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On Movable Singularities of Garnier Systems

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Abstract—We study movable singularities of Garnier systems by using an approach based on the relationship between these systems and Schlesinger isomonodromic deformations of Fuchsian systems, as well as Lauricella hypergeometric equations.

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1. INTRODUCTION

First, consider Painlevé equation VI (Eq. P_{VI}), that is,

$$\frac{d^2u}{dt^2} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\left(\frac{du}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)\frac{du}{dt} + \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{u^2} + \gamma\frac{t-1}{(u-1)^2} + \delta\frac{t(t-1)}{(u-t)^2}\right), \tag{1}$$

which is an ordinary differential equation of second order for a complex function $u(t)$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are constants.

In many cases, instead of the use of the explicit expression in the study of Eq. P_{VI}, it is convenient to regard this equation as

- an equation for an additional (fifth) singularity of the isomonodromic family of scalar Fuchsian equations of second order with the four singular points t, 0, 1, and ∞;
- the most general ordinary differential equation of second order with the Painlevé property;
- an equation describing the Schlesinger isomonodromic deformations of two-dimensional Fuchsian systems with the four singular points t, 0, 1, and ∞.

Let us recall the first two approaches in greater detail (the last one is described in Sec. 3).

The monodromy of the linear differential equation

$$\frac{d^pu}{dz^p} + b_1(z)\frac{d^{p-1}u}{dz^{p-1}} + \cdots + b_p(z)u = 0 \tag{2}$$

of order p with singularities $a_1, \dots, a_n \in \overline{\mathbb{C}}$ (the poles of the coefficients) is defined as follows. In a neighborhood of a nonsingular point z_0, we consider a basis $(u_1, \dots, u_p)$ in the solution space of (2). Analytic continuations of the functions $u_1(z), \dots, u_p(z)$ along an arbitrary loop γ starting at the point z_0 and lying in $\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$ transform the basis $(u_1, \dots, u_p)$ into (generally, some other) basis $(\widetilde{u}_1, \dots, \widetilde{u}_p)$. These two bases are related by some passage matrix G_γ corresponding to the loop γ:

$$(u_1, \dots, u_p) = (\widetilde{u}_1, \dots, \widetilde{u}_p)G_\gamma.$$

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The mapping $[\gamma] \mapsto G_\gamma$ (which depends only on the homotopic class $[\gamma]$ of the loop γ) specifies a representation

$$\chi\colon \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}, z_0) \to \mathrm{GL}(p, \mathbb{C})$$

of the fundamental group of the space $\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$ to the space of nondegenerate complex matrices of dimension p. This representation is called the *monodromy* of Eq. (2). The passage to another basis implies the adjointness of all matrices G_γ to the same matrix, i.e., specifies an equivalent representation.

A singular point a_i of Eq. (2) is said to be *regular* if any solution of the equation has polynomial growth (with respect to $1/|z-a_i|$) in a neighborhood of the point a_i. Linear differential equations having only regular singularities are said to be *Fuchsian*.

Poincaré [1] showed that the number of parameters defining a Fuchsian equation of order p with n singular points is less than the dimension of the space of equivalence classes for the representations χ if $p > 2$ and $n > 2$ or $p = 2$ and $n > 3$ (see also [2, p. 73]). Therefore, in the construction of a Fuchsian equation with given singularities and monodromy, the so-called *additional* singular points appear, at which the coefficients of the equation have poles, but local solutions are unique meromorphic functions. In the case $p = 2$, $n = 4$ ($\{a_1, a_2, a_3, a_4\} = \{t, 0, 1, \infty\}$), there is only one such singularity. If the position of the singular point $z = t$ is slightly changed, so that the monodromy of the equation is preserved (this property, called *isomonodromy*, is rigorously defined in the next section), then the position of the additional (fifth) singularity $w(t)$ will also change so as to satisfy Eq. $\mathrm{P_{VI}}$ (this was first discovered by Fuchs [3]).

Equation (1) has three fixed singular points 0, 1, and ∞. Its movable singularities (which depend on the initial data) can only be poles. In this case, the equation is said to have the *Painlevé property*. In other words, any local solution of Eq. $\mathrm{P_{VI}}$ given in a neighborhood of the point $t_0 \neq 0, 1, \infty$ can be continued to a meromorphic function on the universal covering of the space $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$. The following statement on the movable poles of Eq. (1) is well known: in the case $\alpha \neq 0$, they can only be simple and, in the case $\alpha = 0$, they are of at most second order (see, for example, [4, Chap. VI, Sec. 6]).

Generalizing Fuchs' study to the case of $n+3$ singularities $a_1, \dots, a_n, 0, 1, \infty$, Garnier [5] obtained a system $\mathscr{G}_n(\theta)$ depending on $n+3$ complex parameters $\theta_1, \dots, \theta_{n+2}, \theta_\infty$. This is an integrable system of partial differential equations of second order. Subsequently, it was written by Okamoto [6] in the equivalent Hamiltonian form

$$\frac{\partial u_i}{\partial a_j} = \frac{\partial H_j}{\partial v_i}, \quad \frac{\partial v_i}{\partial a_j} = -\frac{\partial H_j}{\partial u_i}, \qquad i, j = 1, \dots, n, \tag{3}$$

with Hamiltonians $H_j = H_j(a, u, v, \theta)$ depending rationally on

$$a = (a_1, \dots, a_n), \quad u = (u_1, \dots, u_n), \quad v = (v_1, \dots, v_n), \quad \theta = (\theta_1, \dots, \theta_{n+2}, \theta_\infty).$$

Here $u_1(a), \dots, u_n(a)$ are additional singularities of an isomonodromic family of Fuchsian equations of second order depending on the position of the singular points $a_1, \dots, a_n$. In the case $n = 1$, the Garnier system $\mathscr{G}_1(\theta_1, \theta_2, \theta_3, \theta_\infty)$ is an equivalent (Hamiltonian) form of Eq. $\mathrm{P_{VI}}$ (see (1)), where

$$\alpha = \frac{1}{2}\theta_\infty^2, \qquad \beta = -\frac{1}{2}\theta_2^2, \qquad \gamma = \frac{1}{2}\theta_3^2, \qquad \delta = \frac{1}{2}(1-\theta_1^2).$$

For $n > 1$, the Garnier system does not, generally, have the Painlevé property. However, by Garnier's theorem, the elementary symmetric polynomials $\sigma_i(u_1(a), \dots, u_n(a))$ in the local solutions of the Garnier system can be continued to meromorphic functions F_i on the universal covering Z' of the space

$$(\mathbb{C} \setminus \{0, 1\})^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}.$$

The supplement to this theorem proposed in the present paper contains estimates for the orders of the irreducible components of the polar sets of the functions F_i (Theorem 3 and Proposition 2).

