Some Properties of Malgrange Isomonodromic Deformations of Linear 2×2 Systems

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Abstract—We study movable singularities of the Malgrange isomonodromic deformation of a linear differential 2×2 system with two irregular singularities of Poincaré rank 1 and with an arbitrary number of Fuchsian singular points.

1. INTRODUCTION

Consider a *meromorphic linear 2×2 system* on the Riemann sphere $\overline{\mathbb{C}}$, i.e., a system of two linear ordinary differential equations with singularities $a_1^0, \ldots, a_n^0 \in \mathbb{C}$ and possibly ∞. By a conformal mapping one can always arrange that all the singularities are in the complex plane only. This means that one can reduce the system to the form

$$\frac{dy}{dz} = A(z)y, \qquad A(z) = \sum_{i=1}^{n} \sum_{j=1}^{r_i+1} \frac{A_{ij}^0}{(z - a_i^0)^j}, \tag{1}$$

where $y(z) \in \mathbb{C}^2$, A_{ij}^0 are 2×2 matrices and $\sum_{i=1}^{n} A_{i1}^0 = 0$, to ensure that ∞ is not a singular point.

The non-negative integers $r_1, \ldots, r_n$ are called the *Poincaré ranks* of the singularities $a_1^0, \ldots, a_n^0$, respectively. One can assume that $r_1, \ldots, r_m$ are positive and $r_{m+1} = \ldots = r_n = 0$ (that is, the singular points $a_{m+1}^0, \ldots, a_n^0$ are *Fuchsian*) for some $0 \le m \le n$.

We consider the *non-resonant* case (the generic case). This means that the leading term A_{i,r_i+1}^0 of each non-Fuchsian singularity a_i^0, $i = 1, \ldots, m$, has two distinct eigenvalues (thus, the singular points $a_1^0, \ldots, a_m^0$ are irregular).

System (1) can be thought of as a meromorphic connection ∇^0 (more precisely, as an equation for horizontal sections with respect to this connection) on a holomorphically trivial vector bundle E^0 of rank 2 over $\overline{\mathbb{C}}$. As known (see [9, § 21]), in a neighbourhood of each (non-resonant) irregular singularity a_i^0 the local connection form $\omega^0 = A(z)\, dz$ of ∇^0 is formally equivalent to the 1-form

$$\omega_{\Lambda_i^0} = \sum_{j=1}^{r_i+1} \frac{\Lambda_{ij}^0}{(z - a_i^0)^j}\, dz,$$

where $\Lambda_{i1}^0, \ldots, \Lambda_{i,r_i+1}^0$ are diagonal matrices and the leading term Λ_{i,r_i+1}^0 is conjugate to A_{i,r_i+1}^0. This means that there is an invertible formal matrix Taylor series $\widehat{F}(z)$ in $z - a_i^0$ such that the transformation $\widetilde{y} = \widehat{F}^{-1}(z)y$ takes the 1-form ω^0 to $\omega_{\Lambda_i^0}$:

$$\omega_{\Lambda_i^0} = \widehat{F}^{-1}\omega^0\widehat{F} - \widehat{F}^{-1}(d\widehat{F}).$$

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One should note that formally equivalent systems in a neighbourhood $O_{a_i^0}$ of an irregular singularity a_i^0 are not necessary holomorphically or meromorphically equivalent. System (1) has in $O_{a_i^0}$ a formal fundamental matrix of the form

$$\widehat{Y}(z) = \widehat{F}(z)(z - a_i^0)^{\Lambda_{i1}^0} e^{Q(z)}, \qquad Q(z) = -\sum_{j=1}^{r_i} \frac{\Lambda_{i,j+1}^0}{j}(z - a_i^0)^{-j}. \qquad (2)$$

One can cover $O_{a_i^0}$ by a set of sufficiently small sectors $S_1, \ldots, S_N$ with a vertex at a_i^0 such that in each S_k there exists a unique fundamental matrix $Y_k(z) = F_k(z)(z - a_i^0)^{\Lambda_{i1}^0} e^{Q(z)}$ of the system with $F_k(z)$ having $\widehat{F}(z)$ as an asymptotic series in S_k (see [9, § 21]). In every intersection $S_k \cap S_{k+1}$ the fundamental matrices $Y_k(z)$ and $Y_{k+1}(z)$ are connected by a non-singular constant matrix C_k: $Y_{k+1}(z) = Y_k(z)C_k$, which is called a *Stokes matrix*. If a_i^0 is a non-resonant singularity, then two formally equivalent systems are holomorphically equivalent in $O_{a_i^0}$ if and only if they have the same sets of Stokes matrices (see [9, § 21]).

