Manuscript submitted to
AIMS' Journals
Volume 00, Number 0, Xxxx XXXX

Website: http://AIMsciences.org
pp. 000–000

UNBOUNDED SEQUENCES OF CYCLES IN GENERAL AUTONOMOUS EQUATIONS WITH PERIODIC NONLINEARITIES

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Dedicated to Peter P. Zabreiko on the occasion of his 70th birthday

Abstract. Autonomous higher order differential equations with scalar nonlinearities, periodic with respect to the main phase variable under appropriate generic conditions, have an infinite sequence of isolated cycles with amplitudes growing to infinity and periods converging to some specific value T_0.

1. Introduction.

1.1. Investigation of cycles in autonomous systems (existence, stability, number of cycles, bifurcations, numerical computations, simulations, applications, etc.) by various mathematical (analytical, geometrical, topological, fixed point method) approaches is the classical part of mathematics, with a lot of theoretical and applied books and papers devoted to them.

The only simple case is linear: cycles exist if and only if at least one pair of complex conjugate eigenvalues of some matrix (or a pair of roots of some characteristic polynomial) is situated on the imaginary axis. If such a pair is unique and the eigenvalues (the roots) are simple, then all the cycles (in the linear case) are circles, the cycles are not isolated and fill a two dimensional plane in the phase space. Generic nonlinear non-Hamiltonian equations have isolated cycles only.

If a nonlinear differential equation or a dynamical system has a principal (in some appropriate sense) linear part, then sometimes it is also

2000 *Mathematics Subject Classification.* Primary: 34C15, 34C25; Secondary: 93C15.

Key words and phrases. Autonomous equations, cycles, higher order ODE, resonance, unbounded set of solutions, periodic nonlinearities, method of stationary phase.

possible to find round cycles (i.e., close to circles). The classical example is Hopf bifurcation at the origin[1], a similar situation (with a two dimensional manifold filled with round cycles) appears in Hopf bifurcations at infinity.

In [4] we considered single-loop control systems that contain linear parts and nonlinear bounded scalar feedbacks (with and without delays). The principal result concerns the existence of unbounded infinite sequences of isolated cycles (for feedbacks without delays see Statement 1) below. The main condition in [4] has a formally generic form $\limsup X > \liminf X$, however this condition turns out to be valid only for a special class of nonlinearities, containing slowly oscillating components with exponentially increasing intervals between consecutive zeros (such as $\sin(\log(1+|x|))$). In this paper we present results in the spirit of [4] for equations with usual periodic nonlinearities. The results are based on sharp asymptotical representations of projections of periodic nonlinearities. The most cumbersome part of the proofs is related to the Kelvin method of stationary phase ([9], §§11-14).

1.2. Consider a traditional for control theory equation[2]

$$L\left(\frac{d}{dt}\right) x = M\left(\frac{d}{dt}\right) f(x), \tag{1}$$

where L and M are fixed real coprime polynomials of degrees ℓ and m ($\ell > m$), the continuous scalar function $f : \mathbb{R} \to \mathbb{R}$ is uniformly bounded.

Let a value $w_0 > 0$ be a root of the polynomials $\Im(L(wi)M(-wi))$ and $L(wi)$ of the same odd multiplicity K and let $L(kw_0 i) \neq 0$ for $k = 0, 2, 3, 4, \ldots$ Put

$$\Psi(\xi) \stackrel{\text{def}}{=} \int_0^{2\pi} \sin t\, f(\xi \sin t)\, dt = 4 \int_0^{\pi/2} \sin t\, f_{odd}(\xi \sin t)\, dt.$$

This function is odd and it is defined by the odd part $f_{odd}(x) = (f(x) - f(-x))/2$ of the function f. In control theory such functions are called

[1]Consider the system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}, \lambda)$, $\boldsymbol{x} \in \mathbb{R}^d$, $\boldsymbol{f} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ with a real parameter λ. If $\boldsymbol{f}(\boldsymbol{0}, \lambda) \equiv \boldsymbol{0}$ and $\boldsymbol{f}(\boldsymbol{x}, \lambda) = A(\lambda)\boldsymbol{x} + o(\boldsymbol{x})$ at the origin, and if the matrix $A(\lambda)$ has a pair of complex conjugate pure imaginary eigenvalues for some $\lambda = \lambda_0$, then under appropriate assumptions on the spectrum of $A(\lambda)$ for some sufficiently close to λ_0 values of λ (generically, either only for $\lambda < \lambda_0$ or only for $\lambda > \lambda_0$ depending on the small term $o(\boldsymbol{x})$) there exists one small round cycle. In the space $\mathbb{R}^{d+1}$ these small cycles and the origin form a two dimensional manifold.

[2]Equation (1) may be rewritten in the equivalent form $\boldsymbol{z}' = A\boldsymbol{z} + f(\langle \boldsymbol{z}, \boldsymbol{c}\rangle)\boldsymbol{b}$, where $\boldsymbol{z}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^\ell$, and A is an $\ell \times \ell$ matrix. Solutions $x = \langle \boldsymbol{z}, \boldsymbol{c}\rangle$ of (1) are defined for non-smooth f, see manuals on control theory, e.g., [1–3]. All results presented here are new even for usual ordinary differential equations ($M \equiv 1$) of higher order.

describing functions, they are used in stability theory and various other applications. The following statement on the cycles of (1) was proved in [4].

Statement 1. *Let*

$$\Psi^+ \stackrel{\text{def}}{=} \limsup_{\xi\to\infty} \Psi(\xi) > 0 > \liminf_{\xi\to\infty} \Psi(\xi) \stackrel{\text{def}}{=} \Psi^-. \tag{2}$$

Then there exists an infinite sequence x_n of T_n-periodic solutions for equation (1)*, their amplitudes and periods satisfy the relations $\|x_n\|_C \to \infty$, $T_n \to T_0 = 2\pi/\mathrm{w}_0$.*

Let us emphasize that the main part $\Psi^+ > \Psi^-$ of condition (2) is valid for rather specific functions f. For a reasonable f the corresponding describing function Ψ tends to a constant at infinity, the most typical situation $\Psi(\xi) \to 0$ as $\xi \to \infty$ occurs either if f is even, or if f has a sublinear primitive (e.g., f is periodic or almost periodic), or if $f \to 0$ at infinity, or if $f(x) = \operatorname{sign}(x)\sin(|x|^\alpha)$, $\alpha > 0$. If f has a saturation, i.e., if $f(x) \to \pm F \neq 0$ as $x \to \pm\infty$, then $\Psi^\pm = 4F$. The function $f_0(x) = \operatorname{sign}(x)\sin(\ln(1+|x|))$ mentioned above generates Ψ that oscillates at infinity, $\Psi^\pm = \pm\Psi_*$ ($\Psi_* \approx 3.70$), and (2) holds. Since the operation $f \mapsto \Psi$ is linear, condition (2) also holds for Ψ generated by various sums of the type $f_0(x)$ + 'even function' + 'vanishing at infinity function' + 'periodic function' + 'function, oscillating sufficiently fast' etc.

The distances between consecutive zeros of the function f_0 (i.e., between the points $e^{\pi k} - 1$ for $k = 1, 2, \ldots$) are equal to $e^{\pi k}(e^\pi - 1)$ and increase exponentially fast.

Problems on forced periodic oscillations for systems with nonlinearities satisfying $\Psi^+ > \Psi^-$ were considered in [6, 7]. In [6] we found conditions for the existence of sequences of such oscillations with arbitrarily large amplitudes. In [7] we considered equations with a parameter and discovered the existence of infinite sequences of so-called cyclic continuous bounded branches of solutions.

The rest part of the present paper deals with the case of periodic f, in this case always $\Psi^+ = \Psi^- = 0$ and Statement 1 is inapplicable as its main condition (2) is not valid. The paper is organized in the following way. The main result and miscellaneous remarks generalizing and continuing it are given in the next section. Two last sections contain the proofs. The proof of Theorem 2.1 is given in Section 3, in the proof we use auxiliary statements (Lemma 3.4 and Lemma 3.5) from Subsection 3.5, their proofs are presented in the last part of the paper.

