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A MINIMAL APPROACH TO THE THEORY OF GLOBAL ATTRACTORS

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Abstract. For a semigroup $S(t): X \to X$ acting on a metric space (X, d), we give a notion of global attractor based only on the minimality with respect to the attraction property. Such an attractor is shown to be invariant whenever $S(t)$ is *asymptotically closed*. As a byproduct, we generalize earlier results on the existence of global attractors in the classical sense.

1. **Introduction.** Let (X, d) be a metric space, not necessarily complete. A family of maps $S(t): X \to X$ depending on a parameter $t \geq 0$ (conventionally called time) is said to be a *semigroup* or *dynamical system* on X whenever

- $S(0) = \mathrm{id}_X$ (the identity map in X);
- $S(t+\tau) = S(t)S(\tau)$ for all $t, \tau \geq 0$.

The semigroup turns out to be a useful tool in the study of (autonomous) differential equations in normed spaces. Indeed, whenever a Cauchy problem is well-posed for all positive times t and all initial data u_0 taken at $t = 0$, the corresponding solutions $u(t)$ read

$$u(t) = S(t)u_0,$$

where $S(t)$ is uniquely determined by the equation.

A common feature of differential models arising from concrete evolutionary phenomena is the presence of some dissipation mechanism. Mathematically, this translates into the existence of suitably small regions of the phase space which capture all the trajectories at large times. In the standard terminology, these regions are called attracting sets. Besides, it is often possible to locate the smallest attracting set where, roughly speaking, the whole asymptotic dynamics is eventually confined: this is the global attractor.

The theory of dynamical systems, although relatively recent, is nowadays considered a well-established branch of Mathematics, which opened unexplored horizons shedding a new light on the comprehension of evolutionary systems, especially with

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regard to their asymptotic properties, both qualitative and quantitative. For a detailed presentation, we address the reader to the classical textbooks [1, 3, 4, 8] and the subsequent treatises [2, 5], but there are many more.

Developing some insights already present in [2, Chapter XI], the aim of this note is revisiting the basic objects of the theory (such as absorbing and attracting sets, global attractors) only in terms of their attraction properties, and without further assumptions on $S(t)$ other than being a semigroup. As we will see, this is enough to give a sensible definition of global attractor. Hence, the "minimality" we refer to in the title is twofold:

- minimality with respect to the hypotheses;
- minimality with respect to attraction, as the sole characterizing property of the global attractor.

Remark. The minimality property has been shown to play a key role in the study of nonautonomous dynamical systems, where the only way to construct the unique global attractor is finding the smallest attracting set, since there exists no natural notion of invariance under the action of a process (the generalization of the semigroup to nonautonomous evolutions). This strategy has been devised by Haraux in his pioneering works (see [4]), and further enhanced by Chepyzhov and Vishik, with applications to several important nonautonomous equations of Mathematical Physics (see [2]).

Only in a second moment we discuss the invariance of the attractor, which generally requires some kind of continuity of the semigroup. In particular, the global attractor is shown to be invariant provided that $S(t)$ is asymptotically closed, a much weaker condition than strong continuity (see Definition 16). As a byproduct, we improve the known results of existence of global attractors in the classical sense (invariant by definition).

Notation. For every $\varepsilon > 0$, the ε-neighborhood of a set $B \subset X$ is defined as

$$\mathcal{O}_\varepsilon(B) = \bigcup_{x \in B} \big\{ y \in X : \, \mathrm{d}(x,y) < \varepsilon \big\}.$$

We denote the standard Hausdorff semidistance of two (nonempty) sets $B, C \subset X$ by

$$\boldsymbol{\delta}(B,C) = \sup_{x \in B} \mathrm{d}(x,C) = \sup_{x \in B} \inf_{y \in C} \mathrm{d}(x,y).$$

In a completely equivalent manner, we can write

$$\boldsymbol{\delta}(B,C) = \inf \big\{ \varepsilon > 0 : \, B \subset \mathcal{O}_\varepsilon(C) \big\}.$$

2. **Dissipative semigroups.** There are several possibilities to render the concept of dissipation in mathematical terms. Here, in the spirit of the theory of dynamical systems, we adopt a definition of global-geometric flavor, based on the notion of absorbtion.