Further we will focus on deformations of system (1) (of the pair (E^0, ∇^0)) that allow the local formal equivalence class

$$\omega_{\Lambda_i} = \sum_{j=2}^{r_i+1} \frac{\Lambda_{ij}}{(z - a_i)^j}\, dz + \frac{\Lambda_{i1}^0}{z - a_i}\, dz, \qquad i = 1, \ldots, m,$$

to vary in the sense that the diagonal matrices $\Lambda_{i2}, \ldots, \Lambda_{i,r_i+1}$ vary in a neighbourhood of the initial data $\Lambda_{i2}^0, \ldots, \Lambda_{i,r_i+1}^0$ with Λ_{i1}^0 held fixed. Thus for the set $\Lambda_i = \{\Lambda_{i2}, \ldots, \Lambda_{i,r_i+1}\}$ of r_i diagonal matrices we denote by ∇_{Λ_i} the meromorphic connection (on the holomorphically trivial vector bundle of rank 2 over O_{a_i}) whose 1-form is ω_{Λ_i}. To describe the required deformations in more detail, let us begin with a deformation space.

For $k \in \mathbb{N}$ we denote by Z^k the subset of the space $\mathbb{C}^k$ whose points have pairwise distinct coordinates. Then Z^n will be the space of pole locations and

$$\mathcal{C}_i = \underbrace{\mathbb{C}^2 \times \ldots \times \mathbb{C}^2}_{r_i-1} \times Z^2, \qquad i = 1, \ldots, m,$$

will be the space of parameters determining a local formal equivalence class of 1-forms near the singular point a_i (any class is determined by $r_i - 1$ diagonal matrices $\Lambda_{i2}, \ldots, \Lambda_{i,r_i}$ and a diagonal matrix Λ_{i,r_i+1} whose eigenvalues are distinct). Define the deformation space $\mathcal{D}$ as the universal cover of the Cartesian product $Z^n \times \mathcal{C}_1 \times \ldots \times \mathcal{C}_m$, that is,

$$\mathcal{D} = \widetilde{Z}^n \times \widetilde{\mathcal{C}}_1 \times \ldots \times \widetilde{\mathcal{C}}_m.$$

On the set $\mathcal{D}$ one has the standard projections

$$a = (a_1, \ldots, a_n) \colon \mathcal{D} \to Z^n, \qquad \Lambda_i = (\Lambda_{i2}, \ldots, \Lambda_{i,r_i+1}) \colon \mathcal{D} \to \mathcal{C}_i, \quad i = 1, \ldots, m.$$

For every $t \in \mathcal{D}$ we denote by $a_i(t)$ the ith coordinate of the image of t under the first projection and by $\Lambda_i(t)$ the image of t under the second one. Denote then by t^0 the base point of the deformation space $\mathcal{D}$ corresponding to system (1), i.e., $a(t^0) = (a_1^0, \ldots, a_n^0)$ and $\Lambda_i(t^0) = (\Lambda_{i2}^0, \ldots, \Lambda_{i,r_i+1}^0)$. Consider also the singular hypersurfaces

$$X_i = \big\{ (z, t) \in \overline{\mathbb{C}} \times \mathcal{D} \mid z = a_i(t) \big\} \subset \overline{\mathbb{C}} \times \mathcal{D}, \qquad i = 1, \ldots, n.$$

Now for $i = 1, \ldots, m$, consider the fibre bundle $\mathcal{M}_i \to \mathcal{C}_i$ whose fibre over each point $\Lambda_i \in \mathcal{C}_i$ is the space of parameters determining a local *holomorphic* equivalence class of connections that are

all formally equivalent to the connection ∇_{Λ_i}. Every point of this fibre (every local holomorphic equivalence class of connections) is determined by a corresponding set of Stokes matrices. Let $\sigma_i^0 \in \mathcal{M}_i$ denote the holomorphic equivalence class of the connection $\nabla^0|_{O_{a_i^0}} \sim \nabla_{\Lambda_i^0}$, and let σ_i denote the unique horizontal section of the fibre bundle $\mathcal{M}_i \to \mathcal{C}_i$ such that $\sigma_i(\Lambda_i^0) = \sigma_i^0$.

Due to B. Malgrange [12, Theorem 3.1] (see also [14, Theorem 2.9]) the following statement holds.