2. The main result. 2.1.

Let f be a continuous and T-periodic function, let

$$\mu_0 \stackrel{\text{def}}{=} \frac{1}{T}\int_0^T f(x)\,dx = 0. \tag{3}$$

Put $\omega = \frac{2\pi}{T}$ and consider the Fourier series of the function f:

$$f(x) = \sum_{s=1}^{\infty} \mu_s \sin(s\omega\, x + \psi_s), \qquad \mu_s \geq 0. \tag{4}$$

Theorem 2.1. *Let the following conditions be valid:*

1. $\ell > m+1$, *where* $\ell = \deg L$ *and* $m = \deg M$;
2. *The value* w_0 *is a root of the polynomials* $\Im(L(\mathrm{w}i)M(-\mathrm{w}i))$ *and* $L(\mathrm{w}i)$ *of the same odd multiplicity* K;
3. $L(k\mathrm{w}_0 i) \neq 0$ *for* $k = 0, 2, 3, \ldots$;
4. *The Fourier coefficients* μ_s *in* (4) *satisfy*

$$\sum_{s=1}^{\infty} \mu_s \sqrt{s} < \infty; \tag{5}$$

5. *The function* f *is not even.*

Then there exists an infinite sequence x_n *of* T_n*-periodic solutions for equation* (1)*, their amplitudes and periods satisfy* $\|x_n\|_C \to \infty$, $T_n \to T_0 = 2\pi/\mathrm{w}_0$.

The uniform convergence of the series in (4) follows from (5).

The simplest example is the equation $x''' + x'' + x' + x = \sin x$. From Theorem 2.1 it follows the existence of the sequence x_n of T_n-periodic solutions satisfying $\|x_n\|_C \to \infty$, $T_n \to 2\pi$. There are two sequences of large-amplitude cycles: the sequence with stable cycles and the sequence with unstable ones. The stable cycles can be easily found numerically.

2.2. Remarks.

Remark 1. Instead of periodic f it is possible to consider almost periodic sums of two or more periodic functions with (maybe) incommensurable periods and, moreover, Fourier integrals. It is also possible to consider non-periodic f of the form 'periodic term' + 'additional terms', if the additional terms generate their own describing functions of the order less than $\xi^{-1/2}$, (e.g., terms of the form $\mathit{const} \cdot x^{-1/2-\sigma}$, $\sigma > 0$; or rapidly oscillating terms of the type $\sin(x^3)$). Finally, the equations

$$L\left(\frac{d}{dt}\right) x = M\left(\frac{d}{dt}\right) f(x(t), x(t-\Delta)) \tag{6}$$

with delays containing periodic with respect to the both variables functions f can be also considered with similar arguments.

The method of the proof might work for functions f depending on several variables if f is periodic with respect to each of them and, hence, allows for the Fourier expansion in several variables and an analog of Lemma 3.5 is valid.

Remark 2. The periodic solutions x_n from Theorem 1 have the form

$$x_n(t) = \xi_n \sin(2\pi t/T_n) + h_n(t) \tag{7}$$

where h_n are T_n-periodic, $\xi_n \to \infty$, $\|h_n\|_C \to 0$, $T_n \to T_0 = 2\pi/w_0$. The values ξ_n are close to sufficiently large zeros of the T-periodic function

$$g(\xi) = \sum_{s=1}^{\infty} \mu_s \frac{\cos\psi_s \sin(\omega s\xi - \frac{\pi}{4})}{\sqrt{s}}. \tag{8}$$

Moreover, if ξ_* is an isolated zero of the function g and $g(\xi_*+0)g(\xi_*-0) < 0$, then for any sufficiently large integer n there exists a periodic solution of the form (7) and $|\xi_n - \xi_* - nT| \to 0$ as $n \to \infty$. The almost opposite statement is also valid: it is possible to choose the vicinity[3] Ω of the point w_0 such that all large-amplitude cycles with the periods $\tau = 2\pi/w,\ w \in \Omega$ have the round form. More exactly, for any sufficiently small $\varepsilon > 0$ there exists $R(\varepsilon)$ such that any periodic solution $x(t)$ of a period $\tau = 2\pi/w,\ w \in \Omega$, satisfying $\|x\|_C \geq R(\varepsilon)$, has the form $x(t) = \xi\sin(wt+\phi)+h(t)$, $h(t+\tau) \equiv h(t)$, where $\|h\|_C,\ |g(\xi)| \leq \varepsilon$.

It would be interesting to supplement Theorem 2.1 with conditions of uniqueness of periodic solution $\xi\sin(wt) + h(wt)$ where $|\xi - \xi_* - nT|,\ |w - w_0| < \varepsilon$.

The function g is not identically zero ($\mu_s \cos\psi_s \neq 0$ at least for one $s = 1, 2, \ldots$) if and only if Condition 5 is valid. The function g defined by (8) plays essential role in the proofs below, it defines the principal part of the describing function Ψ:

$$\Psi(\xi) = \frac{2\sqrt{2\pi}g(\xi)}{\omega\sqrt{\xi}} + o(\xi^{-1/2})$$

at infinity (see Lemma 3.5 below).

The cycles x_n in the phase space are close to the circles generated by the functions $\xi_n \sin(w_n t)$ with the same ξ_n and w_n. From the proof below it follows that $|w_n - w_0| \leq o(\xi_n^{-(2+\sigma)/K})$, therefore if $K = 1$, then the circles $\xi_n \sin(w_n t)$ are close to the circles $\xi_n \sin(w_0 t)$. These circles for all various n are concentric and belong to the common plane.

Remark 3. Condition 2 means that

$$\lim_{w\to w_0} |w - w_0|^{-K} |\Re(L(wi)M(-wi))| < \infty, \tag{9}$$

[3]The choice of Ω is given in the end of Section 3.1 explicitly.

i.e., either w_0 is a root of the polynomial $\Re(L(wi)M(-wi))$ of a finite multiplicity $K^* \geq K$, or $\Re(L(wi)M(-wi)) \equiv 0$. The assumption $K^* \geq K$ may be slightly weakened; e.g., the inequality $3K \leq 4K^*$ is also sufficient.

Remark 4. Theorem 2.1 can be extended to periodic nonlinearities f with nonzero mean values. Let (instead of (3)) $\mu_0 \neq 0$. The change of variables $x = \mu_0 \frac{M(0)}{L(0)} + y$ transforms (1) into the equation of the same form, with the same polynomials L and M and with the new nonlinearity $f_1(y) = f(\mu_0 \frac{M(0)}{L(0)} + y) - \mu_0$, that is also T-periodic (in y) and has zero mean value:

$$\frac{1}{T}\int_0^T f_1(y)\,dy = 0.$$

Function (8) for this nonlinearity has the Fourier series

$$g_1(y) = \sum_{s=1}^{\infty} \mu_s \frac{\cos(\psi_s + \omega s \mu_0 \frac{M(0)}{L(0)}) \sin(\omega s y - \frac{\pi}{4})}{\sqrt{s}},$$

this function must be not identically zero, i.e., the function f_1 must be not even.

3. **Proof of Theorem 2.1.**

3.1. **Time rescaling.** Let us linearly rescale the time in (1) and consider the equation

$$L(\mathrm{w}\frac{d}{dt})x = M(\mathrm{w}\frac{d}{dt})f(x). \tag{10}$$

Every 2π-periodic solution $x(t)$ of (10) defines the $(2\pi/\mathrm{w})$-periodic solution $x(wt)$ of (1). We look for 2π-periodic solutions of equation (10) in the form

$$x(t) = \xi \sin t + h(t), \tag{11}$$

the Fourier expansion of h does not contain the harmonics $\sin t$ and $\cos t$. We are going to find sequences $\xi_n \to \infty$, $\mathrm{w}_n \to \mathrm{w}_0$ of real numbers and a sequence $h_n(t)$ of 2π-periodic functions such that formula (11) defines the solutions $x_n(t) = \xi_n \sin t + h_n(t)$ of equation (10) with $\mathrm{w} = \mathrm{w}_n$. This would imply the conclusion of Theorem 2.1.