Definition 1. A set $B \subset X$ is an *absorbing set* if for every bounded set $C \subset X$ there is an *entering time* $\tau = \tau(C)$ such that

$$S(t)C \subset B, \quad \forall t \geq \tau.$$

Definition 2. We agree to call $S(t)$ a *dissipative semigroup* whenever there exists a bounded absorbing set.

When a system is characterized by the presence of a sufficiently strong dissipation mechanism, the asymptotic dynamics is generally expected to undergo a substantial loss of degrees of freedom, becoming in a sense much simpler. Unfortunately, since we are mainly interested in subsets of infinite-dimensional normed spaces, bounded sets (e.g. balls) can be to some respect huge objects: independently of their size, they may reflect the structure of the whole space. Therefore, knowing that $S(t)$ is dissipative turns into little information on its longterm behavior. The hope is then proving the existence of suitably small absorbing sets. A good notion of smallness is the one of compactness. Indeed, compact sets are totally bounded and their fractal measure can be finite. Moreover, compact subsets of (infinite-dimensional) normed spaces are thin (nowhere dense), hence negligible in the sense of Baire. In particular, they do not contain balls. At the same time, compactness is a purely topological notion of smallness, and makes perfectly sense when no linear structure is available.

For semigroups generated by differential equations, a natural strategy is looking for absorbing sets which are also bounded (say, balls) in another space Y compactly embedded into X. However, this translates into a regularizing effect on the trajectories, namely, the solution $u(t)$ of the equation becomes smoother than the initial datum $u(0)$ after some time $t > 0$. A quite concrete possibility when dealing with parabolic equations, but completely hopeless for evolutions of hyperbolic type.

In conclusion, aiming to find reasonably good (e.g. compact) sets able to capture and fully describe the asymptotic dynamics, we must be less demanding and rather consider a weaker notion of absorbtion.

Definition 3. A set $K \subset X$ is said to be *attracting* if every ε-neighborhood $\mathcal{O}_\varepsilon(K)$ is an absorbing set.

Remark. The attracting property can be equivalently stated in terms of Hausdorff semidistance: K is an attracting set if the limit

$$\lim_{t\to\infty} \boldsymbol{\delta}(S(t)C, K) = 0$$

holds whenever $C \subset X$ is bounded.

Loosely speaking, attracting sets are in a fact asymptotically absorbing sets. And so is attraction, much more than absorbtion, the correct notion of confinement of the longterm dynamics. Reason why attracting sets play a crucial role in the asymptotic analysis of semigroups. Accordingly, it is convenient to lean on a different (and actually stronger) concept of dissipation.

Definition 4. A semigroup $S(t)$ is said to be *ε-dissipative* if there is a finite set $F = \{x_1, \ldots, x_N\}$ such that $\mathcal{O}_\varepsilon(F)$ is absorbing. The semigroup is called *totally dissipative* whenever is ε-dissipative for all $\varepsilon > 0$.

As a matter of fact, there is a less direct (albeit equivalent) characterization of a totally dissipative semigroup, based on the *Kuratowski measure of noncompactness*

$$\alpha(B) = \inf \big\{ d : B \text{ has a finite cover of balls of } X \text{ of diameter less than } d \big\}$$

of a bounded set $B \subset X$ (see [3]).

Definition 5. A semigroup $S(t)$ fulfills the *Kuratowski property* if there is a bounded absorbing set B for which

$$\lim_{t\to\infty} \alpha(S(t)B) = 0.$$

Indeed, we can state a straightforward proposition.

