Double Exponential Instability of Triangular Arbitrage Systems

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Abstract

If financial markets displayed the informational efficiency postulated in the efficient markets hypothesis (EMH), arbitrage operations would be self-extinguishing. The present paper considers arbitrage sequences in foreign exchange (FX) markets, in which trading platforms and information are fragmented. In Kozyakin et al. (2010) and Cross et al. (2012) it was shown that sequences of triangular arbitrage operations in FX markets containing 4 currencies and trader-arbitrageurs tend to display periodicity or grow exponentially rather than being self-extinguishing. This paper extends the analysis to 5 or higher-order currency worlds. The key findings are that in a 5-currency world arbitrage sequences may also follow an exponential law as well as display periodicity, but that in higher-order currency worlds a double exponential law may additionally apply. There is an "inheritance of instability" in the higher-order currency worlds. Profitable arbitrage operations are thus endemic rather that displaying the self-extinguishing properties implied by the EMH.

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1. Introduction

Arbitrage operations are profitable if a good or asset can be bought for a lower price than that for which it can be sold. Such operations are distinct from speculation in that there is little or no capital or risk exposure. Arbitrage profits are made by trading on prices that are already posted, rather than by trading on speculative guesses about what prices posted in the future might be. The *New Palgrave Dictionary* defines arbitrage as "an investment strategy that guarantees a positive payoff in some contingency with no possibility of negative payoff and with no net investment", Dybvig and Ross (2008).

In the economics and finance literature the exploitation of profitable arbitrage opportunities is postulated to eliminate price discrepancies between goods and assets that are in some sense "identical." This postulate, that arbitrage operations are self-extinguishing, implies the "law of one price", this terminology having been coined in the purchasing power parity explanation of foreign exchange rates, Cassel (1916). A related no-arbitrage condition is that of covered interest parity (CIP), whereby the ratio of forward to spot exchange rates for currency pairs is equal to the ratio of the interest rates on comparable assets denominated in the two currencies over the forward period in question (Keynes, 1923, p. 13). No-arbitrage condition form the bedrock of mainstream theory in finance, as embodied in the Modigliani–Miller theorem regarding corporate capital structure, the Black–Scholes model of option pricing and in the arbitrage pricing model of asset prices, Ross (1978).

The present paper deals with arbitrage processes in foreign exchange (FX) markets. The Bank for International Settlements (BIS, 2011, p. 5) provides a useful classification of arbitrage strategies in such markets. *Classical arbitrage* aims to exploit discrepancies between actual exchange rates and no-arbitrage conditions, such as that the exchange rates for currency pairs are consistent with their cross-exchange rates. *Latency arbitrage* aims to exploit time lags between trades being initiated and FX price quotes being revised. *Liquidity imbalance arbitrage* attempts to take advantage of order book imbalances between different trading platforms. *Complex event arbitrage* is geared towards properties of FX rates, such as mean-reversion and momentum, or towards systematic patterns in the way FX response to new information being released. This paper focuses on triangular arbitrage operations, arising from discrepancies between the exchange rates for currency pairs and the cross-exchange rates for the currencies

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involved. To simplify the analysis we ignore the difference between FX bid and ask prices and do not take account of the cost of executing FX transactions. For evidence that there are arbitrage profits to be made, net of bid-ask spreads and transactions costs, see Marshall et al. (2007) and Akram et al. (2008).

If the efficient market hypothesis (EMH), formulated by Samuelson (1965) and Fama (1970), applied to FX markets, exchange rates would reflect all the information relevant to their determination and there would be no profits to be had from FX arbitrage trades. This would involve an "arbitrage paradox" (Grossman and Stiglitz, 1976): if arbitrage opportunities did not exist, there would be no incentive for FX market participants to monitor exchange rates, so profitable arbitrage opportunities could well arise. There is also the "no trade" problem. If FX markets were informationally efficient, as the EMH postulates, you would expect to observe periods in which FX transactions volumes dwindled when no new information was being generated. Instead FX transactions volumes remain substantial, even during quiescent market conditions.

The microstructure of FX markets is fragmented, implying that information about profitable arbitrage opportunities is also likely to be fragmented. The contrast is between the "lit" areas where information on some of the electronic trading platforms is publicly available, and the "dark pools", where information on FX transactions conducted directly between banks and their end-user clients is initially private. BIS data for 2010 document this FX market fragmentation. Electronic booking systems (EBS), such as the London-based EBS and Reuters, took 18.8% of global turnover; multi-bank electronic communication networks (ECNs), such as the US-based Currenex, Hotspot FX and FXall, accounted for 11.1% of global turnover; and single-bank ECNs took 11.4% of global turnover. Inter-dealer trades, most of which are executed electronically, constituted 18.5% of turnover. The "dark pools" tend to be in the non-electronic execution segments of the market, with customer-direct transactions between banks and end-user clients taking 24.4% of global turnover, and voice-broker trades accounting for 15.9% of global turnover (BIS, 2010, p. 16).

There is also evidence of "home bias" in FX transactions. FX trades initiated in the US and Canada tend to take place during North American trading hours, those initiated in Japan and Australia tend to occur during Asian trading hours, and so on (D'Souza, 2008, Table 2). Estimation of impulse-response functions for the way exchange rates respond to local order flow data indicate that "dealers operating both at the same time and in the same geographic region as fundamentally driven customers have a natural informational advantage" (D'Souza, 2008, pp. 23–24). In a similar vein, Covrig and Melvin (2002) found that Tokyo-based traders had an informational advantage over foreign-based traders regarding the course of the Japanese yen exchange rates.

In previous papers, Kozyakin et al. (2010) and Cross et al. (2012), the implications of this fragmentation of information on FX markets were analysed by a combinatorial analysis of the different possible sequences of arbitrage operations. The "home bias" asymmetry in information is represented by having FX trader-arbitrageurs initially know only the exchange rates involving their own domestic currency. The question is then whether triangular arbitrage transactions are profitable because the cross-exchange rates are misaligned in relation to the exchange rates for the currency pairs. The arbitrage sequences that will be pursued would depend on which trader-arbitrageur first discovers the mis-alignment of the cross-exchange rates. The arbitrage operations are triangular, so, for example, if the US dollar trader were to discover that the euro–sterling rate was out of line with exchange rates for the dollar–euro and dollar–sterling currency pairs, an arbitrage profit could be made by selling dollars for sterling and using the sterling to buy euros.

In the 3-currency case arbitrage operations are reasonable straightforward. The order in which trader-arbitrageurs discover information about cross-exchange rate discrepancies makes a difference to the arbitrage sequences that will be pursued, and to the resulting new no-arbitrage ensemble of exchange rates that will emerge, but one arbitrage transaction suffices to eliminate the arbitrage opportunity. The 4-currency case is significantly more complicated, there being ${}^{4}P_{3} = 24$ possible triangular arbitrage operations. In Kozyakin et al. (2010) and Cross et al. (2012) it was shown that in this 4-currency world arbitrage sequences tend to be periodic in nature or display exponential behaviour, showing no tendency to approach a no-arbitrage ensemble of exchange rates in which there are no profitable opportunities. The intuitive explanation for this periodicity is that the exploitation of one arbitrage opportunity has "ripple effects," disturbing the ensemble of exchange rates and so creating further active arbitrage opportunities.

The task of the present paper is to extend the analysis of FX arbitrage operations to a world of d-currencies, where d is the number of currencies and trade-arbitrageurs involved. One key finding is that in the $(d \geq 5)$-currency case the surprising result is that the arbitrage sequences may follow a double exponential process as well as the periodicity or exponential behaviour observed in the 4-currency case. There is thus an "inheritance of instability" as we move to higher-order currency worlds.

The structure of the paper is as follows. In Section 2 a mathematical formulation of the problem is given. In Section 3 it is shown that the arbitrage dynamics, initially specified in terms of some nonlinear operations, can be described by asynchronous matrix products. For the general case of an arbitrary number of trader-arbitrageurs the properties of the resulting matrices are studied in Section 4. The exposition of Sections 2 and 3 follows the work of Kozyakin et al. (2010). In Section 5 basic results from Kozyakin et al. (2010) and Cross et al. (2012) about the arbitrage dynamics for an FX currency market with 4 traders are recalled. In particular, these results demonstrate that the exchange rates in a foreign exchange currency market with 4 traders, under appropriate choice of the arbitrage sequences, display periodic or exponential behaviour. In Section 6 we construct an example showing that in the case of 5 or more arbitrage traders the situation may be, in a sense, even worse — in this case the exchange rates may change not only periodically or grow exponentially, as in the case of 4 traders, but they may grow in accordance with a double exponential law. In Appendix A and Appendix B an explicit form for all the matrices involved in the description of the arbitrage dynamics for the cases $d = 4$ and $d = 5$, respectively, is presented.