2. ISOMONODROMIC DEFORMATIONS OF FUCHSIAN SYSTEMS

Let us embed the Fuchsian system

$$\frac{dy}{dz} = \biggl(\sum_{i=1}^{n} \frac{B_i^0}{z - a_i^0}\biggr) y, \qquad B_i^0 \in \operatorname{Mat}(p, \mathbb{C}), \quad \sum_{i=1}^{n} B_i^0 = 0, \tag{4}$$

of p equations with singular points $a_1^0, \dots, a_n^0$ in the family

$$\frac{dy}{dz} = \biggl(\sum_{i=1}^{n} \frac{B_i(a)}{z - a_i}\biggr) y, \qquad B_i(a^0) = B_i^0, \quad \sum_{i=1}^{n} B_i(a) = 0, \tag{5}$$

of Fuchsian systems depending holomorphically on the parameter $a = (a_1, \dots, a_n) \in D(a^0)$, where $D(a^0)$ is a disk of small radius centered at the point $a^0 = (a_1^0, \dots, a_n^0)$ of the space $\mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}$.

A family (5) is said to be *isomonodromic* (or is an *isomonodromic deformation* of system (4)) if, for all $a \in D(a^0)$, the representations of the monodromy

$$\chi \colon \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}) \to \operatorname{GL}(p, \mathbb{C})$$

of the corresponding systems are identical.[1] This implies that, for each value of a, there exists a fundamental matrix $Y(z, a)$ of the corresponding system (5), which has the same monodromy for all $a \in D(a^0)$. Such a matrix $Y(z, a)$ is called an *isomonodromic fundamental matrix*. For an isomonodromic deformation, not only the monodromy of the family (5), but also the eigenvalues of the matrix residues $B_i(a)$ are constant (see [2, Lecture 15]).

Can system (4) be always embedded in the isomonodromic family of Fuchsian systems? The answer to this question is positive. For example, if the matrices $B_i(a)$ satisfy the *Schlesinger equation* [7]

$$dB_i(a) = -\sum_{j=1,\, j \neq i}^{n} \frac{[B_i(a), B_j(a)]}{a_i - a_j}\, d(a_i - a_j), \qquad i = 1, \dots, n,$$

then the family (5) is isomonodromic (in this case, it is called an *Schlesinger isomonodromic family*).

By Malgrange's theorem [8], for arbitrary initial conditions $B_i(a^0) = B_i^0$, the Schlesinger equation has a unique solution $\{B_1(a), \dots, B_n(a)\}$ in a disk $D(a^0)$; here the matrices $B_i(a)$ are extended to the universal covering Z of the space $\mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}$ as meromorphic functions. Thus, the Schlesinger equation satisfies the Painlevé property.

Let us recall that a function f is *m*eromorphic on Z if it is holomorphic on $Z \setminus P$, cannot be continued holomorphically to P, and is expressed by the ratio $f(a) = \varphi(a)/\psi(a)$ of holomorphic functions in a neighborhood of each point $a^0 \in P$ (therefore, $\psi(a^0) = 0$) where $P \subset Z$ is an analytic subset of codimension one (locally, it is given by the equation $\psi(a) = 0$ and is called the *polar set* of a meromorphic function f). Points of this set are either *poles* (where the function φ does not vanish) or *indeterminate points* (where $\varphi = 0$).

Let us also recall the notion of a divisor of a meromorphic function. Let $A = N \cup P$ denote the union of the set N of zeros of the function f and its polar set P. Any regular point a^0 of the set A can belong to only one irreducible component of the sets N or P. Thus, the *order* of this component is defined as a power (with the sign "$+$" if $a^0 \in N$ and with the sign "$-$" if $a^0 \in P$) of the corresponding factor in the expansion of the function φ or ψ into irreducible multipliers. By a *divisor* of a meromorphic function f we mean a pair (A, κ), where $\kappa = \kappa(a)$ is an entire function on the set of regular points of the set A; the function $\kappa(a)$ takes a constant value on each of its irreducible components, which is equal to the order

[1] For small changes of the parameter a, there exist canonical isomorphisms of the fundamental groups $\pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\})$ and $\pi_1(\overline{\mathbb{C}} \setminus \{a_1^0, \dots, a_n^0\})$ inducing the canonical isomorphisms

$$\operatorname{Hom}(\pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}), \operatorname{GL}(p, \mathbb{C}))/\operatorname{GL}(p, \mathbb{C}) \cong \operatorname{Hom}(\pi_1(\overline{\mathbb{C}} \setminus \{a_1^0, \dots, a_n^0\}), \operatorname{GL}(p, \mathbb{C}))/\operatorname{GL}(p, \mathbb{C})$$

of the spaces of equivalence classes for the representations of these fundamental groups; this allows us to compare χ for different parameters $a \in D(a^0)$.

of the component. The pair (P, κ) is called the *polar divisor* of the function f. By $(f)_\infty$ we denote the restriction of κ to the set of regular points of the set P.