Proposition 6. *$S(t)$ is totally dissipative if and only if it fulfills the Kuratowski property.*

Remark. If $S(t)$ is a (dissipative) semigroup on a Banach space X with a bounded absorbing set B, a sufficient condition for $S(t)$ to be totally dissipative is the following: for every fixed $\varepsilon > 0$ there exist a decomposition $X = X_1 \oplus X_2$ with $\dim(X_1) < \infty$ and a time $\tau > 0$ such that

$$\sup_{x \in B} \|S(t)x - PS(t)x\| < \varepsilon, \quad \forall t \geq \tau,$$

where P is the canonical projection of X onto X_1. The result, although referred to as "alternative" by some authors, is just an immediate consequence of either Definition 4 or Definition 5.

3. **The global attractor.** Moving from the above discussion, we focus on the family of sets

$$\mathbb{K} = \big\{K \subset X : K \text{ is compact and attracting}\big\}.$$

Definition 7. The semigroup $S(t)$ is called *asymptotically compact* if $\mathbb{K}$ is nonempty.

Remark. It is apparent that an asymptotically compact semigroup is in particular totally dissipative.

With the aim of providing a necessary and sufficient condition in order for a compact set to be in $\mathbb{K}$, let $\mathfrak{C}$ be the collection of all possible sequences of the form

$$y_n = S(t_n)x_n,$$

where x_n is a bounded sequence in X and $t_n \to \infty$. For any $y_n \in \mathfrak{C}$ we denote

$$L(y_n) = \big\{x \in X : y_n \to x \text{ up to a subsequence}\big\},$$

and we define the set

$$A^\star = \bigcup_{y_n \in \mathfrak{C}} L(y_n).$$

Proposition 8. *The following hold:*

(i) *A set $K \subset X$ is attracting if and only if*

$$\mathrm{d}(y_n, K) \to 0, \quad \forall y_n \in \mathfrak{C}.$$

(ii) *$A^\star$ is contained in any closed attracting set.*

(iii) *If $S(t)$ is dissipative, then $A^\star$ coincides with the ω-limit of any bounded absorbing set B, defined as*[1]

$$\omega(B) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} S(\tau)B}.$$

In which case, $A^\star$ is closed.

Proof. Assertions (i) and (iii) are direct consequences of the definitions. In particular, (i) implies that each $L(y_n)$ belongs to the closure of any attracting K. If K is also closed, this is exactly (ii). □

Proposition 9. *Let $K \subset X$ be a compact set. Then $K \in \mathbb{K}$ if and only if*

$$\emptyset \neq L(y_n) \subset K, \quad \forall y_n \in \mathfrak{C}.$$

[1] All bounded absorbing sets share the same ω-limit.

Proof. Let $K \in \mathbb{K}$. Given $y_n \in \mathfrak{C}$, assertions (i)-(ii) above imply

$$L(y_n) \subset K \qquad \text{and} \qquad \mathrm{d}(y_n, \xi_n) \to 0,$$

for some $\xi_n \in K$. Since K is compact, there is $\xi \in K$ such that (up to a subsequence)

$$\xi_n \to \xi \in K \quad \Rightarrow \quad y_n \to \xi \quad \Rightarrow \quad L(y_n) \neq \emptyset.$$

Conversely, if K is not attracting, we deduce from (i) that

$$\mathrm{d}(y_n, K) > \varepsilon,$$

for some $\varepsilon > 0$ and $y_n \in \mathfrak{C}$. Therefore, $L(y_n) \cap K = \emptyset$. □

This simple characterization of $\mathbb{K}$ leads quite naturally to a notion of attractor, different from the classical one [1, 3, 4, 8], based only on the minimality with respect to the attraction property (cf. [2]).

Definition 10. We call *(global) attractor* of $S(t)$ the smallest set $A \in \mathbb{K}$, namely, the compact attracting set A contained in every compact attracting set.

The definition is clearly meaningless if $\mathbb{K}$ is empty. On the contrary, when $\mathbb{K} \neq \emptyset$, Proposition 9 tells that $A^\star$ (closed and contained in every closed attracting set) belongs to $\mathbb{K}$. In summary, we proved

Theorem 11. *The global attractor A exists if and only if $S(t)$ is asymptotically compact. In which case, A coincides with the set $A^\star$ (hence is unique).*

The main task is then showing the existence of sets $K \in \mathbb{K}$. In complete metric spaces, total dissipation turns out to be a sufficient condition as well.