Every non-stationary 2π-periodic solution $x(t)$ of any autonomous equation is included in the continuum $x(t+\phi)$ of shifted solutions; any of them ($\phi \in \mathbb{R}$ or $\phi \in [0, 2\pi)$) defines the same cycle (as a geometric object) in the phase space $\mathbb{R}^\ell$. If x contains the first harmonics, then exactly one solution $x(t+\phi)$ has the form $\xi \sin t + h(t)$ with $\xi > 0$.

The polynomials L and M are coprime, therefore from $L(\mathrm{w}_0 i) = 0$ it follows that $M(\mathrm{w}_0 i) \neq 0$. From Conditions 2 and 3 of Theorem 2.1

it follows that there exists a vicinity $\Omega = (w_1, w_2)$, $w_1 < w_0 < w_2$ of the point w_0 such that the following two assumptions are valid:

- w_0 is an unique root of both the polynomials $\Im(L(wi)M(-wi))$ and $L(wi)$ on $\bar{\Omega}$;
- $L(kwi) \neq 0$ for $k = 0, 2, 3, \ldots$ and $w \in \bar{\Omega}$.

By assumption $\Im(L(w_0 i)M(-w_0 i)) = 0$, the root w_0 has an odd multiplicity K, therefore

$$\Im(L(w_1 i)M(-w_1 i))\ \Im(L(w_2 i)M(-w_2 i)) < 0, \tag{12}$$

moreover $\Im(L(wi)M(-wi)) = (w - w_0)^K N(w)$, where $N(w) \neq 0$ for $w \in \bar{\Omega}$.

3.2. **Linear operators.** We use the spaces C, C^k, L^2 and W_2^1 of functions $x = x(t) : [0, 2\pi] \to \mathbb{R}$ with the usual norms and scalar products and their subspaces C_0, C_0^k of periodic functions, $\|x\|_{W_2^1} = \|x\|_C + \|x'\|_{L^2}$. Denote by $E \subset L^2$ the linear span of the functions $\sin t$ and $\cos t$, denote by $E^\perp \subset L^2$ the orthogonal complement of the plane E. Then

$$\mathcal{P}x(t) = \frac{1}{\pi}\int_0^{2\pi} \cos(t-s)\, x(s)\, ds$$

and $\mathcal{Q} = I - \mathcal{P}$ are orthogonal projectors onto the subspaces E and $E^\perp$ of L^2.

For each $w \in \bar{\Omega}$ denote by $A = A_w$ the linear operator that maps any function $u \in E^\perp \subset L^2$ to an unique solution $x = Au \in E^\perp$ of the linear equation

$$L(w\frac{d}{dt})x = M(w\frac{d}{dt})u. \tag{13}$$

The existence of the solution $x = x(t)$ follows from the relations $u \in E^\perp$ and $L(kwi) \neq 0$ for all $k \neq \pm 1$, $w \in \bar{\Omega}$; the uniqueness follows from $x \in E^\perp$. The projectors $\mathcal{P}$ and $\mathcal{Q}$ commute with differentiation and with the operators A_w in any appropriate spaces.

The operators $A_w : E^\perp \to E^\perp$ are completely continuous in L^2 and in C. The norms of the operators $A_w\mathcal{Q} : C \to C^1$ are uniformly bounded, moreover, $A_w\mathcal{Q}$ acts continuously from C to $C_0^{\ell-m}$. The number

$$\alpha = \sup_{w \in \bar{\Omega}} \|A_w \mathcal{Q}\|_{C \to C^1} < \infty \tag{14}$$

is well-defined. The operator $A_w\mathcal{Q}u : \bar{\Omega} \times C \to C^{\ell-m}$ is completely continuous with respect to the set of its variables (w, u). The operator $A'_w\mathcal{Q} : u(t) \mapsto \frac{d}{dt}A_w\mathcal{Q}u(t)$ is continuous in C.

Consider a function $u \in C$. If its Fourier coefficients ν_k satisfy the estimate $|\nu_k| \le \zeta_k$, then the Fourier coefficients $\tilde{\nu}_k$ and $\tilde{\nu}'_k$ of the functions $A_{\mathrm{w}}\mathcal{Q}u$ and $A'_{\mathrm{w}}\mathcal{Q}u$ satisfy

$$|\tilde{\nu}_k| \le const\, k^{m-\ell}\zeta_k, \qquad |\tilde{\nu}'_k| \le const\, k^{m-\ell+1}\zeta_k. \tag{15}$$

3.3. **Scalar linear equations.**

Lemma 3.1. *The functions* $x(t) = \xi \sin t + h(t)$ $(h \in E^{\perp})$ *and* $u(t) \in C$ *satisfy* (13) *if and only if*

$$\begin{aligned} \pi\, \Re \frac{L(\mathrm{w}i)}{M(\mathrm{w}i)}\, \xi &= \int_0^{2\pi} \sin t\, u(t)\, dt, \\ \pi\, \Im \frac{L(\mathrm{w}i)}{M(\mathrm{w}i)}\, \xi &= \int_0^{2\pi} \cos t\, u(t)\, dt, \\ h &= A_{\mathrm{w}}\mathcal{Q}u. \end{aligned} \tag{16}$$

Proof. By construction equation (13) is equivalent to the system

$$L\big(\mathrm{w}\frac{d}{dt}\big)(\xi \sin t) = M\big(\mathrm{w}\frac{d}{dt}\big)\mathcal{P}u(t), \quad L\big(\mathrm{w}\frac{d}{dt}\big)h = M\big(\mathrm{w}\frac{d}{dt}\big)\mathcal{Q}u(t),$$

The second equation is equivalent to $h = A_{\mathrm{w}}\mathcal{Q}u$, and the first is equivalent to

$$\begin{aligned} &\pi\Re(L(wi))\xi \sin t + \pi\Im(L(wi))\xi \cos t \\ &\qquad = (\Re(M(wi)) \cos t - \Im(M(wi)) \sin t)\int_0^{2\pi} \cos s\, u(s) ds \\ &\qquad + (\Re(M(wi)) \sin t + \Im(M(wi)) \cos t)\int_0^{2\pi} \sin s\, u(s) ds, \end{aligned}$$

that is

$$\pi\Re(L(wi))\xi = -\Im(M(wi)) \int_0^{2\pi} \cos s\ u(s)\, ds \tag{17}$$

$$+ \Re(M(wi)) \int_0^{2\pi} \sin s\, u(s)\, ds, \tag{18}$$

$$\pi\Im(L(wi))\xi = \quad \Re(M(wi)) \int_0^{2\pi} \cos s\ u(s)\, ds \tag{19}$$

$$+ \Im(M(wi)) \int_0^{2\pi} \sin s\, u(s)\, ds. \tag{20}$$

Multiply (17) by $\Re(M(wi))$ and (19) by $\Im(M(wi))$, then sum the products and obtain the first of equations (16). Multiply (17) by $-\Im(M(wi))$ and (19) by $\Re(M(wi))$, then sum the products and obtain the second of equations (16). Since $M(wi) \neq 0$ for $w \in \bar{\Omega}$ all used transformations are equivalent. ∎

3.4. **Topological lemma.** For the sequel, we need the following lemma on the solvability of a system of two scalar equations and an equation in a Banach space H. This lemma contains the sufficient part of more general statements from [5].

Consider the system

$$B_1(w,\xi,h) = 0, \quad B_2(w,\xi,h) = 0, \quad h = B_3(w,\xi,h), \qquad (21)$$

where the unknowns w and ξ are scalar, $w \in \bar{\Omega} = [w_1, w_2]$, $\xi \in \bar{\Xi} = [\xi_1, \xi_2]$, and $h \in H$. Suppose the operators $B_1, B_2 : \bar{\Omega} \times \bar{\Xi} \times H \to \mathbb{R}$ are continuous and the operator $B_3 : \bar{\Omega} \times \bar{\Xi} \times H \to H$ is completely continuous (with respect to the set of their arguments). If B_3 is uniformly bounded

$$\|B_3(w,\xi,h)\|_H \le \rho, \qquad w \in \bar{\Omega},\ \xi \in \bar{\Xi},\ h \in H,$$

then from the Schauder fixed point theorem it follows that the set $\mathcal{H}(w,\xi) = \{h : h = B_3(w,\xi,h)\}$ is non-empty for any $w \in \bar{\Omega},\ \xi \in \bar{\Xi}$. Put $\mathcal{H} = \bigcup_{w\in\bar{\Omega},\xi\in\bar{\Xi}} \mathcal{H}(w,\xi)$.