2. Statement of the Problem

Consider a foreign exchange (FX) currency market involving d currencies involving the exchange rates r_{ij} of the i-th currency to the j-th currency, $j \neq i$. Naturally, the value of r_{ji}, the exchange rate of the j-th currency to the i-th currency, is reciprocal to the value of r_{ij}:

$$r_{ji} = \frac{1}{r_{ij}}.$$

The exchange rates vary with time depending on the state of the market and on the relations between the exchange rates. For instance, if at some moment the trader of the currency i realises that exchange rate of the i-th currency to the j-th currency, via the intermediate currency k, may bring a profit, i.e.,

$$r_{ik} \cdot r_{kj} > r_{ij}, \quad i \neq j, \ k \neq i, j,$$

then he can establish a new exchange rate for the currency

$$r_{ij,\text{new}} = r_{ik} \cdot r_{kj}.$$

So, in what follows, we will suppose that the exchange rates in our arbitrage system are updated in accordance with the following law:

$$r_{ij,\text{new}} = \max\left\{r_{ik} \cdot r_{kj}, \ r_{ij,\text{old}}\right\},$$
$$r_{ji,\text{new}} = \frac{1}{r_{ij,\text{new}}},$$

and simultaneously the exchange rates for only one pair (i, j) of currencies may be updated.

By introducing the auxiliary quantities

$$a_{ij} = \log r_{ij}, \quad \forall i \neq j,$$

it is possible to pass from the "multiplicative" statement of the problem about arbitrage dynamics given above to the "additive" statement of the problem, which in this case will look as follows. Given a skew-symmetric $d \times d$ matrix $A = (a_{ij})$, for a triplet of pairwise distinct indices (i, j, k), $i \neq j$, $k \neq i, j$, the elements a_{ij} and a_{ji} are updated in accordance with the following law:

$$a_{ij,\text{new}} = \max\left\{a_{ik} + a_{kj}, \ a_{ij,\text{old}}\right\}, \tag{1}$$
$$a_{ji,\text{new}} = -a_{ij,\text{new}}. \tag{2}$$

The triplet of indices $\omega = (i, j, k)$, $i \neq j$, $k \neq i, j$, will be called the *arbitrage rule*.

3. Linear Reformulation

Since by (1) and (2) the element a_{ij} of the matrix A is updated "simultaneously" with the updating of the symmetric element a_{ji}, then it is reasonable to speak about updating of the pair (a_{ij}, a_{ji}). Then, according to (1) and (2), the pair of elements (a_{ij}, a_{ji}) can be updated by one of two following scenarios. Either at first the element a_{ij} is updated in accordance with (1), and then the element a_{ji} is updated in accordance with (2), or at first the element a_{ji} is updated by the formula

$$a_{ji,\text{new}} = \max\left\{a_{jk} + a_{ki}, \ a_{ji,\text{old}}\right\},$$

which, in view of the skew-symmetry of the matrix A, leads to the formula

$$a_{ji,\text{new}} = \max\left\{-a_{kj} - a_{ik},\ \ -a_{ij,\text{old}}\right\} = -\min\left\{a_{ik} + a_{kj},\ a_{ij,\text{old}}\right\},$$

and then the element a_{ij} is updated as follows

$$a_{ij,\text{new}} = -a_{ji,\text{new}},$$

which amounts to the final expression for $a_{ij,\text{new}}$:

$$a_{ij,\text{new}} = \min\left\{a_{ik} + a_{kj},\ a_{ij,\text{old}}\right\}. \tag{3}$$

Formulae (1) and (3) show that in the process of updating of the pair (a_{ij}, a_{ji}) the element a_{ij}, independently of the scenario of updating, either is not changed (and then the matrix A is not changed, too) or this element may change its value as follows:

$$a_{ij,\text{new}} = a_{ik} + a_{kj}, \tag{4}$$

whereas the symmetric element a_{ji} changes its value in accordance with (2) which is equivalent to

$$a_{ji,\text{new}} = a_{jk} + a_{ki}. \tag{5}$$

So, the following lemma is proved.

Lemma 1. *If the arbitrage law* (1) *and* (2) *holds then any pair of elements* (a_{ij}, a_{ji}) *either is not changed or is changed by the* ***linear law*** (4), (5)*.*

Clearly, the converse assertion is also true.

Lemma 2. *If a pair of elements* (a_{ij}, a_{ji}) *were to be updated by the law* (4), (5)*, i.e.,*

$$(a_{ij}, a_{ji}) \mapsto (a_{ij,\text{new}}, a_{ji,\text{new}})$$

then one of two scenarios of updating of the pair of elements (a_{ij}, a_{ji}) *can be selected (at first updating* a_{ij} *by formula* (1) *and then adjusting* a_{ji} *by the formula* (2)*; or at first updating* a_{ji} *by the formula* (1) *and then adjusting* a_{ij} *by the formula* (2)*), under which the same pair of new elements* $(a_{ij,\text{new}}, a_{ji,\text{new}})$ *will be obtained. Namely,*

- *if* $a_{ij,\text{new}} > a_{ij}$ *after updating the pair* (a_{ij}, a_{ji}) *then, to interpret such an updating by the arbitrage law* (1) *and* (2)*, one should first apply formula* (1) *and then formula* (2)*;*
- *if* $a_{ij,\text{new}} < a_{ij}$ *after updating the pair* (a_{ij}, a_{ji}) *then, to interpret such an updating by the arbitrage law* (1) *and* (2)*, one should first apply formula* (2) *and then formula* (1)*;*
- *and last, if* $a_{ij,\text{new}} = a_{ij}$ *then the order of updating the elements of the pair* (a_{ij}, a_{ji})*, as well as whether such an updating took place at all, is inessential.*

The previous lemmata indicate that under the max-statement of the problem, as well as under its linear reformulation, it suffices to investigate only the dynamics of the pairs (a_{ij}, a_{ji}) of mutually symmetric elements since such an analysis *allows us to reproduce also the scenarios of intra-pair updating of the elements* a_{ij} *under the max-statement of the problem.*

4. Asynchronous Matrix Products

Define in the space of all skew-symmetric $d \times d$ matrices $A = (a_{ij})$ a basis $\{\boldsymbol{e}_{ij}\}$, $1 \le i < j \le d$, by enumerating in some order the upper off-diagonal elements of such matrices. For example, let us define the element $\boldsymbol{e}_{ij}$, $1 \le i < j \le d$, of this basis as the skew-symmetric matrix whose element a_{ij} is equal to 1 and the others vanish. Then each matrix in this basis can be represented as a column-vector:

$$\boldsymbol{x} = \left\{a_{12}, a_{13}, \ldots, a_{1d}, a_{23}, a_{24}, \ldots\ a_{2d}, \ldots, a_{d-1,d}\right\}^{T} \in \mathbb{R}^{d(d-1)/2}, \tag{6}$$

and computation of the new vector $\boldsymbol{x}_{\text{new}}$ by the old one $\boldsymbol{x}_{\text{old}}$ will be defined by the expression

$$\boldsymbol{x}_{\text{new}} = B_{\omega}\boldsymbol{x}_{\text{old}}. \tag{7}$$

Here the matrix B_ω is determined by the arbitrage rule $\omega = (i,j,k)$ applied to this step of updating and has the form typical in the theory of asynchronous systems, see, e.g., Asarin et al. (1992): all of its rows except one coincide with the rows of the identity matrix while one row contains exactly two non-zero elements specified by formula (4).

If we define on the space of vectors (6) the usual Euclidean inner-product $\langle\cdot,\cdot\rangle$, and then use the matrix B_ω in (7) corresponding to the arbitrage rule $\omega = (i,j,k)$, we will get the following representation[1]:

$$B_\omega \boldsymbol{x} = \begin{cases} \boldsymbol{x} + \langle -\boldsymbol{e}_{ij} - \boldsymbol{e}_{ki} + \boldsymbol{e}_{kj}, \boldsymbol{x}\rangle \boldsymbol{e}_{ij} & \text{for} \quad k<i<j, \\ \boldsymbol{x} + \langle -\boldsymbol{e}_{ij} + \boldsymbol{e}_{ik} + \boldsymbol{e}_{kj}, \boldsymbol{x}\rangle \boldsymbol{e}_{ij} & \text{for} \quad i<k<j, \\ \boldsymbol{x} + \langle -\boldsymbol{e}_{ij} + \boldsymbol{e}_{ik} - \boldsymbol{e}_{jk}, \boldsymbol{x}\rangle \boldsymbol{e}_{ij} & \text{for} \quad i<j<k. \end{cases} \tag{8}$$

Since all the vectors (6) are column-vectors, then the relations (8) can also be rewritten in the following matrix form:

$$B_\omega = \begin{cases} I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{ki}^T - \boldsymbol{e}_{kj}^T) & \text{for} \quad k<i<j, \\ I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T - \boldsymbol{e}_{kj}^T) & \text{for} \quad i<k<j, \\ I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T + \boldsymbol{e}_{jk}^T) & \text{for} \quad i<j<k, \end{cases} \tag{9}$$

where the upper index T denotes transposition of a vector or a matrix.

Let us make some remarks resulting from formulae (8) and (9).