Theorem 12. *Let X be a complete metric space. Then $S(t)$ is asymptotically compact if and only if is totally dissipative.*

Proof. We prove the nontrivial implication, i.e. totally dissipative implies asymptotically compact. For every $\varepsilon > 0$, let F_ε be a finite set such that $\mathcal{O}_\varepsilon(F_\varepsilon)$ is absorbing. Define

$$K = \bigcap_{\varepsilon > 0} B_\varepsilon \qquad \text{where} \qquad B_\varepsilon = \overline{\mathcal{O}_\varepsilon(F_\varepsilon)},$$

and select any $y_n \in \mathfrak{C}$. For an arbitrarily fixed $\varepsilon > 0$, the sequence y_n eventually falls in B_ε (which is obviously absorbing), whereas $K \subset B_\varepsilon$ by construction. Accordingly, both K and $\{y_n\}$ are totally bounded, i.e. coverable by finitely many balls of arbitrarily small radius. In complete metric spaces, this means precompactness. In particular, K is compact (being closed), while $L(y_n)$ is nonempty and contained in every (closed) set B_ε, hence in their intersection K. By Proposition 9, we conclude that $K \in \mathbb{K}$. □

After Proposition 6, the theorem can be equivalently stated in terms of Kuratowski measure of noncompactness (see [3]). Summarizing, if X is a complete metric space, the following statements imply each other:

- there exists the global attractor A;
- $S(t)$ is asymptotically compact;
- $S(t)$ is totally dissipative;
- $S(t)$ fulfills the Kuratowski property.

Anyway, the key point is that nothing more than (enough) dissipation is invoked. In other words, no continuity or continuity-like assumptions at all on $S(t)$ are needed for the existence of the global attractor.

4. **Invariance of the attractor.** Having the attractor A of $S(t)$, it is interesting to see whether or not the following situations occur:

- A is positively invariant: $S(t)A \subset A, \ \forall t \geq 0$;
- A is negatively invariant: $S(t)A \supset A, \ \forall t \geq 0$;
- A is invariant: $S(t)A = A, \ \forall t \geq 0$.

4.1. **A sufficient condition.** We preliminarily observe that negative invariance implies invariance. In fact, a stronger result holds.

Proposition 13. *If $S(\tau)A \supset A$ for some $\tau > 0$, then A is invariant.*

Proof. Let $t \geq 0$ be arbitrarily fixed. For any integer n, we obtain by recursion

$$S(t)A \subset S(t+\tau)A \subset S(t+2\tau)A \subset \cdots \subset S(t+n\tau)A.$$

Consequently,

$$\boldsymbol{\delta}(S(t)A, A) \leq \boldsymbol{\delta}(S(t+n\tau)A, A).$$

Since A is attracting and closed, letting $n \to \infty$ we deduce that

$$\boldsymbol{\delta}(S(t)A, A) = 0 \quad \Rightarrow \quad S(t)A \subset A.$$

Once A is shown to be positively invariant, the former inclusion for $t = 0$ entails

$$A \subset S(n\tau)A \subset A \quad \Rightarrow \quad A = S(n\tau)A.$$

Setting then $t_n = n\tau - t > 0$ (for n large enough), we end up with

$$A = S(n\tau)A = S(t+t_n)A = S(t)S(t_n)A \subset S(t)A \subset A,$$

proving the equality $S(t)A = A$. □

4.2. **The classical definition of attractor.** In the literature, invariance is actually postulated: the global attractor of $S(t)$ is by definition a compact set $A_{\rm cl} \subset X$ which is at the same time attracting and invariant. In particular, being invariant, $A_{\rm cl}$ belongs to any closed attracting set, and therefore is unique. More precisely, whenever exists,

- $A_{\rm cl}$ is the smallest closed attracting set;
- $A_{\rm cl}$ is the largest invariant bounded set.

Assuming the strong continuity of $S(t)$, that is,

$$S(t) \in \mathcal{C}(X, X), \quad \forall t \geq 0,$$

the existence of $A_{\rm cl}$ can be established by means of a classical criterion.