Lemma 3.2. *Suppose*

$$B_1(w_1,\xi,h) \cdot B_1(w_2,\xi,h) < 0, \qquad \xi \in \bar{\Xi},\ h \in \mathcal{H}, \qquad (22)$$

$$B_2(w,\xi_1,h) \cdot B_2(w,\xi_2,h) < 0, \qquad w \in \bar{\Omega},\ h \in \mathcal{H}. \qquad (23)$$

Then system (21) *has at least one solution* $w \in \bar{\Omega}$, $\xi \in \bar{\Xi}$, $h \in H$.

Lemma 3.2 follows from Theorem 2 from [5] that is a generalization of the Rotation Product Formula [8], §7, §23. Under the assumptions of Lemma 3.2 the rotation γ_1 of the infinite dimensional vector field $h - B_3(w,\xi,h) \in H$ with fixed w, ξ on the sphere $\{\|h\|_H = \rho + 1\}$ equals 1. The rotation γ_2 of the two-dimensional vector field $\{B_1(w,\xi,h),\ B_2(w,\xi,h)\}$ with fixed h on the boundary of the rectangular $R = \{w \in (w_1,w_2), \xi \in (\xi_1,\xi_2)\}$ is either 1 or -1. The rotation γ_0 of the field

$$\{B_1(w,\xi,h),\ B_2(w,\xi,h),\ h - B_3(w,\xi,h)\}$$

on the boundary of the domain $R \times \{\|h\|_H < \rho + 1\}$ in the space $\mathbb{R} \times \mathbb{R} \times H$ equals $\gamma_1\gamma_2$ ([5]), i.e., $|\gamma_0| = 1$. Hence there exists a solution of system (21) in this domain. ■

Let us emphasize that (22) and (23) must be checked for $h \in \mathcal{H}$ only, this is the main difference between Lemma 3.2 and more classical variants of the Rotation Product Formula.

3.5. **Estimates of the component h and the main lemma.** In Section 3.6 we rewrite 2π-periodic problem for differential equation (13) in the form of equivalent system (21) of operator equations to apply Lemma 3.2. Lemma 3.5 below allows to obtain the necessary inequality (23), it follows from sharp asymptotic representations for the component $\mathcal{P}f(x)$ as $\xi \to \infty$. Lemmas 3.3 and 3.4 presents the necessary *a priori* estimates for the component $h = \mathcal{Q}x$. Proofs of Lemma 3.4 and 3.5 are given in the next Section 4.

Consider the functions $x(t) = x_{\xi,h}(t) = \xi \sin t + h(t)$, $\xi \geq 1$, $h \in E^{\perp}$. For $\xi \geq 1$ and $\mathrm{w} \in \bar{\Omega}$ put

$$\mathcal{H}(\xi, \mathrm{w}) = \{h :\ h \in C,\ h = A_{\mathrm{w}} \mathcal{Q} f(\xi \sin t + h(t))\} \subset C^1$$

and $\mathcal{H} = \bigcup\limits_{\xi \geq 1,\, \mathrm{w} \in \bar{\Omega}} \mathcal{H}(\xi, \mathrm{w})$.

Lemma 3.3. *The inclusion* $\mathcal{H} \subset B_\gamma = \{y \in C^1 :\ \|y\|_{C^1} \leq \gamma\}$, $\gamma = \alpha \sup|f|$ *is valid, where* α *is the number from* (14).

This is a simple lemma: the operator $x \mapsto f(x(t))$ acts in C and maps C into the ball $\{y \in C :\ \|y\|_C \leq \max|f|\}$, the operators $A_{\mathrm{w}}\mathcal{Q}$ act continuously from C to C^1. ■

The operator $h(t) \mapsto A_{\mathrm{w}} \mathcal{Q} f(\xi \sin t + h(t))$ is completely continuous in C and maps C in a ball, from the Leray-Schauder principle it follows that $\mathcal{H}(\xi, \mathrm{w}) \neq \emptyset$ for any $\xi \geq 1$ and $\mathrm{w} \in \bar{\Omega}$.

Lemma 3.4. *For any* $\varepsilon \in (0, 1/2)$ *there exists* $K = K(\varepsilon)$ *such that the inclusion*

$$\mathcal{H}(\xi, \mathrm{w}) \subset \{y \in W_2^1 :\ \|y\|_{W_2^1} \leq K(\varepsilon)\xi^{-\varepsilon}\}, \qquad \xi \geq 1$$

is valid for all $\mathrm{w} \in \bar{\Omega}$.

Lemma 3.5. *For any* $\rho > 0$ *and* $\varepsilon \in (0, 1/2)$ *there exists* $K_1 = K_1(\rho, \varepsilon)$ *such that for* $\xi \geq 1$

$$\sup_{\|h\|_{W_2^1} \leq \rho\xi^{-\varepsilon}} \left| \sqrt{\xi} \int_0^{2\pi} \sin t\, f(\xi \sin t + h(t))\, dt - \frac{2\sqrt{2\pi}}{\sqrt{\omega}} g(\xi) \right| \leq K_1(1 + \rho)\xi^{-\varepsilon}, \quad (24)$$

and

$$\sup_{\|h\|_{W_2^1} \leq \rho\xi^{-\varepsilon}} \left| \xi^{1+\varepsilon} \int_0^{2\pi} \cos t\, f(\xi \sin t + h(t))\, dt \right| \leq \sqrt{2\pi}\, \rho \max|f|, \quad (25)$$

From these lemmas it follows that for any $h \in \mathcal{H}(\xi, w)$

$$\Big|\sqrt{\xi}\int_0^{2\pi} \sin t\, f(\xi\sin t + h(t))\,dt - \frac{2\sqrt{2\pi}}{\sqrt{\omega}} g(\xi)\Big|,$$
$$\Big|\xi\int_0^{2\pi} \cos t\, f(\xi\sin t + h(t))\,dt\Big| \le c(\varepsilon)\,\xi^{-\varepsilon}.$$

3.6. **Equivalent equations and finalizing of the proof.** From Lemma 3.1 it follows that the function $x(t) = \xi\sin t + h(t)$, $h \in E^\perp$ is a 2π-periodic solution of (10) if and only if it satisfies the system

$$\pi\Re\frac{L(\mathrm{w}i)}{M(\mathrm{w}i)}\,\xi = \int_0^{2\pi}\sin t\, f(x(t))\,dt, \quad \pi\Im\frac{L(\mathrm{w}i)}{M(\mathrm{w}i)}\,\xi = \int_0^{2\pi}\cos t\, f(x(t))\,dt,$$
$$h = A_{\mathrm{w}}\mathcal{Q}f(x(t)).$$

Express the value $\pi(\mathrm{w}-\mathrm{w}_0)^K\xi|M(\mathrm{w}i)|^{-2}$ from the second equation:

$$\pi(\mathrm{w}-\mathrm{w}_0)^K\xi|M(\mathrm{w}i)|^{-2} = \frac{1}{N(\mathrm{w})}\int_0^{2\pi}\cos t\, f(x(t))\,dt,$$

where $N = \Im(L(\mathrm{w}i)M(\mathrm{w}i))(\mathrm{w}-\mathrm{w}_0)^{-K}$ is a polynomial, $N(\mathrm{w}i) \ne 0$. Put this in the first equation, it takes the form

$$\int_0^{2\pi}\sin t\, f(x(t))\,dt = Y(\mathrm{w})\int_0^{2\pi}\cos t\, f(x(t))\,dt,$$

where

$$Y(\mathrm{w}) = \frac{\Re(L(\mathrm{w}i)M(-\mathrm{w}i))}{\Im(L(\mathrm{w}i)M(-\mathrm{w}i))}.$$

According to Condition 2 of the theorem (see also Remark 2 and (9)) the rational function Y is continuous (if $\Re(L(\mathrm{w}i)M(-\mathrm{w}i)) \equiv 0$, then it may be identically zero) and bounded on $\bar\Omega$.