Theorem 1. *There exists a unique[1] isomonodromic deformation (E, ∇) of the pair (E^0, ∇^0), that is, a rank 2 holomorphic vector bundle E over $\overline{\mathbb{C}} \times \mathcal{D}$ and integrable meromorphic connection ∇ with simple type r_i singularities along X_i, $i = 1, \dots, n$, satisfying the following properties:*

- *the restriction of (E, ∇) to $\overline{\mathbb{C}} \times \{t^0\}$ is isomorphic to (E^0, ∇^0);*
- *for any $t \in \mathcal{D}$ the restriction of ∇ to $\overline{\mathbb{C}} \times \{t\}$ is formally equivalent to the local connection $\nabla_{\Lambda_i(t)}$ near $z = a_i(t)$, $i = 1, \dots, m$, and belongs to the local holomorphic equivalence class $\sigma_i(\Lambda_i(t)) \in \mathcal{M}_i$.*

The requirement for ∇ to have simple type r_i singularities along X_i means that near X_i the local connection 1-form Ω of ∇ looks like

$$\Omega = \frac{A_i(z, t)}{(z - a_i(t))^{r_i+1}} \, d(z - a_i(t)) + \sum_k \frac{\widetilde{A}_{ik}(z, t)}{(z - a_i(t))^{r_i}} \, dt_k,$$

where the matrices A_i and $\widetilde{A}_{ik}$ are holomorphic and the eigenvalues of the matrix $A_i(a_i(t), t)$ are distinct $(i = 1, \dots, m)$.

According to the Malgrange–Helminck–Palmer theorem (see [14, §3] or [12, §3]), either the set

$$\Theta = \left\{ t \in \mathcal{D} \mid E|_{\overline{\mathbb{C}} \times \{t\}} \text{ is non-trivial} \right\}$$

is empty or $\Theta \subset \mathcal{D}$ is an analytic subset of codimension 1. If the latter holds, there exists a function τ holomorphic on the whole space $\mathcal{D}$ whose zero set coincides with Θ. The set Θ is called the *Malgrange Θ-divisor*, and the function τ is called the *τ-function* of the isomonodromic deformation.

Thus the Malgrange isomonodromic deformation of the pair (E^0, ∇^0) determines an isomonodromic deformation

$$\frac{dy}{dz} = \left(\sum_{i=1}^n \sum_{j=1}^{r_i+1} \frac{A_{ij}(t)}{(z - a_i(t))^j} \right) y, \qquad A_{ij}(t^0) = A_{ij}^0, \tag{3}$$

of system (1) for $t \in D(t^0)$, where $D(t^0)$ is a neighbourhood of the point t^0 in the space $\mathcal{D}$. The matrix functions $A_{ij}(t)$, holomorphic in $D(t^0)$, can be extended meromorphically to the whole space $\mathcal{D}$ and have Θ as a polar locus.

In the case of a Fuchsian system $(m = 0)$

$$\frac{dy}{dz} = \left(\sum_{i=1}^n \frac{A_i^0}{z - a_i^0} \right) y \tag{4}$$

the best known isomonodromic deformations are the Schlesinger ones [16, 17]. Starting from the initial conditions $A_i(a^0) = A_i^0$, $a^0 = (a_1^0, \dots, a_n^0)$, the residue matrices $A_i(a)$ vary according to the *Schlesinger equation*

$$dA_i(a) = - \sum_{j=1, \, j \neq i}^n \frac{[A_i(a), A_j(a)]}{a_i - a_j} \, d(a_i - a_j), \qquad i = 1, \dots, n,$$

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[1]Under some additional assumption that will be discussed later on.

and they are extended as meromorphic matrix functions to the deformation space $\widetilde{Z}^n$ from a neighbourhood $D(a^0)$ of the initial point a^0.

A.A. Bolibruch [2; 4, Theorem 16.2] obtained the following result concerning the pole orders of the matrices $A_i(a)$.

Theorem 2. *Let the monodromy of the 2×2 system (4) be irreducible, and let $a^* \in \Theta$ be a point of the Θ-divisor such that the restriction $E|_{\overline{\mathbb{C}} \times \{a^*\}}$ is of the form*

$$E|_{\overline{\mathbb{C}} \times \{a^*\}} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1).$$

Then in a neighbourhood $D(a^*)$ of a^* the Θ-divisor is an analytic submanifold and the matrix functions $A_i(a)$ have poles of at most second order along $D(a^*) \cap \Theta$.

In other words, one asserts that the products $\tau^2(a)A_i(a)$ are holomorphic matrix functions in $D(a^*)$. The proof of Theorem 2 (formulated in a more general setting) is also contained in [7].