Remark 1. *Each matrix B_ω is a matrix with* integer *entries. This means that, when considering convergence issues for the matrix products involving such matrices, the convergence, provided that it takes place, will be always achieved by a finite number of steps.* □

Remark 2. *Each matrix B_ω is a* projector. □

Remark 3. *Since the procedure of sequential updating of vectors $\boldsymbol{x}$ according to* (7) *linearly depends on the initial vectors, then to analyse their convergence it suffices to consider only the initial vectors taking the values of the standard basis vectors $\{\boldsymbol{e}_{ij}\}$. Then the whole problem becomes an integer-valued problem, i.e, a* combinatorial *one.* □

Remark 4. *The procedure of constructing the matrices B_ω is very similar to that of constructing the so-called "mixtures" of matrices under investigation in asynchronous systems Asarin et al. (1992). The difference is that in Asarin et al. (1992) each coordinate of the state vector $\boldsymbol{x}_{new}$ in* (7) *is updated by a single rule, whereas in our case each coordinate of the state vector $\boldsymbol{x}_{new}$ in* (7) *may be updated in accordance with several rules — see Section 5 below. This is explained by the fact that the set of all the arbitrage rules $(i,j,\cdot)$ having the same first two indices i,j, in general, contains more than one element.*

Investigation of convergence of the procedures (7) *has very much in common with investigation of the joint/generalized spectral radius of the family of all the matrices B_ω, see, e.g., Jungers (2009) and bibliography therein.*

So, a great variety of results from the theory of asynchronous systems and from the theory of the joint/generalized spectral radius might be helpful here. □

Clearly the arbitrage procedure (7) may have limiting fixed states (with respect to all the matrices B_ω) if and only if the eigenspaces of the matrices B_ω, corresponding to the eigenvalue 1, have nontrivial intersection $\mathbb{F}$. To find this subspace $\mathbb{F}$ it is necessary to solve the following system of linear equations:

$$\boldsymbol{x} = B_\omega \boldsymbol{x}, \quad \forall\ \omega = (i,j,k):\ 1 \le i < j \le d,\ k \ne i,j.$$

Lemma 3. *The subspace $\mathbb{F}$ of common fixed points of all the matrices $\{B_\omega\}$ consists of the column-vectors $\boldsymbol{x} = (x_{ij})^T$ satisfying*

$$x_{ij} = -x_{1i} + x_{1j}, \qquad 2 \le i < j \le d,$$

where $x_{12}, x_{13}, \ldots, x_{1d}$ are free variables.

Proof. By induction with respect to $d \ge 3$. □

[1] Recall that always $i<j$ and $k \ne i,j$.

Remark 5. *By Lemma 3, one can assert that a skew-symmetric matrix A is unchangeable by application of any arbitrage rule if and only if it has the following form:*

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & \dots & a_{1,d-1} & a_{1d} \\ \dots & 0 & a_{13}-a_{12} & a_{14}-a_{12} & \dots & a_{1,d-1}-a_{12} & a_{1d}-a_{12} \\ \dots & \dots & 0 & a_{14}-a_{13} & \dots & a_{1,d-1}-a_{13} & a_{1d}-a_{13} \\ \dots & \dots & \dots & 0 & \dots & a_{1,d-1}-a_{14} & a_{1d}-a_{14} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 & a_{1d}-a_{1,d-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix},$$

where the subdiagonal elements are defined by skew-symmetry and so are ignored. □

Lemma 4. *The matrix*

$$P = \sum_{2\le j\le d} \boldsymbol{e}_{1j}\boldsymbol{e}_{1j}^T + \sum_{2\le i<j\le d} \boldsymbol{e}_{ij}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1i}^T) \tag{10}$$

is a projector on the subspace $\mathbb{F}$. The vectors

$$\boldsymbol{f}_i = P\boldsymbol{e}_{1i} = \sum_{1\le j<i} \boldsymbol{e}_{ji} - \sum_{i<j\le d} \boldsymbol{e}_{ij}, \quad i = 2,3,\dots,d,$$

where the second sum is assumed to vanish when $i = d$, form a basis of $\mathbb{F}$.

Proof. Follows from Lemma 3 and Remark 5. □

Let us define in the space $\mathbb{R}^{d(d-1)/2}$ a new system of coordinates by passing from the vectors $\boldsymbol{x} = (x_{ij})^T$ to the vectors $\boldsymbol{y} = (y_{ij})^T$ in accordance with the rule

$$y_{ij} = \begin{cases} x_{ij} & \text{for} \quad 1 = i < j \le d; \\ x_{ij} + x_{1i} - x_{1j} & \text{for} \quad 2 \le i < j \le d. \end{cases} \tag{11}$$

Then the inverse change of variables is defined as

$$x_{ij} = \begin{cases} y_{ij} & \text{for} \quad 1 = i < j \le d; \\ y_{ij} - y_{1i} + y_{1j} & \text{for} \quad 2 \le i < j \le d. \end{cases} \tag{12}$$

Remark, that in terms of the variables $\boldsymbol{y}$, the subspace $\mathbb{F}$ of common fixed points of the matrices $\{B_\omega\}$ can be characterised as follows:

$$\mathbb{F} = \{\boldsymbol{y} = (y_{ij}) : \ y_{ij} = 0 \text{ for } 2 \le i < j \le d\}.$$

Relations (11) and (12) can be rewritten in matrix form

$$\boldsymbol{y} = Q^{-1}\boldsymbol{x}, \quad \boldsymbol{x} = Q\boldsymbol{y},$$

where

$$Q = I + \sum_{2\le i<j\le d} \boldsymbol{e}_{ij}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1i}^T), \quad Q^{-1} = I - \sum_{2\le i<j\le d} \boldsymbol{e}_{ij}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1i}^T). \tag{13}$$

Since the identity matrix I allows the representation

$$I = \sum_{1\le i<j\le d} \boldsymbol{e}_{ij}\boldsymbol{e}_{ij}^T = \sum_{2\le j\le d} \boldsymbol{e}_{1j}\boldsymbol{e}_{1j}^T + \sum_{2\le i<j\le d} \boldsymbol{e}_{ij}\boldsymbol{e}_{ij}^T,$$

then by (10) we have

$$Q = \sum_{2\le i<j\le d} \boldsymbol{e}_{ij}\boldsymbol{e}_{ij}^T + P.$$

Therefore

$$\boldsymbol{f}_i = P\boldsymbol{e}_{1i} = Q\boldsymbol{e}_{1i}, \quad i = 2,3,\dots,d,$$

and then

$$\boldsymbol{e}_{1i} = Q^{-1}\boldsymbol{f}_i, \quad i = 2, 3, \ldots, d.$$

Because each of the vectors $\boldsymbol{f}_i$, $i = 2, 3, \ldots, d$, is a fixed point of each of the matrices B_ω, i.e.,

$$\boldsymbol{f}_i = B_\omega \boldsymbol{f}_i, \quad i = 2, 3, \ldots, d, \ \forall\, \omega,$$

then each of the matrices $Q^{-1}B_\omega Q$ takes the block-triangular form:

$$D_\omega := Q^{-1}B_\omega Q = \begin{pmatrix} I & F_\omega \\ 0 & G_\omega \end{pmatrix}, \tag{14}$$

where

$$\boldsymbol{e}_{1i} = D_\omega \boldsymbol{e}_{1i}, \quad i = 2, 3, \ldots, d, \ \forall\, \omega,$$

To describe the structure of the matrices D_ω in more detail, make use of the representations (9) and (13), and of the fact that $\boldsymbol{e}_{ij}^T\boldsymbol{e}_{mn} = 1$ if and only if $i = m$ and $j = n$, whereas $\boldsymbol{e}_{ij}^T\boldsymbol{e}_{mn} = 0$ in other cases.