Let us return to the Schlesinger equation. The polar set $\Theta \subset Z$ of extended matrix functions $B_1(a), \dots, B_n(a)$ is called the *Malgrange Θ-divisor*. If system (4) is considered as the equation of horizontal sections of the logarithmic connection ∇_0 (with singularities $a_1^0, \dots, a_n^0$) in a holomorphically trivial vector bundle E_0 of rank p over $\overline{\mathbb{C}}$, then the set Θ corresponds to points a^* for which the bundle E_{a^*} corresponding to the parameter a^* in the isomonodromic deformation $(E_a, \nabla_a)_{a\in Z}$ of the pair (E_0, ∇_0) is not holomorphically trivial (for more details, see [8] and [2, Lecture 16]). Note that the degree of the bundle E_a coincides with the sum of the eigenvalues of matrix residues of connection ∇_a and is zero for all a (this sum is independent of a and is zero for $a = a^0$).

In what follows, we shall use Bolibruch's theorem [9], [10] (the proof is also contained in [11]), which describes the behavior of the general solution of the Schlesinger equation near the Θ-divisor in the case $p = 2$. For the polar set $P \subset Z$ of the function f meromorphic on Z and $a^* \in P$, we denote by $\Sigma_{a^*}(f)$ the sum of the orders of all irreducible components of the set $P \cap D(a^*)$.

Theorem 1. *Suppose that the monodromy of the two-dimensional ($p = 2$) Schlesinger isomonodromic family (5) is irreducible, a^* is an arbitrary point of the Θ-divisor, and $E_{a^*} \cong \mathscr{O}(k) \oplus \mathscr{O}(-k)$. Then*

$$\Sigma_{a^*}(B_i) \geq -2k, \qquad i = 1, \dots, n.$$

Remark 1. It is well known (see, for example, [2, Theorem 11.1]) that $2k \leq n - 2$. Thus, the estimate from Theorem 1 can be rewritten as $\Sigma_{a^*}(B_i) \geq 2 - n$ or, in the case of an odd n, as $\Sigma_{a^*}(B_i) \geq 3 - n$.

In what follows, we shall need the following auxiliary lemma.

Lemma 1. *Consider the two-dimensional Schlesinger isomonodromic family*

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i(a)}{z - a_i}\right) y, \qquad \sum_{i=1}^{n} B_i(a) = K = \operatorname{diag}(\theta, -\theta), \quad \theta \in \mathbb{C},$$

and the function

$$b(a) = \sum_{i=1}^{n} b_i^{12}(a) a_i,$$

where the $b_i^{12}(a)$ are the upper right elements of the matrices $B_i(a)$, respectively. Then the differential of the function $b(a)$ is given by

$$db(a) = (2\theta + 1) \sum_{i=1}^{n} b_i^{12}(a)\, da_i.$$

Proof. The differential $db(a)$ is of the form

$$db(a) = \sum_{i=1}^{n} a_i db_i^{12}(a) + \sum_{i=1}^{n} b_i^{12}(a)\, da_i.$$

In order to find the first of the last two summands, we use the Schlesinger equation for the matrices $B_i(a)$, obtaining

$$\begin{aligned} \sum_{i=1}^{n} a_i\, dB_i(a) &= -\sum_{i=1}^{n} \sum_{j=1,\, j\neq i}^{n} a_i \frac{[B_i(a), B_j(a)]}{a_i - a_j}\, d(a_i - a_j) \\ &= -\sum_{i=1}^{n} \sum_{j>i}^{n} [B_i(a), B_j(a)]\, d(a_i - a_j) \end{aligned}$$

$$= -\sum_{i=1}^{n}\Big[B_i(a), \sum_{j=1,\, j\neq i}^{n} B_j(a)\Big]\, da_i = -\sum_{i=1}^{n}[B_i(a), K]\, da_i.$$

The upper right element of the resulting matrix differential 1-form is

$$\sum_{i=1}^{n} 2\theta b_i^{12}(a)\, da_i;$$

therefore,

$$\sum_{i=1}^{n} a_i db_i^{12}(a) = 2\theta \sum_{i=1}^{n} b_i^{12}(a)\, da_i,$$

and hence

$$db(a) = (2\theta+1)\sum_{i=1}^{n} b_i^{12}(a)\, da_i. \quad \square$$

3. ISOMONODROMIC SCHLESINGER DEFORMATIONS AND GARNIER SYSTEMS

Let us recall the relationship between Schlesinger isomonodromic deformations and Garnier systems.

Consider the two-dimensional Schlesinger isomonodromic family

$$\frac{dy}{dz} = \bigg(\sum_{i=1}^{n+2} \frac{B_i(a)}{z-a_i}\bigg)y, \qquad B_i(a^0) = B_i^0 \in \mathrm{sl}(2,\mathbb{C}), \tag{6}$$

of Fuchsian systems with singular points

$$a_1,\dots,a_n, \quad a_{n+1}=0, \quad a_{n+2}=1, \quad a_{n+3}=\infty,$$

which depends holomorphically on the parameter $a=(a_1,\dots,a_n)\in D(a^0)$, where $D(a^0)$ is a disk of small radius centered at the point a^0 of the space

$$(\mathbb{C}\setminus\{0,1\})^n \setminus \bigcup_{i\neq j}\{a_i=a_j\}.$$

Let $\pm\beta_i$ denote the eigenvalues of the matrices $B_i(a)$, respectively (recall that an isomonodromic deformation preserves the eigenvalues of the matrix residues $B_i(a)$). It follows from the Schlesinger equation that the residue at infinity is a constant matrix. Suppose that this matrix is diagonalizable, i.e.,

$$\sum_{i=1}^{n+2} B_i(a) = -B_\infty = \mathrm{diag}(-\beta_\infty, \beta_\infty).$$

By Malgrange's theorem, the matrix functions

$$B_i(a) = \begin{pmatrix} b_i^{11}(a) & b_i(a) \\ b_i^{21}(a) & b_i^{22}(a)\end{pmatrix}$$

can be continued to the universal covering Z' of the space $(\mathbb{C}\setminus\{0,1\})^n \setminus \bigcup_{i\neq j}\{a_i=a_j\}$ as meromorphic functions (holomorphic outside an analytic subset Θ of codimension one).