Theorem 14 (see [1, 3, 4, 8])**.** *Any asymptotically compact strongly continuous semigroup on a complete metric space X possesses the global attractor $A_{\rm cl}$.*

In a Banach space setting, the strong continuity assumption has been recently relaxed by requiring $S(t)$ to be only weakly continuous, i.e. continuous from X into $X_{\rm w}$ (see [9]). In more generality, Theorem 14 in metric spaces has also been proved under the weaker assumption that $S(t)$ be a closed map [2] for every t (see [6]).

[2] A map $f : X \to X$ is closed if $f(\xi) = \eta$ whenever $x_n \to \xi$ and $f(x_n) \to \eta$.

4.3. **Lack of invariance.** Without additional hypotheses on $S(t)$ (e.g. strong continuity), the attractor A may not fulfill any invariance, even if X is compact. To see that, we define the semigroups $U(t)$ and $V(t)$ on the metric subspaces of the complex plane $\mathbb{C}$

$$\mathbb{U} = \big\{u(\vartheta) = \mathrm{e}^{\mathrm{i}\vartheta},\ \vartheta \in [0, 2\pi)\big\} \qquad \text{and} \qquad \mathbb{V} = \big\{v(r) = \tfrac{2+r}{1+r}\,\mathrm{e}^{\mathrm{i}r},\ r \in [0, \infty)\big\},$$

respectively, acting as (for $t > 0$)

$$U(t)u(\vartheta) = u(2^{-t}(\vartheta - 2\pi) + 2\pi),$$
$$V(t)v(r) = v(r + t).$$

Example I. Choose $X = \mathbb{U}$ (complete metric space) and $S(t) = U(t)$. Then $A = \{u(0)\}$ is the global (and exponential) attractor. Indeed, for $t \gg 0$,

$$\boldsymbol{\delta}(S(t)X, A) = \mathrm{d}(S(t)u(0), u(0)) = \big|\mathrm{e}^{\mathrm{i}2^{1-t}\pi} - 1\big| \leq 2^{1-t}\pi.$$

At the same time, A dramatically fails to be positively invariant, for

$$S(t)A \cap S(\tau)A = \emptyset, \quad \forall t > \tau \geq 0.$$

Example II. Choose $X = \mathbb{U} \cup \mathbb{V}$ and

$$S(t)x = \begin{cases} U(t)x & \text{if } x \in \mathbb{U}, \\ V(t)x & \text{if } x \in \mathbb{V}. \end{cases}$$

Observe that $\mathbb{U} \cap \mathbb{V} = \emptyset$ and $\overline{\mathbb{V}} = X$ (in particular, X is compact). One can easily prove that A exists and coincides with $\mathbb{U}$. On the other hand,

$$S(t)A = \big\{u(\vartheta) :\ \vartheta \in [2\pi(1 - 2^{-t}), 2\pi)\big\}, \quad \forall t > 0.$$

We deduce the strict inclusions

$$S(t)A \subset \overline{S(t)A} = \{u(0)\} \cup S(t)A \subset A, \quad \forall t > 0.$$

Thus A is positively invariant but not invariant.

Remark. In both examples, the semigroup $S(t)$ is continuous in time, that is,

$$t \mapsto S(t)x \in \mathcal{C}([0, \infty), X), \quad \forall x \in X.$$

5. **Recovering Invariance.** The invariance of the attractor (in the sense of Definition 10) is attained whenever $S(t)$ fulfills a suitable continuity-like assumption, actually weaker than those considered in the literature.

5.1. **Recursively closed maps.** We begin with a definition, denoting as usual

$$f^0 = \mathrm{id}_X \qquad \text{and} \qquad f^k = f \circ \cdots \circ f \ \ (k\text{-times}).$$

Definition 15. A map $f : X \to X$ is *recursively closed* if, whenever $f^k(x_n) \to \xi^k$ occurs for every $k \in \mathbb{N}$, we have the equalities $f(\xi^k) = \xi^{k+1}$.