Case $1 = k < i < j$. By (9), in this case $B_\omega = I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{1i}^T - \boldsymbol{e}_{1j}^T)$, from which

$$\begin{aligned}
D_\omega &= Q^{-1}B_\omega Q \\
&= \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{1i}^T - \boldsymbol{e}_{1j}^T)\Big)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(\boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{1i}^T - \boldsymbol{e}_{1j}^T)\Big)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{1i}^T - \boldsymbol{e}_{1j}^T)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{1i}^T - \boldsymbol{e}_{1j}^T) - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{1i}^T - \boldsymbol{e}_{1j}^T)\sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T) \\
&\qquad\qquad = I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{1i}^T - \boldsymbol{e}_{1j}^T) - \boldsymbol{e}_{ij}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1i}^T) = I - \boldsymbol{e}_{ij}\boldsymbol{e}_{ij}^T.
\end{aligned}$$

Case $1 < k < i < j$. By (9), in this case $B_\omega = I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{ki}^T - \boldsymbol{e}_{kj}^T)$, from which

$$\begin{aligned}
D_\omega &= Q^{-1}B_\omega Q \\
&= \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{ki}^T - \boldsymbol{e}_{kj}^T)\Big)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(\boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{ki}^T - \boldsymbol{e}_{kj}^T)\Big)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{ki}^T - \boldsymbol{e}_{kj}^T)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{ki}^T - \boldsymbol{e}_{kj}^T) - \boldsymbol{e}_{ij}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1i}^T) - \boldsymbol{e}_{ij}(\boldsymbol{e}_{1i}^T - \boldsymbol{e}_{1k}^T) + \boldsymbol{e}_{ij}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1k}^T) \\
&\qquad\qquad = I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{ki}^T - \boldsymbol{e}_{kj}^T) = B_\omega.
\end{aligned}$$

Case $1 = i < k < j$. By (9), in this case $B_\omega = I - \boldsymbol{e}_{1j}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1k}^T - \boldsymbol{e}_{kj}^T)$, from which

$$\begin{aligned}
D_\omega &= Q^{-1}B_\omega Q \\
&= \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(I - \boldsymbol{e}_{1j}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1k}^T - \boldsymbol{e}_{kj}^T)\Big)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(\boldsymbol{e}_{1j}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1k}^T - \boldsymbol{e}_{kj}^T)\Big)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(\boldsymbol{e}_{1j}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1k}^T - \boldsymbol{e}_{kj}^T) - \boldsymbol{e}_{1j}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1k}^T)\Big) \\
&\qquad\qquad = I + \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\boldsymbol{e}_{1j}\boldsymbol{e}_{kj}^T = I + \Big(\boldsymbol{e}_{1j} + \sum_{j<n} \boldsymbol{e}_{jn} - \sum_{2\le m<j} \boldsymbol{e}_{mj}\Big)\boldsymbol{e}_{kj}^T.
\end{aligned}$$

Case $1 < i < k < j$. By (9), in this case $B_\omega = I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T - \boldsymbol{e}_{kj}^T)$, from which

$$\begin{aligned}
D_\omega &= Q^{-1}B_\omega Q \\
&= \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T - \boldsymbol{e}_{kj}^T)\Big)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(\boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T - \boldsymbol{e}_{kj}^T)\Big)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T - \boldsymbol{e}_{kj}^T)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T - \boldsymbol{e}_{kj}^T) - \boldsymbol{e}_{ij}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1i}^T) + \boldsymbol{e}_{ij}(\boldsymbol{e}_{1k}^T - \boldsymbol{e}_{1i}^T) + \boldsymbol{e}_{ij}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1k}^T) \\
&\qquad = I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T - \boldsymbol{e}_{kj}^T) = B_\omega.
\end{aligned}$$

Case $1 = i < j < k$. By (9), in this case $B_\omega = I - \boldsymbol{e}_{1j}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1k}^T + \boldsymbol{e}_{jk}^T)$, from which

$$\begin{aligned}
D_\omega &= Q^{-1}B_\omega Q \\
&= \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(I - \boldsymbol{e}_{1j}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1k}^T + \boldsymbol{e}_{jk}^T)\Big)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(\boldsymbol{e}_{1j}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1k}^T + \boldsymbol{e}_{jk}^T)\Big)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(\boldsymbol{e}_{1j}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1k}^T + \boldsymbol{e}_{jk}^T) + \boldsymbol{e}_{1j}(\boldsymbol{e}_{1k}^T - \boldsymbol{e}_{1j}^T)\Big) \\
&\qquad = I - \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\boldsymbol{e}_{1j}\boldsymbol{e}_{jk}^T = I - \Big(\boldsymbol{e}_{1j} + \sum_{j<n} \boldsymbol{e}_{jn} - \sum_{2\le m<j} \boldsymbol{e}_{mj}\Big)\boldsymbol{e}_{jk}^T.
\end{aligned}$$

Case $1 < i < j < k$. By (9), in this case $B_\omega = I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T + \boldsymbol{e}_{jk}^T)$, from which

$$\begin{aligned}
D_\omega &= Q^{-1}B_\omega Q \\
&= \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T + \boldsymbol{e}_{jk}^T)\Big)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \Big(I - \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big)\Big(\boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T + \boldsymbol{e}_{jk}^T)\Big)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T + \boldsymbol{e}_{jk}^T)\Big(I + \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}(\boldsymbol{e}_{1n}^T - \boldsymbol{e}_{1m}^T)\Big) \\
&= I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T + \boldsymbol{e}_{jk}^T) - \boldsymbol{e}_{ij}(\boldsymbol{e}_{1j}^T - \boldsymbol{e}_{1i}^T) + \boldsymbol{e}_{ij}(\boldsymbol{e}_{1k}^T - \boldsymbol{e}_{1i}^T) - \boldsymbol{e}_{ij}(\boldsymbol{e}_{1k}^T - \boldsymbol{e}_{1j}^T) \\
&\qquad = I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T + \boldsymbol{e}_{jk}^T) = B_\omega.
\end{aligned}$$

It is convenient to unite the obtained relations into a single formula:

$$D_\omega = \begin{cases}
I - \boldsymbol{e}_{ij}\boldsymbol{e}_{ij}^T & \text{for} \quad 1 = k < i < j; \\
I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{ki}^T - \boldsymbol{e}_{kj}^T) = B_\omega & \text{for} \quad 1 < k < i < j; \\
I + \Big(\boldsymbol{e}_{1j} + \sum_{j<n} \boldsymbol{e}_{jn} - \sum_{2\le m<j} \boldsymbol{e}_{mj}\Big)\boldsymbol{e}_{kj}^T & \text{for} \quad 1 = i < k < j; \\
I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T - \boldsymbol{e}_{kj}^T) = B_\omega & \text{for} \quad 1 < i < k < j; \\
I - \Big(\boldsymbol{e}_{1j} + \sum_{j<n} \boldsymbol{e}_{jn} - \sum_{2\le m<j} \boldsymbol{e}_{mj}\Big)\boldsymbol{e}_{jk}^T & \text{for} \quad 1 = i < j < k; \\
I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T + \boldsymbol{e}_{jk}^T) = B_\omega & \text{for} \quad 1 < i < j < k.
\end{cases} \tag{15}$$

Now, to compute the matrix F_ω, it is necessary to retain in the right-hand part of (15) only the summands of the form $\boldsymbol{e}_{mn}\boldsymbol{e}_{pq}^T$ in which $m = 1$ and $n, p, q > 1$. Similarly, to compute the matrix G_ω it is necessary to retain in the right-hand part of (15) only the summands of the form $\boldsymbol{e}_{mn}\boldsymbol{e}_{pq}^T$ in which $m, n, p, q > 1$. In doing so one should keep in mind that $I = \sum_{1\le m<n\le d} \boldsymbol{e}_{mn}\boldsymbol{e}_{mn}^T$. This results in

$$F_\omega = \begin{cases} 0 & \text{for} \quad 1 \le k < i < j, \\ \boldsymbol{e}_{1j}\boldsymbol{e}_{kj}^T & \text{for} \quad 1 = i < k < j, \\ 0 & \text{for} \quad 1 < i < k < j, \\ -\boldsymbol{e}_{1j}\boldsymbol{e}_{jk}^T & \text{for} \quad 1 = i < j < k, \\ 0 & \text{for} \quad 1 < i < j < k \end{cases}$$

and

$$G_\omega = \begin{cases} I - \boldsymbol{e}_{ij}\boldsymbol{e}_{ij}^T & \text{for} \quad 1 = k < i < j, \\ I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T + \boldsymbol{e}_{ki}^T - \boldsymbol{e}_{kj}^T) & \text{for} \quad 1 < k < i < j, \\ I + \left(\sum_{j<n} \boldsymbol{e}_{jn} - \sum_{2\le m<j} \boldsymbol{e}_{mj}\right)\boldsymbol{e}_{kj}^T & \text{for} \quad 1 = i < k < j, \\ I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T - \boldsymbol{e}_{kj}^T) & \text{for} \quad 1 < i < k < j, \\ I - \left(\sum_{j<n} \boldsymbol{e}_{jn} - \sum_{2\le m<j} \boldsymbol{e}_{mj}\right)\boldsymbol{e}_{jk}^T & \text{for} \quad 1 = i < j < k, \\ I - \boldsymbol{e}_{ij}(\boldsymbol{e}_{ij}^T - \boldsymbol{e}_{ik}^T + \boldsymbol{e}_{jk}^T) & \text{for} \quad 1 < i < j < k. \end{cases} \tag{16}$$

Naturally, in (16), in contrast to (15), the identity matrix I acts in the subspace spanned over the vectors $\boldsymbol{e}_{mn}$, $2 \le m < n \le d$, and therefore it is representable as $I = \sum_{2\le m<n\le d} \boldsymbol{e}_{mn}\boldsymbol{e}_{mn}^T$.