Since the upper right element $-\sum_{i=1}^{n+2} b_i(a)$ of the matrix B_∞ is zero, the function

$$P_n(z,a) = (z-a_1)\cdots(z-a_{n+2})\sum_{i=1}^{n+2}\frac{b_i(a)}{z-a_i} \tag{7}$$

is a polynomial of degree n in z for each fixed a. Let $u_1(a),\dots,u_n(a)$ denote the roots of this polynomial and consider the functions $v_1(a),\dots,v_n(a)$ defined by

$$v_j(a)=\sum_{i=1}^{n+2}\frac{b_i^{11}(a)+\beta_i}{u_j(a)-a_i},\qquad j=1,\dots,n.$$

Then the following statement is valid (see [6, the proof Proposition 3.1] or [12, Corollary 6.2.2, p. 207]).

Theorem 2. *The pair* $(u(a),v(a))=(u_1,\dots,u_n,v_1,\dots,v_n)$ *satisfies the Garnier system* (3) *with parameters* $2\beta_1,\dots,2\beta_{n+2},2\beta_\infty-1$.

Let us express the coefficients of the polynomial $P_n(z,a)$ via the upper right elements $b_i(a)$ of the matrices $B_i(a)$. Consider the elementary symmetric polynomials

$$\sigma_1(a)=\sum_{i=1}^{n+2}a_i,\quad \sigma_2(a)=\sum_{1\le i<j\le n+2}a_ia_j,\quad \dots,\quad \sigma_{n+1}(a)=a_1\cdots a_n$$

in $a_1,\dots,a_n$, $a_{n+1}=0$, $a_{n+2}=1$ and the polynomial

$$Q(z)=\prod_{i=1}^{n+2}(z-a_i).$$

We have

$$P_n(z,a)=\sum_{i=1}^{n+2}b_i(a)\frac{Q(z)}{z-a_i}=:b(a)z^n+f_1(a)z^{n-1}+\dots+f_n(a)$$

(recall that $\sum_{i=1}^{n+2}b_i(a)=0$). By the Viète theorem, we have

$$b(a)=\sum_{i=1}^{n+2}b_i(a)(-\sigma_1(a)+a_i)=\sum_{i=1}^{n+2}b_i(a)a_i=\sum_{i=1}^{n}b_i(a)a_i+b_{n+2}(a),$$

$$f_1(a)=\sum_{i=1}^{n+2}b_i(a)\biggl(\sigma_2(a)-\sum_{j=1,\,j\ne i}^{n+2}a_ia_j\biggr)=-\sum_{1\le i<j\le n+2}(b_i(a)+b_j(a))a_ia_j.$$

Similarly,

$$f_k(a)=(-1)^k\sum_{1\le i_1<\dots<i_{k+1}\le n+2}(b_{i_1}(a)+\dots+b_{i_{k+1}}(a))a_{i_1}\cdots a_{i_{k+1}}$$

for each $k=1,\dots,n$.

Along with the formulas for the passage from a two-dimensional Schlesinger isomonodromic family with $\mathrm{sl}(2,\mathbb{C})$-residues to the Garnier system, there exist formulas for the inverse passage (see [6, Proposition 3.2]). This allows us to propose a supplement to Garnier's theorem (asserting that the elementary symmetric polynomials $F_i(a)=\sigma_i(u_1(a),\dots,u_n(a))$ in the solutions of the Garnier system are meromorphic on Z').

By the *linear monodromy* of the solution of the Garnier system we mean the monodromy of the corresponding two-dimensional Schlesinger isomonodromic family.

Theorem 3. *Suppose that* $(u(a),v(a))$ *is the solution of the Garnier system* (3) *whose linear monodromy is irreducible and* Δ_i *denotes the polar set of the function* F_i, $i=1,\dots,n$. *Then*

(a) *if* $\theta_\infty=0$ *and* $u_i(a)\not\equiv u_j(a)$ *for* $i\ne j$, *then* $\Sigma_{a^*}(F_i)\ge -n-1$ *for all* $a^*\in\Delta_i$;

(b) *if* $\theta_\infty\ne 0$, *then* $\Sigma_{a^*}(F_i)\ge -n$ *for all* $a^*\in\Delta_i$ *with the exception, perhaps, of a subset* $\Delta^0\subset\Delta_i$ *of positive codimension* (*in any case,* $(F_i)_\infty\ge -n$).

Proof. Consider the family (6) with irreducible monodromy that corresponds to the given solution and the functions $b(a)$, $f_1(a), \dots, f_n(a)$ constructed from the matrix residues $B_i(a)$. By the Viète theorem, we have $F_i(a) = (-1)^i f_i(a)/b(a)$; by Theorem 1 and Remark 1, $\Sigma_{a^*}(f_i) \ge -n-1$ for each function f_i and any point a^* belonging to the Θ-divisor of the family (6).

By Lemma 1, we have

$$db(a) = -\theta_\infty \sum_{i=1}^{n} b_i(a)\, da_i,$$

where $\theta_\infty = 2\beta_\infty - 1$.

(a) If $\theta_\infty = 0$, then $db(a) \equiv 0$ for all $a \in Z'$; therefore, $b(a) \equiv \text{const} \ne 0$. Indeed, if $b(a) \equiv 0$, then $P_n(z,a)$ is a polynomial of degree $n-1$ in z and $u_i(a) \equiv u_j(a)$ for some $i \ne j$, but this contradicts the assumptions of the theorem. Thus,

$$\Sigma_{a^*}(F_i) = \Sigma_{a^*}(f_i) \ge -n-1$$

in the first case.