The final version of the equivalent system has the form

$$\begin{cases} \Im(L(\mathrm{w}i)M(-\mathrm{w}i)) = \dfrac{|M(\mathrm{w}i)|^2}{\pi\xi}\displaystyle\int_0^{2\pi}\cos t\, f(x(t))\,dt, \\ \displaystyle\int_0^{2\pi}\sin t\, f(x(t))\,dt = Y(\mathrm{w})\int_0^{2\pi}\cos t\, f(x(t))\,dt, \\ h = A_{\mathrm{w}}\mathcal{Q}f(x(t)). \end{cases} \tag{26}$$

Consider the T-periodic function g, defined in Remark 2, formula (8). Since g is not identically zero and has zero mean value there exist values ξ_1 and ξ_2 such that $\xi_1 < \xi_2$ and $g(\xi_1)g(\xi_2) < 0$. Fix an $\varepsilon \in (0, 1/2)$, choose sufficiently large integer n (depending on ε), and put

$\xi_1^n = \xi_1 + Tn$, $\xi_2^n = \xi_2 + Tn$. Of course, $g(\xi_1^n)g(\xi_2^n) < 0$ for any integer n. Put $\Xi = [\xi_1^n, \xi_2^n]$, Lemmas 3.3−3.5 imply

$$\int_0^{2\pi} \sin t\, f(x(t))\, dt = \frac{2\sqrt{2\pi}}{\omega} g(\xi)\xi^{-1/2} + O(\xi^{-1/2-\varepsilon}),$$

and

$$\int_0^{2\pi} \cos t\, f(x(t))\, dt = O(\xi^{-1-\varepsilon}),$$

therefore for any $w \in \bar{\Omega}$ and $h \in \mathcal{H}$ for sufficiently large n we have the inequality

$$\Big(\int_0^{2\pi} \sin t\, f\, dt - Y(w)\int_0^{2\pi} \cos t\, f\, dt\Big)\Big|_{\xi=\xi_1^n} \cdot \Big(\int_0^{2\pi} \sin t\, f\, dt - Y(w)\int_0^{2\pi} \cos t\, f\, dt\Big)\Big|_{\xi=\xi_2^n} < 0;$$

and from (12) it follows that (again for sufficiently large integer n)

$$\Big(\Im(L(wi)M(-wi)) - \frac{|M(wi)|^2}{\pi\xi}\int_0^{2\pi} \cos t\, f\, dt\Big)\Big|_{w=w_1} \cdot \Big(\Im(L(wi)M(-wi)) - \frac{|M(wi)|^2}{\pi\xi}\int_0^{2\pi} \cos t\, f\, dt\Big)\Big|_{w=w_2} < 0.$$

These two inequalities play the role of conditions (22) and (23) for system (26). From Lemma 3.4 it follows the relation $A_w\mathcal{Q}f(x(t)) = o(1)$ as $\xi \to \infty$ in W_2^1. The operator $(w, \xi, h) \mapsto f(x)$ is completely continuous as an operator from $\Omega \times \Xi \times C$ to C. For sufficiently large n (i.e., for sufficiently large ξ) Lemma 3.2 is applicable to system (26) on the set $\Omega \times \Xi \times \{\|h\|_C \le \gamma\}$, therefore (26) has a solution. ■

4. Proofs of Lemmas 3.4 and 3.5.

4.1. **Proof of Lemma 3.4.** In the proof of Lemma 3.4 we use auxiliary statements (Lemma 4.1 and Lemma 4.2), their proofs are given in the end of this section.

Lemma 4.1. *For any $\gamma > 0$ for all $k = 0, 1, 2, \ldots$ and $\varphi \in \mathbb{R}$ the estimate*

$$\sup_{\|h\|_{C^1} \le \gamma} \Big| \int_0^{2\pi} e^{i(\xi \sin t + h(t))} \sin(kt + \varphi)\, dt \Big| \le \frac{20}{\sqrt{\xi}} + \frac{4(k+\gamma)\ln\xi}{\xi}, \qquad \xi \ge 1 \tag{27}$$

holds.

From (27) and the trivial relationship

$$\Big|\int_0^{2\pi} e^{i(\xi\sin t+h(t))}\sin(kt+\varphi)\,dt\Big|\le 2\pi \tag{28}$$

(it is valid for all h,k,φ) it follows that

$$\sup_{\|h\|_{C^1}\le\gamma}\Big|\int_0^{2\pi} e^{i(\xi\sin t+h(t))}\sin(kt+\varphi)\,dt\Big| \le \min\Big\{2\pi,\frac{20}{\sqrt{\xi}}+\frac{4(k+\gamma)\ln\xi}{\xi}\Big\} \tag{29}$$

for $\xi\ge 1$. Relation (29) is valid for all ξ,k,γ,φ, put there $\omega s\xi,\omega sh,\omega s\gamma$ instead of ξ,h,γ, where $\omega=\frac{2\pi}{T}>0$ is a real number defined in the beginning of Section 2, $s=1,2,\dots$ is positive integer. We see that the relation

$$\sup_{\|h\|_{C^1}\le\gamma}\Big|\int_0^{2\pi} e^{i\omega s(\xi\sin t+h(t))}\sin(kt+\varphi)\,dt\Big|\le Y(k,s,\xi,\gamma) \stackrel{\rm def}{=} \min\Big\{2\pi,\frac{20}{\sqrt{\omega s\xi}}+\frac{4(k+\gamma\omega s)\ln(\omega s\xi)}{\omega s\xi}\Big\}$$

holds for any non-negative integer s and real $\xi\ge 1$.

Put $\alpha_s(t)=\sin\big(\omega s(\xi\sin t+h(t))+\psi_s\big)$. According to (4) the function $f(\xi\sin t+h(t))$ can be represented as

$$f(\xi\sin t+h(t))=\sum_{s=1}^{\infty}\mu_s\alpha_s(t).$$

Let a_k,c_k,c_k' be the Fourier coefficients of the functions α_s, $H_s=A_{\rm w}\mathcal{Q}\alpha_s$, and $\frac{d}{dt}H_s$. Then $|a_k|\le Y(k,s,\xi,\gamma)$, $k=0,1,2,\dots$, therefore $|c_0|\le const\cdot\ln s\cdot(\omega s\,\xi)^{-1/2}$ and

$$|c_k|\le const\cdot Y(k,s,\xi,\gamma)\,k^{m-\ell},\ \ |c_k'|\le const\cdot Y(k,s,\xi,\gamma)\,k^{m-\ell+1}$$

for $k=1,2,\dots$ (see (15)). In particular, from $m-\ell+1\le -1$ it follows that

$$|c_k'|\le const\cdot Y(k,s,\xi,\gamma)k^{-1}. \tag{30}$$

Lemma 4.2. *Let $\ell\ge m+2$. For any $\varepsilon\in(0,\frac12)$ the estimate $\|H_s\|_{W_2^1}\le const(\varepsilon)\xi^{-\varepsilon}\ln s$ holds.*

Lemma 4.2 is proved in Subsection 4.4. Now $h\in\mathcal{H}$, the relation $h=\sum_{s=1}^{\infty}\mu_sH_s$ implies

$$\|h\|_{W_2^1}\le const\sum_{s=1}^{\infty}\mu_s\|H_s\|_{W_2^1}\le const\cdot\xi^{-\varepsilon}\sum_{s=1}^{\infty}\mu_s|\ln s|\le const\cdot\xi^{-\varepsilon},$$

and Lemma 3.4 follows from Condition 4 of Theorem 1. ■

4.2. **Proof of Lemma 3.5.**

Lemma 4.3. *For any ε there exists some $\beta = \beta(\varepsilon) > 0$ such that for any $\rho > 0$ the relation*

$$\sup_{\|h\|_{W_2^1} \le \rho\xi^{-\varepsilon}} \Big| \sqrt{\xi} \int_0^{2\pi} e^{i(\xi \sin t + h(t))} \sin t \, dt - \Delta(\xi) i \Big| \le \beta(1+\rho)\xi^{-\varepsilon} \quad (31)$$

holds, where $\Delta(\xi) = \Im\Big(e^{(\xi-\pi/4)i}\sqrt{2\pi} - e^{-(\xi-\pi/4)i}\sqrt{2\pi}\Big) = 2\sin(\xi - \pi/4)\sqrt{2\pi} = 2\sqrt{\pi}(\sin\xi - \cos\xi)$.