Remark 6. *In formulae* (15) *and* (16) *the sums corresponding to an empty set of indices are assumed to vanish. For instance,* $\sum_{j<n} \boldsymbol{e}_{jn} = 0$ *if* $j = d$ *or* $\sum_{2\le m<j} \boldsymbol{e}_{mj} = 0$ *if* $j = 2$. □

Remark 7. *The idea of highlighting the matrices* G_ω *is to lower the dimension of the studied matrices since, according to* (14)*, the behaviour of the products of matrices* G_ω *essentially determines the properties of the products of matrices* D_ω *and* B_ω. □

5. Case $d = 4$

The detailed analysis of the FX currency market with 4 trader-arbitragers, i.e., when $d = 4$, was fulfilled in Kozyakin et al. (2010) and Cross et al. (2012). Recall some related results. In this case the matrix A is of the form

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{14} & -a_{34} & 0 \end{pmatrix}.$$

In terms of the upper off-diagonal elements, the arbitrage rules can be written as follows:

$$\begin{cases} (123): & a_{12,\text{new}} & = & a_{13} - a_{23}, & \text{other elements unchanged}, \\ (124): & a_{12,\text{new}} & = & a_{14} - a_{24}, & \text{other elements unchanged}, \\ (132): & a_{13,\text{new}} & = & a_{12} + a_{23}, & \text{other elements unchanged}, \\ (134): & a_{13,\text{new}} & = & a_{14} - a_{34}, & \text{other elements unchanged}, \\ (142): & a_{14,\text{new}} & = & a_{12} + a_{24}, & \text{other elements unchanged}, \\ (143): & a_{14,\text{new}} & = & a_{13} + a_{34}, & \text{other elements unchanged}, \\ (231): & a_{23,\text{new}} & = & -a_{12} + a_{13}, & \text{other elements unchanged}, \\ (234): & a_{23,\text{new}} & = & a_{24} - a_{34}, & \text{other elements unchanged}, \\ (241): & a_{24,\text{new}} & = & -a_{12} + a_{14}, & \text{other elements unchanged}, \\ (243): & a_{24,\text{new}} & = & a_{23} + a_{34}, & \text{other elements unchanged}, \\ (341): & a_{34,\text{new}} & = & -a_{13} + a_{14}, & \text{other elements unchanged}, \\ (342): & a_{34,\text{new}} & = & -a_{23} + a_{24}, & \text{other elements unchanged}. \end{cases}$$

The subspace of fixed points $\mathbb{F}$ consists of all the vectors $\boldsymbol{x} = (x_{ij})^T$, coordinates of which satisfy

$$\begin{cases} x_{12} = x_{13} - x_{23}, & x_{12} = x_{14} - x_{24}, & x_{13} = x_{12} + x_{23}, & x_{13} = x_{14} - x_{34}, \\ x_{14} = x_{12} + x_{24}, & x_{14} = x_{13} + x_{34}, & x_{23} = -x_{12} + x_{13}, & x_{23} = x_{24} - x_{34}, \\ x_{24} = -x_{12} + x_{14}, & x_{24} = x_{23} + x_{34}, & x_{34} = -x_{13} + x_{14}, & x_{34} = -x_{23} + x_{24}. \end{cases}$$

This system has redundant equations. In particular, all the equations containing the minus sign are redundant. By removing them we obtain:

$$x_{13} = x_{12} + x_{23}, \quad x_{14} = x_{12} + x_{24}, \quad x_{14} = x_{13} + x_{34}, \quad x_{24} = x_{23} + x_{34}.$$

Here one of the equations determining x_{14} is also redundant. By removing it, we obtain the final set of independent equations defining the subspace $\mathbb{F}$:

$$x_{13} = x_{12} + x_{23}, \quad x_{14} = x_{13} + x_{34}, \quad x_{24} = x_{23} + x_{34}.$$

By choosing x_{12}, x_{13} and x_{14} as independent variables, the remaining variables can be defined by the equalities

$$x_{23} = -x_{12} + x_{13}, \quad x_{24} = -x_{12} + x_{14}, \quad x_{34} = -x_{13} + x_{14}.$$

The form of the matrices B_ω, D_ω and G_ω for this case is given in Appendix A.

As was noted in Kozyakin et al. (2010) and Cross et al. (2012), in the case $d = 4$ *the arbitrage process may be non-convergent for some periodical sequence of the arbitrage rules*. To support this claim let us consider the set of all vectors each of which is mapped by some of the matrices G_ω to zero. Direct calculations show that, up to non-negative factors, this set consists of the following 12 vectors:

$$\boldsymbol{s}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ \boldsymbol{s}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \boldsymbol{s}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \ \boldsymbol{s}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \boldsymbol{s}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \boldsymbol{s}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\boldsymbol{s}_7 = -\boldsymbol{s}_1, \quad \boldsymbol{s}_8 = -\boldsymbol{s}_2, \quad \boldsymbol{s}_9 = -\boldsymbol{s}_3, \quad \boldsymbol{s}_{10} = -\boldsymbol{s}_4, \quad \boldsymbol{s}_{11} = -\boldsymbol{s}_5, \quad \boldsymbol{s}_{12} = -\boldsymbol{s}_6,$$

Moreover, as is easily verified, the set

$$\mathbb{S} = \mathrm{co}\,\{\pm\boldsymbol{s}_1, \pm\boldsymbol{s}_2, \pm\boldsymbol{s}_3, \pm\boldsymbol{s}_4, \pm\boldsymbol{s}_5, \pm\boldsymbol{s}_6\}$$

is mapped by each of the matrices G_ω into itself and is a body, i.e., contains interior points. Hence, $\mathbb{S}$ is a unit ball of some norm $\|\cdot\|_*$ for which

$$\|G_\omega\|_* \le 1, \quad \forall\omega. \tag{17}$$

So, the set of matrices $\{G_\omega\}$ is *neutrally stable* and, it seems, there is the possibility that all products of matrices G_ω are convergent. In Sections 5.1 and 5.2 it is shown that this is not the case.

Remark 8. *By* (14) *the following representation is valid*

$$B_\omega = Q \begin{pmatrix} I & F_\omega \\ 0 & G_\omega \end{pmatrix} Q^{-1}.$$

From here, and from the estimate (17)*, it then follows that the norm of any products of the matrices B_ω may grow no faster than cn with some constant c, where n is the number of factors B_ω in the product. Example 5.2 demonstrates that such a rate of growth of products of the matrices B_ω, in the case $d = 4$, is really achievable.* □

5.1. Example

Consider the following sequence of the arbitrage rules

$$\begin{array}{rrrr} \omega(1) = (1,4,2), & \omega(2) = (1,2,3), & \omega(3) = (3,4,1), & \omega(4) = (1,4,2), \\ \omega(5) = (1,3,4), & \omega(6) = (2,4,3), & \omega(7) = (2,3,1), & \omega(8) = (3,4,2), \\ \omega(9) = (2,4,1), & \omega(10) = (1,3,4), & \omega(11) = (3,4,2), & \omega(12) = (1,4,3), \\ \omega(13) = (2,3,4), & \omega(14) = (1,3,2), & \omega(15) = (1,2,4), & \omega(16) = (1,4,3), \end{array}$$

and extend it by periodicity. Set also $\boldsymbol{x}_0 = \boldsymbol{s}_1$, and build recurrently the sequence

$$\boldsymbol{x}_n = G_{\omega(n)}\boldsymbol{x}_{n-1}, \quad n = 1, 2, \ldots, 16.$$

Then

$$
\begin{aligned}
\boldsymbol{x}_0 &= \boldsymbol{s}_1, & \boldsymbol{x}_1 &= G_{\omega(1)}\boldsymbol{x}_0 = G_{(142)}\boldsymbol{x}_0 = \boldsymbol{s}_2,\\
\boldsymbol{x}_2 &= G_{\omega(2)}\boldsymbol{x}_1 = G_{(123)}\boldsymbol{x}_1 = -\boldsymbol{s}_3, & \boldsymbol{x}_3 &= G_{\omega(3)}\boldsymbol{x}_2 = G_{(341)}\boldsymbol{x}_2 = \boldsymbol{s}_5,\\
\boldsymbol{x}_4 &= G_{\omega(4)}\boldsymbol{x}_3 = G_{(142)}\boldsymbol{x}_3 = \boldsymbol{s}_6, & \boldsymbol{x}_5 &= G_{\omega(5)}\boldsymbol{x}_4 = G_{(134)}\boldsymbol{x}_4 = \boldsymbol{s}_4,\\
\boldsymbol{x}_6 &= G_{\omega(6)}\boldsymbol{x}_5 = G_{(243)}\boldsymbol{x}_5 = \boldsymbol{s}_1, & \boldsymbol{x}_7 &= G_{\omega(7)}\boldsymbol{x}_6 = G_{(231)}\boldsymbol{x}_6 = -\boldsymbol{s}_5,\\
\boldsymbol{x}_8 &= G_{\omega(8)}\boldsymbol{x}_7 = G_{(342)}\boldsymbol{x}_7 = \boldsymbol{s}_3, & \boldsymbol{x}_9 &= G_{\omega(9)}\boldsymbol{x}_8 = G_{(241)}\boldsymbol{x}_8 = \boldsymbol{s}_6,\\
\boldsymbol{x}_{10} &= G_{\omega(10)}\boldsymbol{x}_9 = G_{(134)}\boldsymbol{x}_9 = \boldsymbol{s}_4, & \boldsymbol{x}_{11} &= G_{\omega(11)}\boldsymbol{x}_{10} = G_{(342)}\boldsymbol{x}_{10} = \boldsymbol{s}_2,\\
\boldsymbol{x}_{12} &= G_{\omega(12)}\boldsymbol{x}_{11} = G_{(143)}\boldsymbol{x}_{11} = \boldsymbol{s}_1, & \boldsymbol{x}_{13} &= G_{\omega(13)}\boldsymbol{x}_{12} = G_{(234)}\boldsymbol{x}_{12} = \boldsymbol{s}_1,\\
\boldsymbol{x}_{14} &= G_{\omega(14)}\boldsymbol{x}_{13} = G_{(132)}\boldsymbol{x}_{13} = \boldsymbol{s}_3, & \boldsymbol{x}_{15} &= G_{\omega(15)}\boldsymbol{x}_{14} = G_{(124)}\boldsymbol{x}_{14} = -\boldsymbol{s}_2,\\
\boldsymbol{x}_{16} &= G_{\omega(16)}\boldsymbol{x}_{15} = G_{(143)}\boldsymbol{x}_{15} = -\boldsymbol{s}_1.
\end{aligned}
$$