(b) If $\theta_\infty \ne 0$, then

$$\begin{gathered} b_i(a) = -\frac{1}{\theta_\infty}\frac{\partial b(a)}{\partial a_i}, \qquad i = 1, \dots, n, \\ b_{n+2}(a) = b(a) - \sum_{i=1}^{n} b_i(a) a_i, \qquad b_{n+1}(a) = -b_{n+2}(a) - \sum_{i=1}^{n} b_i(a). \end{gathered} \tag{8}$$

Therefore, if the function b is holomorphic at the point $a' \in Z'$, then so are the functions b_i, $i = 1, \dots, n+2$, and hence the functions f_i. Therefore, the points $a^* \in \Delta_i$ can be of two types: $b(a^*) = 0$ (and hence $\Sigma_{a^*}(F_i) \ge -1$, because the function b is irreducible[2]) and those belonging to the polar set $\Delta \subset \Theta$ of the function b.

Let $\Delta^0 \subset \Delta$ denote the set of indeterminate points of the function b. Then, in a neighborhood of any point $a^* \in \Delta \setminus \Delta^0$, b can be expressed as

$$b(a) = \frac{h(a)}{\tau_1^{j_1}(a) \cdots \tau_r^{j_r}(a)}, \qquad j_1 \ge 1, \quad \dots, \quad j_r \ge 1, \tag{9}$$

where the τ_l, $l = 1, \dots, r$, h are functions holomorphic near a^*, $h(a^*) \ne 0$, and the τ_l are irreducible at the point a^*, while

$$f_i(a) = \frac{g_i(a)}{\tau_1^{k_1}(a) \cdots \tau_r^{k_r}(a)}, \qquad k_1 + \dots + k_r \le n+1, \tag{10}$$

where the g_i, $i = 1, \dots, n$, are functions holomorphic near a^*. Thus,

$$\frac{f_i(a)}{b(a)} = \frac{g_i(a)}{\tau_1^{k_1}(a) \cdots \tau_r^{k_r}(a)} \Big/ \frac{h(a)}{\tau_1^{j_1}(a) \cdots \tau_r^{j_r}(a)} = \frac{g_i(a)/h(a)}{\tau_1^{k_1-j_1}(a) \cdots \tau_r^{k_r-j_r}(a)};$$

therefore,

$$\Sigma_{a^*}(F_i) = -\sum_{\alpha} (k_\alpha - j_\alpha) \ge -n$$

(the summation is over indices α for which $k_\alpha - j_\alpha > 0$); this proves the first part of assertion (b).

In a neighborhood of the point $a^* \in \Delta^0$, the expansions (9) and (10) for the functions b and f_i, respectively, are also valid, but $h(a^*) = 0$. However, in view of the fact that the function b is irreducible,

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[2]Indeed, if $db(a') \equiv 0$ for some $a' \in \{b(a) = 0\}$, then $\sum_{i=1}^{n} b_i(a')\, da_i \equiv 0$ and $b_1(a') = \dots = b_n(a') = 0$. Taking relations (8) into account, we find that $b_{n+2}(a') = 0$ and $b_{n+1}(a') = 0$. But this contradicts the fact that the monodromy of the family (6) is irreducible.

all irreducible factors in the expansion $h(a) = h_1(a) \cdots h_s(a)$ are different in a neighborhood of the point a^* (we can also assume that none of the functions h_i coincides with any one of the functions τ_l). Since $k_l - j_l \le n$ for all $l = 1, \dots, r$, the second part of assertion (b) follows from the expansion

$$\frac{f_i(a)}{b(a)} = \frac{g_i(a)}{h_1(a) \cdots h_s(a)\, \tau_1^{k_1 - j_1}(a) \cdots \tau_r^{k_r - j_r}(a)}. \quad \square$$

Remark 2. It follows from Remark 1 that, for even n, we can replace n by $n-1$ in all the estimates from Theorem 3.

In particular, the polar sets of the functions

$$F_1(a) = u_1(a) + u_2(a) \qquad \text{and} \qquad F_2(a) = u_1(a) u_2(a),$$

where (u_1, u_2, v_1, v_2) is the solution of the Garnier system $\mathscr{G}_2(\theta_1, \dots, \theta_4, \theta_\infty)$ corresponding to the two-dimensional Schlesinger isomonodromic family with five singular points and an irreducible monodromy are analytic submanifolds, and $(F_i)_\infty \ge -2$. (Note that the bundle E_{a^*} corresponding to the point a^* of the Θ-divisor of this family is of the form $E_{a^*} \cong \mathscr{O}(1) \oplus \mathscr{O}(-1)$, which implies the regularity of the Θ-divisor; see [2, Theorem 16.2] or [11].)

The solutions of the Garnier system (3) whose linear monodromy is *reducible* can be expressed in terms of solutions of hypergeometric partial differential equations. A more detailed discussion of this fact discovered by Mazzocco [13] is presented in the next section.

4. GARNIER SYSTEMS AND LAURICELLA HYPERGEOMETRIC EQUATIONS

Consider the *Lauricella hypergeometric equation* $E_D(\alpha, \beta_1, \dots, \beta_n, \gamma)$, by which term we mean the following system of linear partial differential equations of second order for the complex function $u(a_1, \dots, a_n)$:

$$\begin{aligned} (1 - a_i) \sum_{j=1}^n a_j \frac{\partial^2 u}{\partial a_i \partial a_j} &+ (\gamma - (\alpha + 1) a_i) \frac{\partial u}{\partial a_i} \\ &- \beta_i \sum_{j=1}^n a_j \frac{\partial u}{\partial a_j} - \alpha \beta_i u = 0, \qquad i = 1, \dots, n, \\ (a_i - a_j) \frac{\partial^2 u}{\partial a_i \partial a_j} &+ \beta_i \frac{\partial u}{\partial a_j} - \beta_j \frac{\partial u}{\partial a_i} = 0, \qquad i, j = 1, \dots, n, \end{aligned} \tag{11}$$

where $\alpha, \beta_1, \dots, \beta_n, \gamma \in \mathbb{C}$ are constants. This system can be regarded as a natural generalization of the hypergeometric equation

$$z(1 - z) \frac{d^2 u}{dz^2} + (\gamma - (\alpha + \beta + 1) z) \frac{du}{dz} - \alpha \beta u = 0, \qquad \alpha, \beta, \gamma \in \mathbb{C}, \tag{12}$$

for the complex function $u(z)$, because for $n = 1$ ($a_1 = z$, $\beta_1 = \beta$) system (11) coincides with (12). In addition, it is also easy to see that the solution $u(z)$ of Eq. (12) generates the solution $u(a_1, \dots, a_n) = u(a_i)$ of the equation $E_D(\alpha, \beta_1, \dots, \beta_n, \gamma)$, where $\beta_i = \beta$ and $\beta_j = 0$ for $j \ne i$.