Lemma 4.3 is proved in the end of the paper.

Put in (31) the expressions $\omega s\xi, \omega s h$ instead of ξ and h and rewrite the obtained relation

$$\sup_{\|\omega s h\|_{W_2^1} \le \rho(\omega s\xi)^{-\varepsilon}} \Big| \sqrt{\omega s \xi} \int_0^{2\pi} e^{i\omega s(\xi \sin t + h(t))} \sin t \, dt - \Delta(\omega s\xi) i \Big|$$
$$\le \beta(1+\rho)(\omega s\xi)^{-\varepsilon},$$

in the form

$$\sup_{\|h\|_{W_2^1} \le \rho\xi^{-\varepsilon}} \Big| \sqrt{\xi} \int_0^{2\pi} e^{i\omega s(\xi \sin t + h(t))} \sin t \, dt - \frac{\Delta(\omega s\xi)}{\sqrt{\omega s}} i \Big|$$
$$\le \beta\sqrt{\omega s}(w^{-1-\varepsilon} + \rho)\xi^{-\varepsilon},$$

replacing $\rho(\omega s)^{-1-\varepsilon}$ by ρ. The last inequality can be rewritten as two real relationships

$$\sup_{\|h\|_{W_2^1} \le \rho\xi^{-\varepsilon}} \Big| \sqrt{\xi} \int_0^{2\pi} \cos(\omega s(\xi \sin t + h(t))) \sin t \, dt \Big| \le \beta\sqrt{\omega s}(\omega^{-1-\varepsilon} + \rho)\xi^{-\varepsilon},$$

$$\sup_{\|h\|_{W_2^1} \le \rho\xi^{-\varepsilon}} \Big| \sqrt{\xi} \int_0^{2\pi} \sin(\omega s(\xi \sin t + h(t))) \sin t \, dt - \frac{\Delta(\omega s\xi)}{\sqrt{\omega s}} \Big|$$
$$\le \beta\sqrt{\omega s}(\omega^{-1-\varepsilon} + \rho)\xi^{-\varepsilon},$$

therefore,

$$\sup_{\|h\|_{W_2^1} \le \rho\xi^{-\varepsilon}} \Big| \sqrt{\xi} \int_0^{2\pi} \sin(\omega s(\xi \sin t + h(t)) + \psi_s) \sin t \, dt - \frac{\cos\psi_s \Delta(\omega s\xi)}{\sqrt{\omega s}} \Big|$$
$$\le \beta\sqrt{\omega s}(\omega^{-1-\varepsilon} + \rho)\xi^{-\varepsilon}.$$

If we sum the obtained inequalities for various $s = 1, 2, \dots$ with the coefficients μ_s, we obtain

$$\sup_{\|h\|_{W_2^1} \le \rho\xi^{-\varepsilon}} \Big| \sqrt{\xi} \int_0^{2\pi} f(\xi \sin t + h(t)) \sin t \, dt - \sum_{s=1}^{\infty} \mu_s \frac{\cos\psi_s \Delta(\omega s \xi)}{\sqrt{\omega s}} \Big| \le \beta\sqrt{\omega}(\omega^{-1-\varepsilon} + \rho)\xi^{-\varepsilon} \sum_{s=1}^{\infty} \mu_s \sqrt{s}.$$

The definition of the function g:

$$\sum_{s=1}^{\infty} \frac{\mu_s \cos\psi_s \Delta(\omega s \xi)}{\sqrt{\omega s}} = 2\sqrt{\frac{2\pi}{\omega}} g(s)$$

proves the first part of Lemma 3.5. The second part (25) follows from the identities

$$\int_0^{2\pi} f(\xi \sin t + h(t)) \cos t \, dt$$
$$= \frac{1}{\xi} \int_0^{2\pi} f(\xi \sin t + h(t)) \, d(\xi \sin t + h(t)) - \frac{1}{\xi} \int_0^{2\pi} h'(t) f(\xi \sin t + h(t)) \, dt$$

and

$$\int_0^{2\pi} f(x(t)) \, x'(t) \, dt \equiv 0,$$

and the estimates

$$\Big| \int_0^{2\pi} f(\xi \sin t + h(t)) \cos t \, dt \Big| = \Big| \frac{1}{\xi} \int_0^{2\pi} h'(t) f(\xi \sin t + h(t)) \, dt \Big| \le \frac{\sqrt{2\pi} \sup|f|}{\xi} \|h'\|_{L^2} \le \frac{\sqrt{2\pi}\rho \sup|f|}{\xi^{1+\varepsilon}}.$$

Lemma 3.5 is completely proved. ■

4.3. **Proof of Lemma 4.1.** Put

$$q(t) = \sin(kt+\varphi)e^{ih(t)}, \ \ q'(t) = k\cos(kt+\varphi)e^{ih(t)} + i\sin(kt+\varphi)e^{ih(t)}h'(t);$$

obviously, $\|q\|_C \le 1$, $\|q_t'\|_C \le k + \gamma$. Let us estimate the value

$$I(\xi) = I^{\xi}_{(0,\pi/2)} = \int_0^{\pi/2} e^{i(\xi \sin t + h(t))} \sin(kt + \varphi) \, dt = \int_0^{\pi/2} e^{i\xi \sin t} q(t) dt,$$

analogous integrals $I^{\xi}_{(\pi/2,\pi)}$, $I^{\xi}_{(\pi,3\pi/2)}$, and $I^{\xi}_{(3\pi/2,2\pi)}$ along the corresponding intervals (the function $\sin t$ is monotone on each such interval) can be considered with the use of the same scheme. After the change of variables $v = \sin t$ in the integral $I(\xi)$ we have

$$I(\xi) = \int_0^1 e^{iv\xi} W(v)dv, \qquad W(v) = \frac{q(\arcsin v)}{\sqrt{1-v^2}}.$$

The function W is continuous on $[0,1)$, $|W(v)| \le 1/\sqrt{1-v}$ and

$$|W'(v)| \le \Big|\frac{q'(\arcsin v)}{1-v^2}\Big| + \Big|\frac{v\, q(\arcsin v)}{(\sqrt{1-v^2})^3}\Big| \le \frac{k+\gamma}{1-v} + \frac{1}{\sqrt{(1-v)^3}}\,.$$

Furthermore, $I(\xi) = I_1(\xi) + I_2(\xi)$;

$$I_1(\xi) = \int_0^{1-\xi^{-1}} e^{i\xi v} W(v)\, dv, \quad I_2(\xi) = \int_{1-\xi^{-1}}^1 e^{i\xi v} W(v)\, dv.$$

Now let us estimate the integrals I_1 and I_2 separately. First of all

$$|I_2(\xi)| = \Big|\int_{1-\xi^{-1}}^1 e^{i\xi v} W(v)\, dv\Big| \\ \le \int_{1-\xi^{-1}}^1 \frac{dv}{\sqrt{1-v}} = -2\sqrt{1-v}\,\Big|_{1-\xi^{-1}}^1 = \frac{2}{\sqrt{\xi}},$$

then

$$\begin{aligned} |I_1(\xi)| &= \Big|\int_0^{1-\xi^{-1}} e^{i\xi v} W(v)\, dv\Big| = \frac{1}{\xi}\Big|\int_0^{1-\xi^{-1}} W(v)\, d(e^{i\xi v})\Big| \\ &\le \Big|\Big(\frac{e^{i\xi v}W(v)}{\xi}\Big|_0^{1-\xi^{-1}}\Big)\Big| + \frac{1}{\xi}\Big|\int_0^{1-\xi^{-1}} e^{i\xi v}\, W'_v(v)\, dv\Big| \\ &\le \frac{1}{\xi} + \frac{1}{\sqrt{\xi}} + \frac{1}{\xi}\int_0^{1-\xi^{-1}} \frac{k+\gamma}{1-v}\, dv + \frac{1}{\xi}\int_0^{1-\xi^{-1}} \frac{dv}{\sqrt{(1-v)^3}} \\ &= \frac{1}{\xi} + \frac{1}{\sqrt{\xi}} + \frac{(k+\gamma)\ln\xi}{\xi} + \frac{2}{\xi\sqrt{1-v}}\Big|_0^{1-\xi^{-1}} \\ &\le \frac{1}{\xi} + \frac{1}{\sqrt{\xi}} + \frac{(k+\gamma)\ln\xi}{\xi} + \frac{2\sqrt{\xi}}{\xi} - \frac{2}{\xi} \le \frac{3}{\sqrt{\xi}} + \frac{(k+\gamma)\ln\xi}{\xi}\,. \end{aligned}$$