Continuing building, we will obtain a 32-periodic sequence $\boldsymbol{x}_n$. It is not difficult to see that the sequence of products of the corresponding matrices B_ω:

$$H_n = B_{\omega(n)}H_{n-1}, \qquad H_0 = I,\ n = 1, 2, \dots,$$

will be also 32**-periodic** and **non-convergent to any limit**.

5.2. Example

Consider the following sequence of the arbitrage rules

$$
\begin{array}{llll}
\omega(1) = (1,4,3), & \omega(2) = (3,4,1), & \omega(3) = (3,4,2), & \omega(4) = (1,4,2),\\
\omega(5) = (1,2,4), & \omega(6) = (2,3,1), & \omega(7) = (1,3,2), & \omega(8) = (2,4,3),\\
\omega(9) = (1,3,4), & \omega(10) = (2,4,1), & \omega(11) = (1,2,3), & \omega(12) = (2,3,4),
\end{array}
$$

and extend it by periodicity. Set also $\boldsymbol{x}_0 = \boldsymbol{s}_1$, and build recurrently the sequence

$$\boldsymbol{x}_n = G_{\omega(n)}\boldsymbol{x}_{n-1}, \quad n = 1, 2, \dots, 12.$$

Then

$$
\begin{aligned}
\boldsymbol{x}_0 &= \boldsymbol{s}_1, & \boldsymbol{x}_1 &= G_{\omega(1)}\boldsymbol{x}_0 = G_{(143)}\boldsymbol{x}_0 = \boldsymbol{s}_1,\\
\boldsymbol{x}_2 &= G_{\omega(2)}\boldsymbol{x}_1 = G_{(341)}\boldsymbol{x}_1 = \boldsymbol{s}_1, & \boldsymbol{x}_3 &= G_{\omega(3)}\boldsymbol{x}_2 = G_{(342)}\boldsymbol{x}_2 = \boldsymbol{s}_1,\\
\boldsymbol{x}_4 &= G_{\omega(4)}\boldsymbol{x}_3 = G_{(142)}\boldsymbol{x}_3 = \boldsymbol{s}_2, & \boldsymbol{x}_5 &= G_{\omega(5)}\boldsymbol{x}_4 = G_{(124)}\boldsymbol{x}_4 = \boldsymbol{s}_2,\\
\boldsymbol{x}_6 &= G_{\omega(6)}\boldsymbol{x}_5 = G_{(231)}\boldsymbol{x}_5 = -\boldsymbol{s}_6, & \boldsymbol{x}_7 &= G_{\omega(7)}\boldsymbol{x}_6 = G_{(132)}\boldsymbol{x}_6 = -\boldsymbol{s}_6,\\
\boldsymbol{x}_8 &= G_{\omega(8)}\boldsymbol{x}_7 = G_{(243)}\boldsymbol{x}_7 = -\boldsymbol{s}_3, & \boldsymbol{x}_9 &= G_{\omega(9)}\boldsymbol{x}_8 = G_{(134)}\boldsymbol{x}_8 = -\boldsymbol{s}_1,\\
\boldsymbol{x}_{10} &= G_{\omega(10)}\boldsymbol{x}_9 = G_{(241)}\boldsymbol{x}_9 = -\boldsymbol{s}_4, & \boldsymbol{x}_{11} &= G_{\omega(11)}\boldsymbol{x}_{10} = G_{(123)}\boldsymbol{x}_{10} = -\boldsymbol{s}_5,\\
\boldsymbol{x}_{12} &= G_{\omega(12)}\boldsymbol{x}_{11} = G_{(234)}\boldsymbol{x}_{11} = \boldsymbol{s}_1.
\end{aligned}
$$

Continuing building, we will obtain a 12-periodic sequence $\boldsymbol{x}_n$. The sequence of the corresponding products of the matrices $G_{\omega(n)}$ is **bounded** by (17) whereas the sequence of products of the matrices B_ω:

$$H_n = B_{\omega(n)}H_{n-1}, \qquad H_0 = I,\ n = 1, 2, \dots,$$

is **divergent**, and the norms of matrices H_n **have the linear rate of growth**, see Remark 8.

6. Case $d = 5$

The full set of the matrices B_ω, D_ω and G_ω for this case is presented in Appendix B. Direct calculations show that the matrix

$$H = B_{(143)}B_{(231)}B_{(245)}B_{(342)}B_{(451)}B_{(124)}B_{(453)}$$

is as follows

$$
H = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

and its spectral radius $\rho(H)$ is strictly greater than 1:

$$\rho(H) = \frac{1+\sqrt{5}}{2} \simeq 1.618.$$

To ascertain this it suffices to note that the matrix H has the following eigenvalues (ordered by decreasing absolute values):

$$-\frac{1+\sqrt{5}}{2},\ 1,\ 1,\ 1,\ 1,\ 1,\ \frac{\sqrt{5}-1}{2},\ 0,\ 0,\ 0.$$

Consider the following sequence of the arbitrage rules

$$\begin{array}{llll} \omega(1) = (4,5,3), & \omega(2) = (1,2,4), & \omega(3) = (4,5,1), & \omega(4) = (3,4,2), \\ \omega(5) = (2,4,5), & \omega(6) = (2,3,1), & \omega(7) = (1,4,3), & \end{array}$$

and extend it by periodicity. Then, as is known from the linear algebra, for almost all initial vectors[2] $\boldsymbol{x}_0$, the sequence of vectors

$$\boldsymbol{x}_n = B_{\omega(n)}\boldsymbol{x}_{n-1}, \quad n = 1, 2, \dots,$$

obtained by sequential application of the arbitrage rules $\omega(n)$ will diverge with the rate $\left(\frac{1+\sqrt{5}}{2}\right)^{n/7}$, i.e.,

$$c\left(\frac{1+\sqrt{5}}{2}\right)^{n/7}\|\boldsymbol{x}_0\| \le \|\boldsymbol{x}_n\| \le C\left(\frac{1+\sqrt{5}}{2}\right)^{n/7}\|\boldsymbol{x}_0\|,$$

where c, C are some positive constants. Taking into account that $1.071 \le \left(\frac{1+\sqrt{5}}{2}\right)^{1/7} \le 1.072$, the inequalities obtained may be represented in a more descriptive form:

$$c\,1.071^n\|\boldsymbol{x}_0\| \le \|\boldsymbol{x}_n\| \le C\,1.072^n\|\boldsymbol{x}_0\|. \tag{18}$$

Remark 9. *Relations* (18) *imply that for $d = 5$, in contrast to the case $d = 4$, the norms of the vectors $\{\boldsymbol{x}_n\}$ as well as the norms of the matrix products*

$$H_n = B_{\omega(n)}H_{n-1}, \qquad H_0 = I,\ n = 1, 2, \dots,$$

may grow with the exponential rate. □

Remark 10. *As was shown in Remark 9, in the case $d = 5$, the norms of the matrix products may grow with the exponential rate whereas in the case $d = 4$, according to Section 5, the norms of the matrix products may have at most a linear rate of growth. Any sequence of arbitrage rules involving* 4 *traders may be treated as a particular case of a sequence of arbitrage rules for* 5 *traders. Therefore in the case $d = 5$ one may observe also the sequences of matrices whose norms of the products have a linear rate of growth, as well as sequences of matrices (vectors) varying periodically, see Section 5.*

When $d > 5$ we face an effect of "inheritance of instability" taking place in the cases $d = 4, 5$. In this case one may observe all the types of behaviour of the matrix products described above as well as of the corresponding state vectors $\boldsymbol{x}_n$: periodicity, growth at a linear rate and growth at an exponential rate. □

Remark 11. *If we return back from the "artificial" quantities a_{ij} to the true exchange rates*

$$r_{ij} = e^{a_{ij}}, \tag{19}$$

we arrive to a rather unexpected conclusion.