Adapting Bolibruch's ideas to the case of linear systems with irregular singularities, we propose a local description of the Θ-divisor of the Malgrange isomonodromic deformation and a generalization of Theorem 2 to the case when the initial system has at most two irregular singularities and their Poincaré ranks are equal to 1 (Theorem 3).

2. HOLOMORPHIC VECTOR BUNDLES AND THE RIEMANN–HILBERT PROBLEM

The fact $t^* \in \Theta$ means that the restriction $E|_{\overline{\mathbb{C}} \times \{t^*\}}$ of the holomorphic vector bundle E described in Theorem 1 is not holomorphically trivial. This restriction belongs to the family $\mathcal{F}$ of holomorphic vector bundles over the Riemann sphere endowed with meromorphic connections. The family $\mathcal{F}$ occurs in the investigation of the corresponding Riemann–Hilbert problem, the question on existence of a global meromorphic linear system with the singular points $a_1^* = a_1(t^*), \ldots, a_n^* = a_n(t^*)$ of Poincaré ranks $r_1, \ldots, r_n$, respectively, that

(i) has the same monodromy as the initial system and

(ii) is meromorphically equivalent to the local system

$$dy = \omega_i^* y \tag{5}$$

determined by the local holomorphic equivalence class $\sigma_i(\Lambda_i(t^*))$ near each irregular singular point a_i^*.

The Riemann–Hilbert problem under consideration has a positive answer (it is sufficient that one of the irregular singularities be non-resonant for the positive solution in the two-dimensional case, see [5]). This means that there is a holomorphically trivial holomorphic vector bundle in the family $\mathcal{F}$. Thus we are coming to the point where it is natural to recall briefly the construction of the family $\mathcal{F}$ (see details in [5]).

By the monodromy representation (generated by the monodromy matrices $G_1, \ldots, G_n$) of the initial system (1) one constructs a rank 2 holomorphic vector bundle $\widetilde{F}$ over the punctured Riemann sphere $\overline{\mathbb{C}} \setminus \{a_1^*, \ldots, a_n^*\}$ with a holomorphic connection $\widetilde{\nabla}$ having the prescribed monodromy. This bundle is defined by a set $\{U_\alpha\}$ of sufficiently small discs covering $\overline{\mathbb{C}} \setminus \{a_1^*, \ldots, a_n^*\}$ and a set $\{g_{\alpha\beta}\}$ of constant matrices defining a gluing cocycle which corresponds to this covering. The connection $\widetilde{\nabla}$ is defined by a set $\{\omega_\alpha\}$ of matrix differential 1-forms $\omega_\alpha \equiv 0$. So in the intersections $U_\alpha \cap U_\beta \neq \varnothing$ the gluing conditions

$$\omega_\alpha = (dg_{\alpha\beta})g_{\alpha\beta}^{-1} + g_{\alpha\beta}\omega_\beta g_{\alpha\beta}^{-1}$$

hold.

Further one extends the pair $(\widetilde{F}, \widetilde{\nabla})$ to the whole Riemann sphere. In neighbourhoods $O_{a_i^*}$ of the irregular singular points a_i^*, $i = 1, \ldots, m$, the extension of $\widetilde{\nabla}$ is determined by the corresponding local matrix differential 1-forms ω_i^* of the coefficients of systems (5), while in neighbourhoods $O_{a_i^*}$ of the Fuchsian singular points a_i^*, $i = m+1, \ldots, n$, the extension of $\widetilde{\nabla}$ is determined by the matrix differential 1-forms $E_i\, dz/(z - a_i^*)$. Here $E_i = (2\pi\sqrt{-1})^{-1} \ln G_i$ is a normalized logarithm of the monodromy matrix G_i and its branch is chosen so that the eigenvalues ρ_i^k of E_i satisfy the condition

$$0 \le \operatorname{Re} \rho_i^k < 1. \tag{6}$$

This is the so-called *canonical* extension $(\widetilde{F}^0, \widetilde{\nabla}^0)$ of the pair $(\widetilde{F}, \widetilde{\nabla})$ in the sense of Malgrange [13] (and in the sense of Deligne [8] for the Fuchsian case).