System (11) is defined on the space

$$B = (\mathbb{C} \setminus \{0, 1\})^n \setminus \bigcup_{i \ne j} \{a_i = a_j\}.$$

As shown in [12, Proof of Proposition 9.1.4, p. 249], the vector function

$$y(a) = \left(u, a_1 \frac{\partial u}{\partial a_1}, \dots, a_n \frac{\partial u}{\partial a_n}\right)^\top$$

of a variable $a = (a_1, \dots, a_n) \in B$ satisfies the completely integrable linear *Pfaffian system*

$$dy = \omega y, \qquad y(a) \in \mathbb{C}^{n+1}, \tag{13}$$

where ω is a matrix differential 1-form holomorphic on B. Therefore, the set of local solutions of the Lauricella hypergeometric equation near each point $a \in B$ constitutes an $(n+1)$-dimensional vector space, and the solutions can be extended holomorphically to the universal covering Z' of the space B.

Further, we recall how some solutions of certain Garnier systems are related to Lauricella hypergeometric equations and study the movable singularities of these solutions.

As mentioned above, for $n > 1$, the Garnier system (3) does not, generally, satisfy the Painlevé property (the coordinates $u_1, \dots, u_n$ of the solution of the system are the roots of a polynomial of degree n); there exists a (symplectic) transformation

$$(a, u, v, H) \mapsto (s, q, p, \widetilde{H}), \qquad \sum_{i=1}^{n}(p_i\,dq_i - \widetilde{H}_i\,ds_i) = \sum_{i=1}^{n}(v_i\,du_i - H_i\,da_i),$$

of this system into a Hamiltonian system $\mathscr{H}_n(\theta)$ of the form

$$\frac{\partial q_j}{\partial s_i} = \frac{\partial \widetilde{H}_i}{\partial p_j}, \quad \frac{\partial p_j}{\partial s_i} = -\frac{\partial \widetilde{H}_i}{\partial q_j}, \qquad i, j = 1, \dots, n,$$

having the Painlevé property (see [12, Chap. III, Sec. 7]). Let us recall the definition of this transformation. Consider the functions

$$\begin{aligned} M_i(a,u) &= -\frac{(a_i - u_1)\cdots(a_i - u_n)}{\prod_{j=1, j\neq i}^{n+2}(a_i - a_j)}, \\ M^{k,i}(a,u) &= \frac{u_k(u_k - 1)(u_k - a_1)\cdots(u_k - a_n)}{(u_k - a_i)\prod_{j=1,\, j\neq k}^{n}(u_k - u_j)}, \end{aligned} \qquad i, k = 1, \dots, n. \tag{14}$$

The transformation $(a, u, v) \mapsto (s, q, p)$ is given by

$$s_i = \frac{a_i}{a_i - 1}, \quad q_i = -a_i M_i, \quad p_i = (1 - a_i)\sum_{k=1}^{n}\frac{M^{k,i} v_k}{u_k(u_k - 1)}, \qquad i = 1, \dots, n; \tag{15}$$

here

$$v_i = \sum_{k=1}^{n}\frac{q_k p_k}{u_i - a_k}, \qquad i = 1, \dots, n, \tag{16}$$

while the new Hamiltonians

$$\widetilde{H}_i = -(1 - a_i)^2\biggl(H_i + \sum_{j=1}^{n} p_j \frac{\partial q_j}{\partial a_i}\biggr) = \frac{1}{s_i(s_i - 1)}\biggl(\sum_{j,k=1}^{n} E^i_{jk}(s,q)p_j p_k - \sum_{j=1}^{n} F^i_j(s,q)p_j + \varkappa q_i\biggr)$$

are polynomials in the variables $(q, p) = (q_1, \dots, q_n, p_1, \dots, p_n)$; here $E^i_{jk}(s,q)$ and $F^i_j(s,q)$ are polynomials in q of the third and the second degree, respectively, and

$$\varkappa = \frac{1}{4}\biggl(\biggl(\sum_{i=1}^{n+2}\theta_i - 1\biggr)^2 - \theta_\infty^2\biggr).$$

If $\varkappa = 0$, i.e.,

$$\sum_{i=1}^{n+2}\theta_i - 1 = \pm\theta_\infty,$$

then it follows from the form of the functions $\widetilde{H}_i$ that the system $\mathscr{H}_n(\theta)$ has solutions of the form $(q, 0)$, where q is the solution of the system

$$s_i(s_i - 1)\frac{\partial q_j}{\partial s_i} = -F^i_j(s,q), \qquad i, j = 1, \dots, n. \tag{17}$$

Since the right-hand side of this system consists of polynomials of the second degree in q, Eqs. (17) can be regarded as a generalization of the classical Riccati equation to the case of several variables. Just as in the classical case, system (17) can be linearized by a suitable change of the unknown. Namely, in view of [12, Theorem 9.2.1, p. 252], the solution $(q_1, \dots, q_n)$ of system (17) can be expressed as

$$q_i(s) = \frac{s_i(s_i-1)}{\sum_{j=1}^{n+2} \theta_j - 1}\left(\frac{\theta_i}{s_i-1} + \frac{1}{f}\frac{\partial f}{\partial s_i}\right), \qquad i = 1, \dots, n,$$

where $f(s)$ is an arbitrary solution of the Lauricella hypergeometric equation

$$E_D\left(1-\theta_{n+2}, \theta_1, \dots, \theta_n, \sum_{j=1}^{n+1} \theta_j\right).$$

Note that

$$\sum_{j=1}^{n+2} \theta_j - 1 \neq 0 \qquad \text{if} \quad \theta_\infty \neq 0.$$

The function f is irreducible at its zeros;[3] therefore, any solution $(q_1, \dots, q_n)$ of system (17), as well as the linear combination Q of the coordinates q_i (with holomorphic coefficients), is meromorphic on Z'; further, the polar set of the function Q is an analytic submanifold, and $(Q)_\infty = -1$ (because the function fQ is holomorphic on Z').