Combining the obtained estimate for $I^{\xi}_{(0,\pi/2)}$ with the same estimates for the integrals $I^{\xi}_{(\pi/2,\pi)}$, $I^{\xi}_{(\pi,3\pi/2)}$, and $I^{\xi}_{(3\pi/2,2\pi)}$ we have (27). ∎

4.4. **Proof of Lemma 4.2.** From Parseval's Formula

$$\|H_s'\|_{L^2}^2 = \sum_{k=1}^{\infty} |c_k'|^2$$

and (30) it follows the estimate

$$\|H_s'\|_{L^2}^2 \le const \sum_{k=1}^{\infty} |Y(k,s,\xi,\gamma)k^{-1}|^2.$$

Split for any ξ the last series into two parts: a finite part for[4] $k \le [\xi]$ and an infinite rest part for $k > [\xi]$. Then

$$\sum_{k=[\xi]+1}^{\infty} \Big(\frac{Y(k,s,\xi,\gamma)}{k}\Big)^2 \le \sum_{k=[\xi]+1}^{\infty} \frac{4\pi^2}{k^2} \le \sum_{k=[\xi]+1}^{\infty} \frac{4\pi^2}{([\xi]+1)^{2\varepsilon} k^{2-2\varepsilon}} \le \frac{1}{\xi^{2\varepsilon}} \sum_{k=1}^{\infty} \frac{4\pi^2}{k^{2-2\varepsilon}}$$

($\varepsilon \in (0, \frac{1}{2}) \Rightarrow 2 - 2\varepsilon > 1$) and, since $(3-2\varepsilon)/4 + \varepsilon + (1/2 - \varepsilon)/2 = 1$ and $\varepsilon \in (0, \frac{1}{2}) \Rightarrow (3-2\varepsilon)/4 > \frac{1}{2}$,

$$\begin{aligned}
\sum_{k=1}^{[\xi]} \Big(\frac{Y(k,s,\xi,\gamma)}{k}\Big)^2 &\le \sum_{k=1}^{[\xi]} \left(\frac{20}{k\sqrt{\omega s\xi}} + \frac{4(k+\gamma\omega s)\ln(\omega s\xi)}{k\omega s\xi}\right)^2 \\
&\le \sum_{k=1}^{[\xi]} \left(\frac{20}{k\sqrt{\omega s\xi}} + \frac{4(k+\gamma\omega s)}{k\omega s}\,\frac{\ln(\omega s\xi)}{\xi^{(1/2-\varepsilon)/2}}\,\frac{1}{k^{1-\varepsilon-(1/2-\varepsilon)/2}\xi^{\varepsilon}}\right)^2 \\
&\le \sum_{k=1}^{\infty} \left(\frac{20}{k\sqrt{\omega s\xi}} + \frac{4(k+\gamma\omega s)}{k\omega s}\,\frac{\ln(\omega s\xi)}{\xi^{(1/2-\varepsilon)/2}}\,\frac{1}{k^{(3-2\varepsilon)/4}\xi^{\varepsilon}}\right)^2 \\
&\le \frac{\ln^2 s}{\xi^{2\varepsilon}} \sum_{k=1}^{\infty} \left(\frac{20}{k} + \frac{const}{k^{(3-2\varepsilon)/4}}\right)^2 = \frac{c\ln^2 s}{\xi^{2\varepsilon}}.
\end{aligned}$$

We proved the estimate $\|H_s'\|_{L^2} \le c(\varepsilon)\xi^{-\varepsilon}\ln s$, it implies $\|H_s'\|_{L^1} \le \tilde{c}_1(\varepsilon)\xi^{-\varepsilon}\ln s$. Any continuous periodic function H_s always takes its mean value c_0 that is its zero harmonics, it satisfies $|c_0| \le \tilde{c}(\varepsilon)\,\xi^{-\varepsilon}\ln s$, let $H_s(t_0) = c_0$. The estimates for the values $\|H_s\|_C$ follow from

$$|H_s(t)| \le |c_0| + \int_{t_0}^{t} |H_s'(t)|\,dt,$$

therefore, $\|H_s\|_C \le \big(\tilde{c}(\varepsilon) + \tilde{c}_1(\varepsilon)\big)\,\xi^{-\varepsilon}\ln s$. ∎

[4]We denote the integer part as $[\cdot]$.

4.5. **Proof of Lemma 4.3.** Let us slightly change (compare with the proof of Lemma 4.1) some denominations:

$$q(t) = \sin t\, e^{ih(t)}, \quad q'(t) = \cos t\, e^{ih(t)} + i \sin t\, e^{ih(t)} h'(t).$$

Obviously, $\|q\|_C \le 1$, $\|q'\|_C \le 1+\gamma$. Consider in detail the integral

$$I(\xi) = \int_0^{\pi/2} e^{i(\xi \sin t + h(t))} \sin t\, dt = \int_0^{\pi/2} e^{i\xi \sin t} q(t)\, dt,$$

the integrals

$$\int_{\pi/2}^{\pi} \dots, \quad \int_{\pi}^{3\pi/2} \dots, \quad \int_{3\pi/2}^{2\pi} \dots$$

can be considered in a similar way. The statement of Lemma 4.3 follows from the relations

$$I(\xi) = \int_{\pi/2}^{\pi} \dots = \sqrt{\frac{\pi}{2\xi}}\, e^{(\xi - \frac{\pi}{4})i} + O(\xi^{-\frac{1}{2}-\varepsilon}),$$

$$\int_{\pi}^{3\pi/2} \dots = \int_{3\pi/2}^{2\pi} \dots = -\sqrt{\frac{\pi}{2\xi}}\, e^{-(\xi - \frac{\pi}{4})i} + O(\xi^{-\frac{1}{2}-\varepsilon}),$$

we prove the first one (concerning $I(\xi)$) only. Put $v = \sin t$ in the integral $I(\xi)$:

$$I(\xi) = \int_0^1 e^{iv\xi} W(v) dv, \quad W(v) = \frac{e^{ih(\arcsin(v))} v}{\sqrt{1-v^2}}\,.$$

The function W is continuous on $[0,1)$,

$$|W(v)| \le \frac{1}{\sqrt{1-v}}\,,$$

$$|W'(v)| \le \Big|\frac{e^{iq(\dots)}}{\sqrt{1-v^2}}\Big| + \Big|\frac{e^{iq(\dots)} h'(\dots) v}{1-v^2}\Big| + \Big|\frac{e^{iq(\dots)} v^2}{2(\sqrt{1-v^2})^3}\Big| \le \frac{2+\gamma}{\sqrt{(1-v)^3}}\,.$$

Obviously,

$$I(\xi) = J(\xi) + \int_0^1 e^{iv\xi} U(v) dv$$

where

$$J(\xi) = \int_0^1 e^{iv\xi} \frac{dv}{\sqrt{2(1-v)}},\ U(v) = W(v) - \frac{1}{\sqrt{2(1-v)}}.$$