Finally, consider a formal fundamental matrix (see (2))

$$\widehat{Y}_i(z) = \widehat{F}_i(z)(z - a_i^*)^{\Lambda_{i1}^0} e^{Q_i(z)}, \qquad Q_i(z) = -\sum_{j=1}^{r_i} \frac{\Lambda_{i,j+1}^*}{j}(z - a_i^*)^{-j}, \qquad \Lambda_{i,j+1}^* = \Lambda_{i,j+1}(t^*),$$

of each local irregular system (5), $i = 1, \ldots, m$, and write it in the form

$$\widehat{Y}_i(z) = \widehat{F}_i(z)(z - a_i^*)^{D_i^0}(z - a_i^*)^{\widehat{E}_i} e^{Q_i(z)}, \qquad D_i^0 = [\operatorname{Re} \Lambda_{i1}^0]. \tag{7}$$

The diagonal elements of the integer matrix D_i^0 are referred to as the *formal valuations* of the system. As follows, the diagonal elements ρ_i^k of the matrix $\widehat{E}_i$ satisfy condition (6). By an analogue of Sauvage's lemma (see [15, Lemma 11.2]) for formal matrix series, for any diagonal integer matrix D_i there exists a matrix $\Gamma_i'(z)$ meromorphically invertible in $O_{a_i^*}$ such that

$$\Gamma_i'(z)\widehat{F}_i(z)(z - a_i^*)^{D_i^0 - D_i} = (z - a_i^*)^{\widetilde{D}_i}\widehat{H}_i(z), \tag{8}$$

where $\widetilde{D}_i$ is a diagonal integer matrix and $\widehat{H}_i(z)$ is an invertible formal (matrix) Taylor series in $z - a_i^*$.

Now one constructs the family $\mathcal{F}$ of extensions of the pair $(\widetilde{F}, \widetilde{\nabla})$ by replacing the 1-form ω_i^* in the construction of $(\widetilde{F}^0, \widetilde{\nabla}^0)$ with the 1-form

$$\omega^{D_i} = (d\Gamma_i)\Gamma_i^{-1} + \Gamma_i \omega_i^* \Gamma_i^{-1}, \qquad \Gamma_i(z) = (z - a_i^*)^{-\widetilde{D}_i}\Gamma_i'(z), \qquad i = 1, \ldots, m,$$

and the 1-form $E_i\, dz/(z - a_i^*)$ with the 1-form

$$\omega^{D_i} = (d\Gamma_i)\Gamma_i^{-1} + \Gamma_i \frac{E_i\, dz}{z - a_i^*}\Gamma_i^{-1}, \qquad \Gamma_i(z) = (z - a_i^*)^{D_i}C_i, \qquad i = m+1, \ldots, n,$$

where D_i is a diagonal integer matrix whose diagonal elements (for $i = m+1, \ldots, n$) form a non-increasing sequence, and C_i is a non-singular matrix reducing the matrix E_i to an upper triangular form $E_i' = C_i E_i (C_i)^{-1}$. As follows from (7), (8), a formal fundamental matrix of the local irregular system $dy = \omega^{D_i} y$, $i = 1, \ldots, m$, is of the form

$$\widehat{Y}_i'(z) = \Gamma_i(z)\widehat{Y}_i(z) = \widehat{H}_i(z)(z - a_i^*)^{D_i}(z - a_i^*)^{\widehat{E}_i} e^{Q_i(z)}. \tag{9}$$

Its singular point $z = a_i^*$ is of Poincaré rank r_i again. At the same time, the local system $dy = \omega^{D_i} y$, $i = m+1, \ldots, n$, is Fuchsian:

$$\omega^{D_i} = \left(\frac{D_i}{z - a_i^*} + (z - a_i^*)^{D_i}\frac{E_i'}{z - a_i^*}(z - a_i^*)^{-D_i}\right)dz.$$

Let us call the matrices $D_1, \dots, D_n$ and $C_{m+1}, \dots, C_n$ involved in the construction above the *admissible* matrices. Thus the family $\mathcal{F}$ consists of the pairs $(F^{D,C}, \nabla^{D,C})$ obtained by using all sets $(D, C) = \{D_1, \dots, D_n, C_{m+1}, \dots, C_n\}$ of admissible matrices. Though the matrices $\Gamma_1'(z), \dots, \Gamma_m'(z)$ (see (8)) are also involved in the construction of the pair $(F^{D,C}, \nabla^{D,C})$, one should note that in our (non-resonant) case the bundle $F^{D,C}$ does not depend on them (for a fixed set (D, C)).