In the case $d = 4$ the exchange rates may vary periodically or grow exponentially which follows from (19) *and the results of Section 5.*

[2] To be more specific, for all initial vectors not belonging to the linear invariant subspace of the matrix H corresponding to the set of eigenvalues distinct from $-\frac{1+\sqrt{5}}{2}$.

In the case $d \geq 5$ the exchange rates also may vary periodically or grow exponentially but also, what is surprising, they ***may grow in accordance with the double exponential law****. For instance, in the case $d = 5$ formula* (18) *implies*

$$e^{\tilde{c}\,1.071^n} \leq \|\boldsymbol{r}_n\| \leq e^{\tilde{C}\,1.072^n},$$

where $\boldsymbol{r}_n$ is the exchange rate vector

$$\{r_{12}, r_{13}, \ldots, r_{1d}, r_{23}, r_{24}, \ldots r_{2d}, \ldots, r_{d-1,d}\}^T$$

at the moment n. □

Remark 12. *As follows from the analysis undertaken, after introducing new variables bringing the matrices B_ω to block-triangular form, one obtains a set of variables, which is updated only by matrices G_ω, and the behaviour of which is thus determined only by* ***its own history****. These variables may be called, in a sense, "leading exchange rates indices." For $d = 4$, this set is constituted of the variables*

$$a_{12} - a_{13} + a_{23}, \quad a_{12} - a_{14} + a_{24}, \quad a_{13} - a_{14} + a_{34}, \tag{20}$$

which, in terms of exchange rates, corresponds to the set of variables

$$\frac{r_{12} \cdot r_{23}}{r_{13}} = r_{12} \cdot r_{23} \cdot r_{31}, \quad \frac{r_{12} \cdot r_{24}}{r_{14}} = r_{12} \cdot r_{24} \cdot r_{41}, \quad \frac{r_{13} \cdot r_{34}}{r_{14}} = r_{13} \cdot r_{34} \cdot r_{41}. \tag{21}$$

For $d = 5$, the related set is constituted of the variables

$$a_{12} - a_{13} + a_{23}, \quad a_{12} - a_{14} + a_{24}, \quad a_{12} - a_{15} + a_{25}, \quad a_{13} - a_{14} + a_{34}, \quad a_{13} - a_{15} + a_{35}, \quad a_{14} - a_{15} + a_{45}, \tag{22}$$

and to it, in terms of exchange rates, corresponds the set of variables

$$r_{12} \cdot r_{23} \cdot r_{31}, \quad r_{12} \cdot r_{24} \cdot r_{41}, \quad r_{12} \cdot r_{25} \cdot r_{51}, \quad r_{13} \cdot r_{34} \cdot r_{41}, \quad r_{13} \cdot r_{35} \cdot r_{51}, \quad r_{14} \cdot r_{45} \cdot r_{51}. \tag{23}$$

For $d = 4$, the variables (20) *and* (21) *always* ***vary boundedly*** *over time, while, for $d \geq 5$, the variables* (22) *and* (23) *may grow unboundedly. At the same time, the original set of exchange rates $\{r_{ij}\}$ and the corresponding set of "artificial" log-variables $\{a_{ij}\}$ for $d = 4$, as well as for $d \geq 5$,* ***may grow unboundedly****.*

The intuition underlying the "leading exchange rates indices" in (23) can be explained as follows. The term $r_{12}r_{23}r_{31}$ can be interpreted in the following way: the 1st trader is selling the currency 1 for the currency 2, then uses the currency 2 to buy currency 3 and then uses the currency 3 to buy his "home currency" 1. So, the term $r_{12}r_{23}r_{31}$ is the "arbitrage profit factor" for the 1st trader when the trades involve the "most simple" cyclic triangular path:

$$1 \to 2 \to 3 \to 1.$$

If this factor $r_{12}r_{23}r_{31}$ is equals 1 then such "cyclic triangular trades" bring no profit to the 1st trader, while when $r_{12}r_{23}r_{31} \neq 1$ then such "cyclic triangular trades" bring profit (or loss) to the 1st trader. Hence the condition

$$r_{12}r_{23}r_{31} = 1$$

is the "no-arbitrage" condition for trading by the cyclic path $1 \to 2 \to 3 \to 1$, **for the 1st trader**. This "unit circle" no-arbitrage condition was pointed out in Cournot (1929, first published in 1838).

Then (23) is simply the full set of factors whose simultaneous equality to 1 represents the no-arbitrage conditions (NACs) for the whole set of currencies in the $d = 5$ case:

$$r_{12}r_{23}r_{31} = 1, \quad r_{12}r_{24}r_{41} = 1, \quad r_{12}r_{25}r_{51} = 1, \quad r_{13}r_{34}r_{41} = 1, \quad r_{13}r_{35}r_{51} = 1, \quad r_{14}r_{45}r_{51} = 1. \tag{24}$$

It appears that different currencies are represented in (24) non-symmetrically. But in fact there are different NACs which are equivalent to each other.

We may introduce a measure, say,

$$Q = |r_{12}r_{23}r_{31} - 1| + |r_{12}r_{24}r_{41} - 1| + |r_{12}r_{25}r_{51} - 1| + |r_{13}r_{34}r_{41} - 1| + |r_{13}r_{35}r_{51} - 1| + |r_{14}r_{45}r_{51} - 1|,$$

which can be interpreted as an "accumulative" NAC. The meaning is that the equality $Q = 0$ involves the absence of profitable arbitrage opportunities in the whole FX market system, while the quantity $Q \neq 0$ characterises the extent of profitable arbitrage opportunities in the FX market system.

7. Concluding Remarks

The EMH implies that arbitrage operations in markets such as those for FX should be self-extinguishing. This simplistic view of arbitrage processes may reflect a failure to consider the complex chains of arbitrage possible in financial markets, such as those for FX, that have segmented trading platforms and fragmented information. In Kozyakin et al. (2010) and Cross et al. (2012) it was shown, applying combinatorial analysis and asynchronous systems theory, that arbitrage sequences in a 4-currency world tend to display periodicity or exponential growth rather than being self-extinguishing. The present paper has used a log-linear reformulation of the analysis to show that in a ($d \geq 5$)-currency world, as well as inheriting the periodicity or exponential growth found in the 4-currency case, arbitrage sequences can be augmented by a double-exponential process. Thus there appears to be a cumulative "inheritance of instability".

The results may help to explain why extensive arbitrage trading is endemic in FX markets. The fact that the number of currencies makes a difference is interesting: periodicity or exponential behaviour arise when you move from 3 to 4 currencies; double exponential behaviour as well, when you move beyond 4 currencies. The *Financial Times* provides daily quotes for 52 currencies against the US dollar, euro and pound sterling. But some currencies are more important than others. BIS data indicate that trades involving the US dollar accounted for 85/200 of all FX trades in 2010, followed by the euro (39/200), Japanese yen (19/200) and sterling (13/200), so trades involving these four key currencies accounted for 78% of global FX turnover (BIS, 2010, p. 12). In terms of currency pairs the leading trades are the dollar–euro, which accounted for 28% of global turnover, followed by the dollar–yen (14%), dollar–sterling (9%), euro–yen (3%), euro–sterling (3%) and sterling–yen (3%) — see BIS (2010, p. 15). In the terminology of Section 6 of this paper, these four currencies may be called "leading currencies." In a world containing just these four leading currencies, arbitrage operations can display periodicity, thus being bounded, or "mild" exponential growth. Once we move to account for triangular arbitrage outside these four leading currencies the arbitrage sequences may be unbounded, displaying "explosive" double exponential behaviour. In practice central banks often intervene to attempt to prevent exchange rates exceeding certain bounds. The history of financial crises, however, is replete with examples of central bank interventions in FX markets being unsuccessful, the central banks being eventually obliged to abandon their exchange rate pegs or target ranges.

The "leading exchange rate indices," discussed at the end of Section 6 of this paper, indicate arbitrage profit opportunities expressed in an FX trader's "home currency" unit of account. The August 15 1971 abandonment of the convertibility of the US dollar into gold at a fixed price of 35$ per ounce, and the breakdown of the Bretton Woods system whereby IMF member currencies were pegged to the US dollar, left the international monetary system without a well-defined numeraire or unit of account. To fill this gap, currency-specific effective exchange rate (EER) indices are calculated by national central banks and finance ministries, and by international organisations such as the IMF and BIS. An EER is a geometrically weighted average of the bilateral exchange rates for a particular currency. So, for example, the BIS currently publishes broad EERs covering 61 currencies, and narrow EERs covering 27 currencies, using 2008–2010 international trade flow data for weights (Klau and Fung, 2006; BIS, 2012). The various EERs use the "home currency" unit of account to provide exchange rate indices designed for a world in which there is no gold or single reserve currency anchor.