Now consider the solution $(u, 0)$ of the Garnier system (3) corresponding to the solution $(q, 0)$ of the system $\mathscr{H}_n(\theta)$, where

$$\sum_{i=1}^{n+2} \theta_i - 1 = \pm\theta_\infty$$

(note that $v = 0$ if and only if $p = 0$ in view of (15) and (16)). The elementary symmetric polynomials $F_i(a) = \sigma_i(u_1(a), \dots, u_n(a))$ are linear combinations (with holomorphic coefficients) of functions q_i. Indeed, suppose that

$$Q_i(a) := \prod_{j=1,\, j\neq i}^{n+2} (a_i - a_j), \qquad i = 1, \dots, n+2.$$

Then formulas (14) and (15) imply that

$$a_i^n - F_1(a)a_i^{n-1} + \dots + (-1)^n F_n(a) = \frac{Q_i(a)}{a_i} q_i, \qquad i = 1, \dots, n.$$

Therefore, the vector $(-F_1, \dots, (-1)^n F_n)$ is the solution of a system of linear equations with coefficient matrix whose determinant is the Vandermonde determinant.

The arguments given above lead to the following statement.

Proposition 1. *Suppose that* $(u(a), 0)$ *is a solution of the Garnier system* (3)*, where*

$$\sum_{i=1}^{n+2} \theta_i - 1 = \pm\theta_\infty \neq 0.$$

Then the polar sets of the functions F_i are analytic submanifolds and $(F_i)_\infty = -1$.

This proposition will be used in the study of singularities of certain solutions of Garnier systems whose linear monodromy is *reducible*.

[3] If $f(s^*) = 0$ and $(\partial f/\partial s_1)(s^*) = \dots = (\partial f/\partial s_n)(s^*) = 0$, then $f \equiv 0$ because of the uniqueness of the solution of system (13).

5. SOLUTIONS OF THE GARNIER SYSTEM WHOSE LINEAR MONODROMY IS REDUCIBLE

Consider the solution $(u(a), v(a))$ of the system $\mathscr{G}_n(\theta)$ which corresponds to the Schlesinger isomonodromic family (6) with reducible monodromy. The eigenvalues of the matrix residues $B_i(a)$ of this family are $\pm\theta_i/2$, respectively, for $i = 1, \dots, n+2$, and

$$B_\infty = -\sum_{i=1}^{n+2} B_i(a) = \frac{1}{2}\operatorname{diag}(\theta_\infty + 1, -\theta_\infty - 1)$$

is the residue at infinity. Since the monodromy is reducible, it follows that the product of the eigenvalues

$$e^{\pi\sqrt{-1}\varepsilon_1\theta_1}, \quad \dots, \quad e^{\pi\sqrt{-1}\varepsilon_{n+2}\theta_{n+2}}, \quad e^{\pi\sqrt{-1}\varepsilon_\infty(\theta_\infty+1)}$$

of the monodromy matrices $G_1, \dots, G_{n+2}, G_\infty$, respectively, is equal to 1 (under some choice of the numbers $\varepsilon_k \in \{1, -1\}$), i.e., the sum

$$\varepsilon_1\theta_1 + \dots + \varepsilon_{n+2}\theta_{n+2} + \varepsilon_\infty(\theta_\infty + 1) \tag{18}$$

is an even integer.

Lemma 2. *If the sum* (18) *is zero, then all matrices* $B_i(a)$ *are upper-triangular.*

Proof. The assumptions of the lemma mean that, for $a = a^0$ (as well as for any $a \in D(a^0)$), the corresponding Fuchsian system (6) specifies a connection in the holomorphically trivial vector bundle E_0 of rank 2 over $\overline{\mathbb{C}}$ having a linear subbundle E_0' of degree zero (it is, therefore, holomorphically trivial), which is stabilized by the connection.

Consider the fundamental matrix $Y(z)$ of the system whose columns contain the coordinates of the horizontal sections h_1 and h_2 of the connection such that h_1 is the section of the subbundle E_0'. (The coordinates in question correspond to the basis (s_1, s_2) of the global holomorphic sections of the bundle E_0; in this basis, the matrix differential 1-form of the connection coincides with the 1-form of the coefficients of the system.) Then the coordinates of these sections in the basis $(s_1', s_2') = (s_1, s_2)C^{-1}$, where s_1' is the section of the subbundle E_0', constitute the fundamental matrix $Y'(z) = CY(z)$ of upper-triangular form. Therefore, all matrix residues $B_i' = CB_i^0C^{-1}$ of the corresponding Fuchsian system are upper-triangular.

Let us now show that if all matrices $\widetilde{B}_i(a) = CB_i(a)C^{-1}$ are upper-triangular for $a = a^0$, then they are upper-triangular for any value of the parameter a.