The value $J(\xi)$ contains the principal term that can be computed explicitly:

$$J(\xi) = \frac{e^{i\xi}}{\sqrt{2}} \int_0^1 \frac{e^{-iu\xi}du}{\sqrt{u}} = \frac{e^{i\xi}}{\sqrt{2}} \overline{\int_0^1 \frac{e^{iu\xi}du}{\sqrt{u}}}$$
$$= \frac{e^{i\xi}}{\sqrt{2}} \overline{\int_0^\infty \frac{e^{iu\xi}du}{\sqrt{u}}} - \frac{e^{i\xi}}{\sqrt{2}} \overline{\int_1^\infty \frac{e^{iu\xi}du}{\sqrt{u}}}.$$

Now Lemma 12.1 from [9] (page 100, formula (12.01)) implies

$$\int_0^\infty \frac{e^{iu\xi}du}{\sqrt{u}} = \frac{1}{\sqrt{\xi}} \int_0^\infty \frac{e^{iu}du}{\sqrt{u}} = \frac{e^{\frac{\pi i}{4}}\Gamma(\frac{1}{2})}{\sqrt{\xi}} = \frac{e^{\frac{\pi i}{4}}\sqrt{\pi}}{\sqrt{\xi}}.$$

Obviously

$$\Big| \int_1^\infty \frac{e^{iu\xi}du}{\sqrt{u}} \Big| = \frac{1}{\xi} \Big| \int_1^\infty \frac{de^{iu\xi}}{\sqrt{u}} \Big| \le \frac{1}{\xi} + \frac{1}{2\xi} \int_1^\infty \frac{du}{\sqrt{u^3}} \le \frac{1}{\xi} + \frac{1}{\xi} = \frac{2}{\xi}.$$

Therefore,

$$\Big| J(\xi) - \sqrt{\frac{\pi}{2\xi}}\, e^{(\xi - \frac{\pi}{4})i} \Big| \le \frac{\sqrt{2}}{\xi}.$$

Now eliminate the principal term from the function $U(v)$:

$$U(v) = \frac{e^{ih(\arcsin(v))}v}{\sqrt{1-v^2}} - \frac{1}{\sqrt{2(1-v)}} = E(v) - \frac{(1 - e^{ih(\arcsin(v))})v}{\sqrt{1-v^2}}.$$

The function

$$E(v) \overset{\text{def}}{=} \frac{v}{\sqrt{1-v^2}} - \frac{1}{\sqrt{2(1-v)}} = \frac{v\sqrt{2} - \sqrt{1+v}}{\sqrt{2(1-v^2)}}$$
$$= \frac{(v\sqrt{2} + \sqrt{1+v})(v\sqrt{2} - \sqrt{1+v})}{(v\sqrt{2} + \sqrt{1+v})\sqrt{2(1-v^2)}}$$
$$= \frac{(v-1)(2v+1)}{(v\sqrt{2} + \sqrt{1+v})\sqrt{2(1-v^2)}} = \frac{\sqrt{1-v}(2v+1)}{(v\sqrt{2} + \sqrt{1+v})\sqrt{2(1+v)}}$$

is continuous on $[0,1]$, the derivative E' is continuous on $[0,1)$, $|E'(v)| \approx const/\sqrt{1-v}$ at $v=1$, therefore $\int_0^1 |E'(v)|dv = E_0 < \infty$. Now we have

$$\Big| \int_0^1 e^{iv\xi} E(v)\, dv \Big| = \Big| \frac{1}{i\xi} \int_0^1 E(v)\, d\left(e^{iv\xi}\right) \Big|$$
$$\le \frac{|E(0)|}{\xi} + \frac{1}{\xi} \Big| \int_0^1 E'(v)\, e^{iv\xi} dv \Big| \le \frac{|E(0)| + E_0}{\xi}.$$

Consider the integral

$$J_0(\xi) = \int_0^1 e^{iv\xi} \frac{(1 - e^{ih(\arcsin(v))})v\, dv}{\sqrt{1-v^2}} = \int_0^{1-\xi^{-1}} \dots + \int_{1-\xi^{-1}}^1 \dots$$

It is simple to estimate the second term here: since[5] $|1-e^{ir}|\le |r|$, we have

$$\Big|\int_{1-\xi^{-1}}^{1} e^{iv\xi}\frac{(1-e^{ih(\arcsin(v))})v\,dv}{\sqrt{1-v^2}}\Big| \le \|h\|_C\int_{1-\xi^{-1}}^{1}\frac{dv}{\sqrt{1-v}} = 2\|h\|_C\sqrt{\xi^{-1}} \le 2\rho\xi^{-\frac12-\varepsilon}.$$

To estimate the first term, let us firstly estimate the values

$$\begin{aligned}
G_1 &= \Big|\int_0^{1-\xi^{-1}} e^{iv\xi}\left(\frac{(-e^{ih(\arcsin(v))})vh'(\arcsin(v))}{1-v^2}\,dv\right)\Big| \\
&\le \int_0^{1-\xi^{-1}}\Big|\frac{h'(\arcsin(v))}{1-v^2}\Big|\,dv \\
&\le \int_0^{1-\xi^{-1}}\left(\max_{v\in[0,\xi^{-1}]}\frac{1}{\sqrt{1-v^2}}\right)\ \Big|\frac{h'(\arcsin(v))}{\sqrt{1-v^2}}\Big|\,dv \\
&\le \sqrt{\xi}\int_0^{1-\xi^{-1}}\Big|\frac{h'(\arcsin(v))}{\sqrt{1-v^2}}\Big|\,dv \le \sqrt{\xi}\int_0^1\frac{|h'(\arcsin(v))|}{\sqrt{1-v^2}}\,dv \\
&= \sqrt{\xi}\int_0^{\pi/2}|h'(t)|\,dt \le \sqrt{\frac{\pi}{2}}\sqrt{\xi}\|h'\|_{L^2} \le \sqrt{\frac{\pi}{2}}\rho\xi^{\frac12-\varepsilon},
\end{aligned}$$

$$\begin{aligned}
G_2 &= \Big|\int_0^{1-\xi^{-1}} e^{iv\xi}\frac{(1-e^{ih(\arcsin(v))})v^2}{\sqrt{(1-v^2)^3}}\,dv\Big| \\
&\le \|h\|_C\int_0^{1-\xi^{-1}}\frac{dv}{\sqrt{(1-v)^3}} = \frac12\|h\|_C\sqrt{\xi} \le \frac12\rho\xi^{\frac12-\varepsilon},
\end{aligned}$$

$$\begin{aligned}
G_3 &= \Big|\int_0^{1-\xi^{-1}} e^{iv\xi}\frac{(1-e^{ih(\arcsin(v))})}{\sqrt{1-v^2}}\,dv\Big| \\
&\le \|h\|_C\int_0^{1-\xi^{-1}}\frac{dv}{\sqrt{1-v}} = 2\|h\|_C \le 2\rho\xi^{-\varepsilon}.
\end{aligned}$$

Now

$$\begin{aligned}
\Big|\int_0^{1-\xi^{-1}} e^{iv\xi}\frac{(1-e^{ih(\arcsin(v))})v\,dv}{\sqrt{1-v^2}}\Big| &\le \frac1\xi\Big|\int_0^{1-\xi^{-1}}\frac{(1-e^{ih(\arcsin(v))})v}{\sqrt{1-v^2}}\,d\big(e^{iv\xi}\big)\Big| \\
&\le \frac{\|h\|_C}{\sqrt{\xi}}+\frac1\xi\Big|\int_0^{1-\xi^{-1}}e^{iv\xi}\frac{d}{dv}\left(\frac{(1-e^{ih(\arcsin(v))})v}{\sqrt{1-v^2}}\right)\,dv\Big| \\
&\le \frac{\|h\|_C}{\sqrt{\xi}}+\frac1\xi(G_1+G_2+G_3) \le 5\rho\xi^{-\frac12-\varepsilon}.
\end{aligned}$$

[5] An arc is always longer than its chord.

We proved that $J_0(\xi) \leq 7\rho\xi^{-\frac{1}{2}-\varepsilon}$, by construction

$$I(\xi) = J(\xi) + \int_0^1 e^{iv\xi} E(v)\, dv - J_0(\xi).$$

Lemma 4.3 is proved. ■

Acknowledgments. The author is supported by the Russian Foundation for Basic Researches, Grant 10-01-93112.

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Received xxxx 20xx; revised xxxx 20xx.

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