Special Drawing Rights (SDRs) were created by the IMF in 1969 to supplement the gold and foreign exchange reserves available to member central banks. The SDR was initially defined as equivalent to 0.888671 grams of fine gold, which could be exchanged for 1 US dollar in 1969, but is now defined as a weighted value of the US dollar, euro, pound sterling and Japanese yen, with the weights being changed every five years, expressed in US dollars (IMF, 2012). A European Currency Unit (ECU), conceived by European Economic Community (EEC, later the EU) in 1979, served as an EU unit of account, being a weighted value of EU member currencies, until the euro was launched in 1999.

The governor of the People's Bank of China recently advocated that the SDR be adopted as a global, tradeable reserve currency (Xiaochuan, 2009). The SDR is a dollar–euro–sterling–yen currency index, expressed in US dollars, so this is akin to the "leading exchange rate" index faced by a dollar FX trader-arbitrageur in the $d = 4$ case considered earlier in the present paper. Accordingly, the periodicity and exponential growth features of arbitrage sequences involving these four currencies would apply. This suggests that central banks holding the SDR as a tradeable reserve currency would be faced with endemic, rather than self-extinguishing, opportunities for arbitrage profits on their FX reserve portfolios. The proposal that "the basket of currencies forming the basis for SDR valuation should be expanded to include currencies of all major economics" (Xiaochuan, 2009, p. 3) would exacerbate the arbitrage problem, introducing the "inheritance of instability" arising in the $d \geq 5$ case considered in the present paper.

Appendix

Appendix A. Matrices $B_{(ijk)}, D_{(ijk)}$ and $G_{(ijk)}$ in the Case $d = 4$

The matrices B_ω:

$$B_{(123)} = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{(124)} = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{(132)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{(134)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{(142)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{(143)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{(231)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{(234)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{(241)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{(243)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{(341)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_{(342)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

The matrices Q, Q^{-1} and $D_\omega = Q^{-1} B_\omega Q$ take the form:

$$Q = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right), \quad Q^{-1} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right),$$

$$D_{(123)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right), \quad D_{(124)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$D_{(132)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right), \quad D_{(134)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$D_{(142)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right), \quad D_{(143)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$D_{(231)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right), \quad D_{(234)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$
D_{(241)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right), \quad D_{(243)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),
$$

$$
D_{(341)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad D_{(342)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{array}\right).
$$

The matrices G_ω:

$$
G_{(123)} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_{(124)} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_{(132)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},
$$

$$
G_{(134)} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{(142)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad G_{(143)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
G_{(231)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_{(234)} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_{(241)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

$$
G_{(243)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_{(341)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{(342)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}.
$$

Appendix B. Matrices $B_{(ijk)}, D_{(ijk)}$ and $G_{(ijk)}$ in the case $d = 5$

The matrices B_ω:

$$
B_{(123)} = \begin{pmatrix}
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
B_{(124)} = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
B_{(125)} = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
B_{(132)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
B_{(134)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
B_{(135)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
B_{(142)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
B_{(143)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$B_{(145)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{(152)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{(153)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{(154)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{(231)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{(234)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{(235)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{(241)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{(243)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{(245)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{(251)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{(253)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{(254)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{(341)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{(342)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_{(345)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{(351)} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&-1&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}, \quad B_{(352)} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix},$$

$$B_{(354)} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}, \quad B_{(451)} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&-1&1&0&0&0&0&0&0 \end{pmatrix},$$

$$B_{(452)} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&1&0&0&0 \end{pmatrix}, \quad B_{(453)} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1&1&0 \end{pmatrix}.$$

The matrices Q, Q^{-1} and $D_\omega = Q^{-1}B_\omega Q$ take the form:

$$Q = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline -1&1&0&0&1&0&0&0&0&0\\ -1&0&1&0&0&1&0&0&0&0\\ -1&0&0&1&0&0&1&0&0&0\\ 0&-1&1&0&0&0&0&1&0&0\\ 0&-1&0&1&0&0&0&0&1&0\\ 0&0&-1&1&0&0&0&0&0&1 \end{array}\right), \quad Q^{-1} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 1&-1&0&0&1&0&0&0&0&0\\ 1&0&-1&0&0&1&0&0&0&0\\ 1&0&0&-1&0&0&1&0&0&0\\ 0&1&-1&0&0&0&0&1&0&0\\ 0&1&0&-1&0&0&0&0&1&0\\ 0&0&1&-1&0&0&0&0&0&1 \end{array}\right),$$

$$D_{(123)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&1&0&0&0&0\\ 0&0&0&0&-1&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right), \quad D_{(124)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&-1&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&-1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right),$$

$$D_{(125)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&-1&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&-1&0&0&0\\ 0&0&0&0&0&1&-1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right), \quad D_{(132)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right),$$

$$D_{(134)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&-1&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right), \quad D_{(135)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&-1&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&-1&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right),$$

$$D_{(142)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-1&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0&1 \end{array}\right), \quad D_{(143)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&-1&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0&1 \end{array}\right),$$

$$D_{(145)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&-1\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&1\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0 \end{array}\right), \quad D_{(152)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&1&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&-1&0&1&0\\ 0&0&0&0&0&0&-1&0&0&1 \end{array}\right),$$

$$D_{(153)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&1&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&-1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1&1 \end{array}\right), \quad D_{(154)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&1\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&-1\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&-1\\ 0&0&0&0&0&0&0&0&0&0 \end{array}\right),$$

$$D_{(231)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right), \quad D_{(234)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&0&1&0&-1&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right),$$

$$D_{(235)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&1&0&-1&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right), \quad D_{(241)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right),$$

$$D_{(243)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right), \quad D_{(245)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&-1\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right),$$

$$D_{(251)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right), \quad D_{(253)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right),$$

$$D_{(254)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&1\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right), \quad D_{(341)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right),$$

$$D_{(342)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&-1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right), \quad D_{(345)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&-1\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right),$$

$$D_{(351)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right), \quad D_{(352)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right),$$

$$D_{(354)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right), \quad D_{(451)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0 \end{array}\right),$$

$$D_{(452)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&1&0&0&0 \end{array}\right), \quad D_{(453)} = \left(\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1&1&0 \end{array}\right).$$

The matrices G_ω:

$$G_{(123)} = \begin{pmatrix} 0&0&0&0&0&0\\ -1&1&0&0&0&0\\ -1&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad G_{(124)} = \begin{pmatrix} 1&-1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&-1&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix},$$

$$G_{(125)} = \begin{pmatrix} 1&0&-1&0&0&0\\ 0&1&-1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad G_{(132)} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&0&1&0&0\\ 1&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix},$$

$$G_{(134)} = \begin{pmatrix} 1&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&-1&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad G_{(135)} = \begin{pmatrix} 1&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&-1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix},$$

$$G_{(142)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&-1&0&1&0&0\\ 0&0&0&0&1&0\\ 0&1&0&0&0&1 \end{pmatrix}, \quad G_{(143)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&-1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&1 \end{pmatrix},$$

$$G_{(145)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&1\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0 \end{pmatrix}, \quad G_{(152)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&-1&0&1&0\\ 0&0&-1&0&0&1 \end{pmatrix},$$

$$G_{(153)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&-1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&-1&1 \end{pmatrix}, \quad G_{(154)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&-1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&-1\\ 0&0&0&0&0&0 \end{pmatrix},$$

$$G_{(231)} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad G_{(234)} = \begin{pmatrix} 0&1&0&-1&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix},$$

$$G_{(235)} = \begin{pmatrix} 0&0&1&0&-1&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad G_{(241)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix},$$

$$G_{(243)} = \begin{pmatrix} 1&0&0&0&0&0\\ 1&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad G_{(245)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&1&0&0&-1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix},$$

$$G_{(251)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad G_{(253)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix},$$

$$G_{(254)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&1&0&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad G_{(341)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix},$$

$$G_{(342)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ -1&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad G_{(345)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&-1\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix},$$

$$G_{(351)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad G_{(352)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ -1&0&1&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix},$$

$$G_{(354)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&0&1\\ 0&0&0&0&0&1 \end{pmatrix}, \quad G_{(451)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0 \end{pmatrix},$$

$$G_{(452)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&-1&1&0&0&0 \end{pmatrix}, \quad G_{(453)} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&-1&1&0 \end{pmatrix}.$$

The spectral radius of the matrix

$$G_{(143)}G_{(231)}G_{(245)}G_{(342)}G_{(451)}G_{(124)}G_{(453)} = \begin{pmatrix} 0&0&0&0&0&0\\ 1&-2&1&0&0&0\\ 0&-1&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ -1&1&0&0&0&0 \end{pmatrix},$$

is $\frac{1+\sqrt{5}}{2}$, and its eigenvalues ordered by decreasing of their modulus are:

$$-\frac{1+\sqrt{5}}{2},\ 1,\ \frac{\sqrt{5}-1}{2},\ 0,\ 0,\ 0.$$

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