Any closed map is evidently recursively closed, but not the other way around. For instance, $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = -x\chi_{(-\infty,0)}(x) + \chi_{\{0\}}(x) + \frac{1}{x}\chi_{(0,\infty)}(x)$$

is recursively closed but not closed: we cannot exhibit any (nontrivial) sequence $x_n \to 0$ such that $f^k(x_n)$ converges for every $k \in \mathbb{N}$, whereas

$$x_n \uparrow 0 \quad \Rightarrow \quad f(x_n) = |x_n| \to 0 \neq f(0).$$

Remark. In a metric space X, the three continuity-like notions encountered so far can be reformulated in terms of the property

$$f(\overline{C}) \subset \overline{f(C)}, \quad C \subset X. \tag{P}$$

It is indeed standard matter verifying the mutual implications:

- f is continuous $\Leftrightarrow$ (P) holds whenever C is precompact;
- f is closed $\Leftrightarrow$ (P) holds whenever $f^k(C)$ are precompact for $k = 0, 1$;
- f is recursively closed $\Leftrightarrow$ (P) holds whenever $f^k(C)$ are precompact for all $k \in \mathbb{N}$.

In particular, the three notions coincide when X is a compact space.

5.2. **Asymptotically closed semigroups.** Dealing with semigroups, we can give a more general definition.

Definition 16. A semigroup $S(t)$ on X is said to be *asymptotically closed* if there exists a sequence of times $0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 \dots$ with the following property: whenever the convergence $S(\tau_k)x_n \to \xi^k$ occurs for every $k \in \mathbb{N}$ we have the equalities $S(\tau_k)\xi^0 = \xi^k$.

Remark. If for some fixed $\tau > 0$ the map $S(\tau)$ is recursively closed, then $S(t)$ is an asymptotically closed semigroup. Simply observe that the sequence $\tau_k = k\tau$ complies with Definition 16.

We are now in a position to state our invariance criterion.

Theorem 17. *Let $S(t)$ have the attractor A. If $S(t)$ is asymptotically closed, then A is invariant (hence an attractor in the classical sense).*

Proof. Let τ_k be any sequence complying with Definition 16. In light of Proposition 13, it is enough showing the inclusion $A \subset S(\tau_1)A$. To this end, select an arbitrary $x \in A$. Then, we learn from Theorem 11 that

$$y_n \to x \qquad \text{for some} \qquad y_n = S(t_n)x_n \in \mathfrak{C}.$$

Define the family of sequences depending on $k \in \mathbb{N}$ (with $n \gg 1$ to ensure $t_n \geq \tau_1$)

$$\eta_n^k = S(t_n + \tau_k - \tau_1)x_n \in \mathfrak{C}.$$

By Proposition 9, for every $k \in \mathbb{N}$ there is $\xi^k \in A$ such that $\eta_n^k \to \xi^k$ up to a subsequence. Applying a standard diagonalization method, we extract from each η_n^k a subsequence (that we keep calling η_n^k) so to have

$$S(\tau_k)\eta_n^0 = \eta_n^k \to \xi^k.$$

Since $S(t)$ is asymptotically closed, we draw the equalities

$$S(\tau_k)\xi^0 = \xi^k.$$

At this point, we merely observe that η_n^1 is a subsequence of the original y_n. Thus, for $k = 1$,

$$x = \xi^1 = S(\tau_1)\xi^0 \in S(\tau_1)A,$$

yielding the desired inclusion. □

One may wonder if in the hypotheses of Theorem 17 the restriction $S(t) : A \to A$ becomes continuous.

Proposition 18. *Let $S(t)$ be an asymptotically closed semigroup possessing the attractor A. Then the map $S(\tau)$ is continuous on A whenever τ is a finite sum of terms of the sequence τ_k.*

Proof. We can clearly assume $\tau = \tau_k$. Given $x \in A$, let $x_n \in A$ be any sequence converging to x. Then, by the same argument of the previous proof, for every fixed $k \in \mathbb{N}$ there is $\xi^k \in A$ such that (up to a subsequence)

$$S(\tau_k)x_n \to \xi^k \quad \Rightarrow \quad S(\tau_k)\xi^0 = \xi^k,$$

as $S(t)$ is asymptotically closed. Since $\xi^0 = x$, we deduce by compactness the convergence of the whole sequence $S(\tau_k)x_n$ to $S(\tau_k)x$, establishing the continuity of $S(\tau_k)$. □

However, the result is generally false for t arbitrary.