Now the restriction $(E, \nabla)|_{\overline{\mathbb{C}} \times \{t^*\}}$ can be thought of as an element of the family $\mathcal{F}$:

$$(E, \nabla)|_{\overline{\mathbb{C}} \times \{t^*\}} \cong \left(F^{D^0, C^0}, \nabla^{D^0, C^0} \right), \qquad D^0 = \{D_1^0, \dots, D_n^0\}, \quad C^0 = \{C_{m+1}^0, \dots, C_n^0\},$$

where the matrices $D_1^0, \dots, D_m^0$ are defined in (7), and the sets of the (admissible) matrices $D_{m+1}^0, \dots, D_n^0$ and $C_{m+1}^0, \dots, C_n^0$ come from the *Levelt* decompositions [11] of a fundamental matrix $Y(z)$ of the initial system (1) at the corresponding Fuchsian singularities $a_{m+1}^0, \dots, a_n^0$:

$$Y(z) = U_i(z)(z - a_i^0)^{D_i^0} C_i^0 (z - a_i^0)^{E_i}, \qquad i = m+1, \dots, n,$$

where the matrix $U_i(z)$ is holomorphically invertible at the point a_i^0. The matrices $D_{m+1}^0, \dots, D_n^0$ are preserved along the deformation (see [3]). One usually requires the matrices $C_{m+1}^0, \dots, C_n^0$ to be also preserved, to ensure that the Malgrange deformation is a unique isomonodromic deformation of the pair (E^0, ∇^0) (see Theorem 1).

3. THEOREM ON Θ-DIVISOR

Now let us consider a linear meromorphic 2×2 system with n singular points such that $m \leq 2$ of them are irregular and their Poincaré ranks are equal to 1, i.e., a system of the form (1) with $r_{1,2} \leq 1$ and $r_3 = \dots = r_n = 0$:

$$\frac{dy}{dz} = \left(\frac{A_{12}^0}{(z - a_1^0)^2} + \frac{A_{22}^0}{(z - a_2^0)^2} + \sum_{i=1}^n \frac{A_{i1}^0}{z - a_i^0} \right) y. \tag{10}$$

The Θ-divisor and the coefficient matrices $A_{ij}(t)$ of the Malgrange isomonodromic deformation (3) of such a system possess the following properties.

Theorem 3. *Let the monodromy representation of the 2×2 system (10) be irreducible, and let $t^* \in \Theta$ be a point of the Θ-divisor such that*

$$E|_{\overline{\mathbb{C}} \times \{t^*\}} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1).$$

Then in a neighbourhood $D(t^*)$ of t^* the Θ-divisor is an analytic submanifold and the matrix functions $A_{ij}(t)$ have poles of at most second order along $D(t^*) \cap \Theta$.

Before proving Theorem 3 let us recall a calculation algorithm for the local τ-function of the Malgrange isomonodromic deformation (E, ∇) of system (10).

Consider a point $t^* \in \Theta$. Though the corresponding pair $(E, \nabla)|_{\overline{\mathbb{C}} \times \{t^*\}} \cong (F^{D^0, C^0}, \nabla^{D^0, C^0})$ is such that the bundle

$$F^{D^0, C^0} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$$

is not holomorphically trivial, one can construct an auxiliary linear meromorphic system

$$\frac{dy}{dz} = \left(\frac{B_{12}^0}{(z - a_1^*)^2} + \frac{B_{22}^0}{(z - a_2^*)^2} + \sum_{i=1}^n \frac{B_{i1}^0}{z - a_i^*} \right) y \tag{11}$$

with irregular non-resonant singular points $a_1^* = a_1(t^*)$, $a_2^* = a_2(t^*)$ of Poincaré rank 1 and Fuchsian singular points $a_3^* = a_3(t^*), \ldots, a_n^* = a_n(t^*)$. This system is holomorphically equivalent to the local system determined by the connection ∇^{D^0,C^0} in a neighbourhood of each singular point, but it has an *apparent* Fuchsian singularity at infinity (i.e., the monodromy at this point is trivial). Its fundamental matrix is of the form $Y^*(z) = U(z)z^K$ near infinity, where

$$U(z) = I + U_1 \frac{1}{z} + U_2 \frac{1}{z^2} + \ldots, \qquad K = \mathrm{diag}(-1, 1). \tag{12}$$

Therefore, the residue matrix at infinity is equal to $-K$, and $\sum_{i=1}^n B_{i1}^0 = K$ (the existence of such a system in the Fuchsian case is explained, for example, in the proof of Theorem 2 from [6]; an explanation here is the same).