The set $\{\widetilde{B}_1(a), \dots, \widetilde{B}_{n+2}(a)\}$ is a solution of the Schlesinger equation, and all matrices $\widetilde{B}_i(a^0)$ are upper-triangular, i.e., the lower left elements $\widetilde{b}_i^{21}(a)$ of all matrices $\widetilde{B}_i(a)$ vanish for $a = a^0$. Using the Schlesinger equation, we obtain

$$d\widetilde{b}_i^{21} = 2\sum_{j=1,\, j\neq i}^{n+2} \frac{\widetilde{b}_i^{11}\widetilde{b}_j^{21} - \widetilde{b}_j^{11}\widetilde{b}_i^{21}}{a_i - a_j}\, d(a_i - a_j)$$

(recall that the traces of all matrices $\widetilde{B}_i(a)$ are zero); therefore, the partial derivatives of the functions $\widetilde{b}_i^{21}$ are of the form

$$\frac{\partial\widetilde{b}_i^{21}}{\partial a_i} = \alpha_i\widetilde{b}_i^{21} + \sum_{j=1,\, j\neq i}^{n+2} \alpha_{ij}\widetilde{b}_j^{21}, \qquad i = 1, \dots, n,$$

$$\frac{\partial\widetilde{b}_i^{21}}{\partial a_j} = -\alpha_{ij}\widetilde{b}_j^{21} + \beta_{ij}\widetilde{b}_i^{21}, \qquad i = 1, \dots, n+2, \quad j = 1, \dots, n, \quad j \neq i,$$

where α_i, α_{ij}, β_{ij} are holomorphic functions in $D(a^0)$. Thus, the partial derivatives of any order m of each function $\widetilde{b}_i^{21}$ are expressed linearly (with holomorphic coefficients) in terms of the derivatives of lower order of the functions $\widetilde{b}_1^{21}, \dots, \widetilde{b}_{n+2}^{21}$. But since $\widetilde{b}_1^{21}(a^0) = \dots = \widetilde{b}_{n+2}^{21}(a^0) = 0$, it follows that the partial derivatives of any order of all functions $\widetilde{b}_i^{21}$ vanish for $a = a^0$. Therefore, these functions are identically zero.

Thus, there exists a matrix $C = (c_{ij}) \in \mathrm{GL}(2, \mathbb{C})$ such that

$$C \begin{pmatrix} b_i^{11}(a) & b_i^{12}(a) \\ b_i^{21}(a) & b_i^{22}(a) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\varepsilon_i\theta_i & \widetilde{b}_i^{12}(a) \\ 0 & -\frac{1}{2}\varepsilon_i\theta_i \end{pmatrix} C, \qquad i = 1, \dots, n+2, \tag{19}$$

$$C \begin{pmatrix} \frac{1}{2}(\theta_\infty + 1) & 0 \\ 0 & -\frac{1}{2}(\theta_\infty + 1) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\varepsilon_\infty(\theta_\infty + 1) & * \\ 0 & -\frac{1}{2}\varepsilon_\infty(\theta_\infty + 1) \end{pmatrix} C. \tag{20}$$

If $\varepsilon_\infty = 1$, then it follows from relation (20) (by comparing the lower left elements) that $c_{21} = 0$, i.e., the matrix C is upper-triangular. Therefore, all matrices $B_i(a) = C^{-1}\widetilde{B}_i(a)C$ are upper-triangular.

If $\varepsilon_\infty = -1$, then it follows from relation (20) (by comparing the lower right elements) that $c_{22} = 0$. But, in this case, it follows from (19) (by comparing the lower right elements) that all $b_i^{12}(a)$ are zero, i.e., all matrices $B_i(a)$ are lower-triangular. This contradicts the fact that the original family (6) is constructed from the solution of the Garnier system and the function

$$(z - a_1) \cdots (z - a_{n+2}) \sum_{i=1}^{n+2} \frac{b_i^{12}(a)}{z - a_i}$$

is a polynomial of degree n in z (see (7)). □

Proposition 1 and Lemma 2 imply the following statement.

Proposition 2. *If $(u(a), v(a))$ is the solution of the Garnier system (3), whose linear monodromy is reducible, the corresponding sum (18) is zero, and $\theta_\infty \neq 0$, then the polar sets of the functions $F_i = \sigma_i(u_1, \dots, u_n)$ are analytic submanifolds and $(F_i)_\infty = -1$.*

Proof. By Lemma 2, the matrix residues $B_i(a)$ of the Schlesinger isomonodromic family (6) corresponding to the solution $(u(a), v(a))$ are of the form

$$B_i(a) = \begin{pmatrix} \frac{1}{2}\varepsilon_i\theta_i & b_i^{12}(a) \\ 0 & -\frac{1}{2}\varepsilon_i\theta_i \end{pmatrix}, \qquad i = 1, \dots, n+2,$$

$$B_\infty = \begin{pmatrix} \frac{1}{2}(\theta_\infty + 1) & 0 \\ 0 & -\frac{1}{2}(\theta_\infty + 1) \end{pmatrix}$$

($\varepsilon_\infty = 1$). Therefore, by Theorem 2, the pair $(u(a), 0)$ is a solution of the Garnier system $\mathscr{G}_n(\widetilde{\theta})$ with parameters

$$(\widetilde{\theta}_1, \dots, \widetilde{\theta}_{n+2}, \widetilde{\theta}_\infty) = (-\varepsilon_1\theta_1, \dots, -\varepsilon_{n+2}\theta_{n+2}, \theta_\infty),$$

satisfying the following relation from Proposition 1:

$$\sum_{i=1}^{n+2} \widetilde{\theta}_i - 1 = -\sum_{i=1}^{n+2} \varepsilon_i\theta_i - 1 = (\theta_\infty + 1) - 1 = \widetilde{\theta}_\infty.$$

Hence Proposition 1 can be applied to the functions $F_i = \sigma_i(u_1, \dots, u_n)$. □

Thus, if the parameters of the Garnier system $\mathscr{G}_n(\theta)$ are such that the sum (18) is not an even integer under any choice of the numbers $\varepsilon_k \in \{1, -1\}$, then the linear monodromy of any solution of this system is irreducible and Theorem 3 is applicable. But if, under some choice of the numbers $\varepsilon_k \in \{1, -1\}$, the sum (18) is an even integer, then the system may have solutions with reducible linear monodromy. Further, if the sum (18) corresponding to a subrepresentation of the monodromy is zero, then we can apply Proposition 2 to the corresponding solution.

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