Example. Let $\psi : \mathbb{R} \to \mathbb{R}$ be the periodic extension of the identity map on $(0, 1]$. Consider the semigroup $S(t)$ on $X = [0, 1]$ given by

$$S(t)x = \begin{cases} \psi(x+t) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

It is readily seen that the global attractor A exists and coincides with the entire space X. Moreover, $S(k) = \mathrm{id}_X$ for every $k \in \mathbb{N}$. In particular, the semigroup fits Definition 16 with $\tau_k = k$, and so is asymptotically closed. At the same time, for any $t \notin \mathbb{N}$ and any strictly positive sequence $x_n \to 0$, we have

$$S(t)x_n = \psi(x_n + t) \to \psi(t) \neq 0 = S(t)0.$$

Hence $S(t)$ fails to be continuous at $x = 0$.

6. An Application to PDE.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$, and consider the wave equation with nonlinear damping

$$\begin{cases} u_{tt} + \sigma(u)u_t - \Delta u + \varphi(u) = 0, \\ u_{|\partial\Omega} = 0, \\ u(0) = u_0, \\ u_t(0) = u_1, \end{cases}$$

modeling a vibrating membrane in a stratified viscous medium. For simplicity, we assume here the nonlinearity φ and the displacement-dependent damping coefficient σ of the forms

$$\varphi(u) = u^3 - u \qquad \text{and} \qquad \sigma(u) = 1 + u^2.$$

As shown in [7], for every $T > 0$ and every initial data $x = (u_0, u_1)$ in the weak energy space

$$X = H_0^1(\Omega) \times L^2(\Omega),$$

there is a unique variational solution

$$u \in \mathcal{C}([0,T], H_0^1(\Omega)) \cap \mathcal{C}^1([0,T], L^2(\Omega)).$$

Accordingly, the equation generates a semigroup $S(t) : X \to X$. The same paper shows the existence of a (compact) attracting set for $S(t)$, which is bounded in the more regular space

$$Y = \left[H^2(\Omega) \cap H_0^1(\Omega)\right] \times H_0^1(\Omega) \Subset X.$$

Hence, the semigroup is asymptotically compact, and by Theorem 11 we infer the existence of the global attractor A (in the sense of Definition 10) bounded in Y. Nonetheless, calling

$$W = L^2(\Omega) \times H^{-1}(\Omega),$$

we can only prove the following continuous dependence estimate with respect to the initial data (see [7, Proposition 2.5]).

Proposition 19. *Given any $\varrho \geq 0$ there is $\kappa = \kappa(\varrho) \geq 0$ such that*

$$\|S(t)x_1 - S(t)x_2\|_W \leq \kappa \mathrm{e}^{\kappa t} \|x_1 - x_2\|_X, \quad \forall t > 0,$$

for all $x_1, x_2 \in X$ of norm not exceeding ϱ.

In other words, for every fixed $t > 0$, the semigroup fulfills the weaker continuity property

$$S(t) \in \mathcal{C}(X, W),$$

not enough to apply the classical Theorem 14, yielding the invariant global attractor. For any $t > 0$, it is however apparent that the simultaneous occurrences

$$x_n \xrightarrow{X} \xi \qquad \text{and} \qquad S(t)x_n \xrightarrow{X} \eta$$

imply the equality

$$S(t)\xi = \eta.$$

Indeed, by the continuity of $S(t)$, we know that

$$x_n \xrightarrow{X} \xi \quad \Rightarrow \quad S(t)x_n \xrightarrow{W} S(t)\xi,$$

allowing us to identify the limit. Therefore, $S(t)$ is a closed (hence recursively closed) semigroup, and the invariance of A follows from Theorem 17.

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