–Stokes systems, reaction-diffusion equations, the Ginzburg–Landau equation, and dissipative hyperbolic equations. For each particular equation or system we define the symbol space and the trajectory space. Then we prove the theorem on the existence of a uniform trajectory and global attractors. We study both the equations without uniqueness and the equations with unique solvability of the Cauchy problem. The latter equations admit the attraction to the trajectory and global attractors in stronger topologies.

In Chapter XVI we study the approximation of trajectory and global attractors by trajectory and global attractors of the corresponding Galerkin systems. We prove that the trajectory and global attractors of Galerkin systems converge to the trajectory and global attractors of the original PDE in the corresponding spaces.

In Chapter XVII we briefly consider the theory of perturbation of trajectory attractors and global attractors for non-autonomous partial differential equations. The corresponding theory for autonomous equations was studied in Chapter XIII in greater detail.

The last Chapter XVIII deals with averaging of attractors of evolution equations with rapidly oscillating terms. We study functions that oscillate rapidly with respect to spatial or time variables. We consider the basic autonomous and non-autonomous evolution equations of mathematical physics from the previous chapters. For each equation with oscillating coefficients we define an averaged equation. We prove that the trajectory and global attractors of initial equations converge to the trajectory and global attractors of the averaged equations respectively in the corresponding spaces.

The book ends with two Appendices providing the proofs of some general theorem formulated in different chapters of the book.

Notice that the book [**89**] by A.Haraux contains chapters that are devoted to the study of attractors of processes generated by non-autonomous partial differential equations with almost periodic in time coefficients and terms. This book played a stimulated role for the authors in the study of non-autonomous evolution equations. The paper G.R.Sell [**145**] is devoted to the construction of an attractor of the 3D Navier–Stokes system. Other papers and books devoted to the study of attractors of autonomous and non-autonomous equations of mathematical physics are cited in the text of the book and in Bibliography.

In this book the authors have tried to treat systematically some questions related to the theory of attractors of autonomous and non-autonomous evolution equations of mathematical physics. The authors' interest for the subject of this book was stimulated by seminars at the Moscow State University and the Institute for Information Transmission Problems (Russian Academy of Sciences).

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