The columns of the fundamental matrix $Y^*(z)$ of system (11) determine a basis of horizontal (with respect to ∇^{D^0,C^0}) sections of the bundle F^{D^0,C^0} over $\mathbb{C}$. Consider a matrix $V(z)$ holomorphically invertible in a neighbourhood O_∞ of infinity whose columns determine this basis over O_∞. Then the quotient $Y^*(z)V^{-1}(z) = g_{0\infty}$ is a cocycle of the bundle F^{D^0,C^0}. On the other hand, the cocycle of the bundle F^{D^0,C^0} is z^K; hence

$$U(z)z^K = z^K V(z). \tag{13}$$

Let us include the auxiliary system (11) in the Malgrange isomonodromic family

$$\frac{dy}{dz} = \left(\frac{B_{12}(t)}{(z - a_1(t))^2} + \frac{B_{22}(t)}{(z - a_2(t))^2} + \sum_{i=1}^n \frac{B_{i1}(t)}{z - a_i(t)} \right) y, \qquad B_{ij}(t^*) = B_{ij}^0. \tag{14}$$

An appropriate matrix meromorphic differential 1-form determining this family (see [10, Ch. 4, § 1]) has the form

$$\omega = \sum_{i=1}^2 \frac{B_{i2}(t)}{(z - a_i(t))^2} \, d(z - a_i(t)) + \sum_{i=1}^n \frac{B_{i1}(t)}{z - a_i(t)} \, d(z - a_i(t)) + (d\Lambda)\text{-part}. \tag{15}$$

Observe that the equality $\sum_{i=1}^n B_{i1}(t) = K$ holds. Indeed, the differential 1-form ω satisfies the Frobenius integrability condition ($d\omega = \omega \wedge \omega$). One can directly check that the residue (in the sense of Leray) of $\omega \wedge \omega$ along $\{z = \infty\}$ is equal to zero and the residue of the 2-form $d\omega$ along $\{z = \infty\}$ is equal to $d\sum_{i=1}^n B_{i1}(t)$.

Let $Y(z,t)$ be the fundamental matrix of the Pfaffian system $dy = \omega y$ of the form

$$Y(z,t) = U(z,t)z^K, \qquad U(z,t) = I + U_1(t)\frac{1}{z} + U_2(t)\frac{1}{z^2} + \ldots, \tag{16}$$

at infinity, and $Y(z,t^*) = Y^*(z)$ (by analogy with the Fuchsian case [1]).

As follows from (15),

$$\frac{\partial Y}{\partial a_i} Y^{-1} = -\sum_{j=1}^{r_i+1} \frac{B_{ij}(t)}{(z - a_i)^j} = -\sum_{j=1}^{r_i+1} \frac{B_{ij}(t)}{z^j \left(1 - \frac{a_i}{z}\right)^j}. \tag{17}$$

Expanding the left- and the right-hand sides of (17) in Taylor series near infinity, one gets

$$\frac{\partial U_1(t)}{\partial a_i} \frac{1}{z} + o(z^{-1}) = \left(-B_{i1}(t)\frac{1}{z} + o(z^{-1}) \right) \left(I + U_1(t)\frac{1}{z} + o(z^{-1}) \right);$$

therefore,

$$\frac{\partial U_1(t)}{\partial a_i} = -B_{i1}(t), \qquad i = 1, \ldots, n. \tag{18}$$

Further the relation

$$\frac{\partial Y}{\partial z} Y^{-1} = \sum_{i=1}^{n} \sum_{j=1}^{r_i+1} \frac{B_{ij}(t)}{z^j \left(1 - \frac{a_i}{z}\right)^j}$$

implies

$$-U_1(t)\frac{1}{z^2} + o(z^{-2}) + \left(I + U_1(t)\frac{1}{z} + o(z^{-1})\right)\frac{K}{z}$$

$$= \left(\frac{K}{z} + \left(\sum_{i=1}^{n} B_{i1}(t)a_i + B_{12}(t) + B_{22}(t)\right)\frac{1}{z^2} + o(z^{-2})\right)\left(I + U_1(t)\frac{1}{z} + o(z^{-1})\right);$$

hence

$$-U_1 + [U_1, K] = \sum_{i=1}^{n} B_{i1}(t)a_i + B_{12}(t) + B_{22}(t).$$

Thus the upper right element $u_1(t)$ of the matrix $U_1(t)$ coincides with the same element of the matrix $\sum_{i=1}^{n} B_{i1}(t)a_i + B_{12}(t) + B_{22}(t)$.

Lemma 1. *The function $u_1(t)$ is not equal to zero identically and vanishes at the point $t = t^*$.*

Proof. Since the matrix $U_1(t^*)$ is the matrix U_1 from (12), the vanishing of $u_1(t)$ at the point t^* follows from relation (13).

Now let us explain that the function $u_1(t)$ is not equal to zero identically. We denote by $b_{ij}(t)$ the upper right elements of the matrices $B_{ij}(t)$. Then

$$u_1(t) = b_{12}(t) + b_{22}(t) + \sum_{i=1}^{n} b_{i1}(t)a_i$$

and, as follows from (18),

$$\frac{\partial u_1(t)}{\partial a_i} = -b_{i1}(t), \qquad i = 1, \ldots, n.$$

Thus the equality $u_1(t) \equiv 0$ implies

$$b_{i1}(t) \equiv 0, \quad i = 1, \ldots, n, \qquad b_{12}(t) + b_{22}(t) \equiv 0.$$

We will show that $b_{12}(t) = b_{22}(t) \equiv 0$ as well, which contradicts the irreducibility of the monodromy of the family (14).

Let us turn to a new independent variable $\xi = z^{-1}$ to examine the matrix differential 1-form $B(z, t) dz$ of the coefficients of the family (14) near the apparent singularity $z = \infty$ (respectively, near the apparent singularity $\xi = 0$ after the change of the variable):

$$B(z, t) dz = -\frac{B(\xi^{-1}, t)}{\xi^2} d\xi,$$

$$-\frac{B(\xi^{-1},t)}{\xi^2} = -\sum_{i=1}^{2}\frac{B_{i2}(t)}{(1-a_i\xi)^2} - \sum_{i=1}^{n}\frac{B_{i1}(t)}{\xi(1-a_i\xi)}$$

$$= -\frac{1}{\xi}\left(K + \sum_{i=1}^{n}B_{i1}(t)a_i\xi + \sum_{i=1}^{n}B_{i1}(t)a_i^2\xi^2 + o(\xi^2)\right) - \left(\sum_{i=1}^{2}B_{i2}(t) + 2\sum_{i=1}^{2}B_{i2}(t)a_i\xi + o(\xi)\right)$$

$$= -\frac{1}{\xi}K - \left(\sum_{i=1}^{n}B_{i1}(t)a_i + \sum_{i=1}^{2}B_{i2}(t)\right) - \left(\sum_{i=1}^{n}B_{i1}(t)a_i^2 + 2\sum_{i=1}^{2}B_{i2}(t)a_i\right)\xi + o(\xi)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\xi} + \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} + \begin{pmatrix} * & -2\sum_{i=1}^{2}b_{i2}(t)a_i \\ * & * \end{pmatrix}\xi + o(\xi).$$

The gauge transformation $\widetilde{y} = \xi^K y$ changes the latter matrix into a new one of the form

$$\frac{1}{\xi}\begin{pmatrix} 0 & -2\sum_{i=1}^{2}b_{i2}(t)a_i \\ 0 & 0 \end{pmatrix} + O(1).$$

The monodromy matrix of the Fuchsian singular point $\xi = 0$ of the transformed system is the identity matrix. On the other hand, both eigenvalues of its residue matrix are zeros. Thus the monodromy matrix is equal to the exponential of the residue matrix, i.e.,

$$\exp\left\{2\pi\sqrt{-1}\begin{pmatrix} 0 & -2\sum_{i=1}^{2}b_{i2}(t)a_i \\ 0 & 0 \end{pmatrix}\right\} = I.$$

Then the equality $b_{12}(t)a_1 + b_{22}(t)a_2 \equiv 0$ holds, which (together with the equality $b_{12}(t) + b_{22}(t) \equiv 0$) implies $b_{12}(t) = b_{22}(t) \equiv 0$. $\quad\square$

Lemma 2. *The function $u_1(t)$ is a local τ-function of the Malgrange isomonodromic deformation of system* (10); *i.e., it locally determines the Θ-divisor near the point $t^* \in \Theta$.*

Proof. If $u_1(t) \neq 0$, then we can consider a matrix

$$\Gamma_1'(z,t) = \begin{pmatrix} 1 & 0 \\ -\frac{z}{u_1(t)} & 1 \end{pmatrix},$$

which is holomorphically invertible (with respect to z) in $\mathbb{C}$. By construction the matrix $U'(z,t) = \Gamma_1'(z,t)U(z,t)$ is of the form

$$U'(z,t) = \left(U_0'(t) + U_1'(t)\frac{1}{z} + \dots\right)z^{-K}, \qquad U_0'(t) = \begin{pmatrix} 0 & u_1(t) \\ -\frac{1}{u_1(t)} & \frac{f(t)}{u_1(t)} \end{pmatrix},$$

where $f(t)$ is a holomorphic function at the point $t = t^*$.

The gauge transformation

$$y_1 = \Gamma_1(z,t)y, \qquad \Gamma_1(z,t) = U_0'(t)^{-1}\Gamma_1'(z,t), \tag{19}$$

transforms system (14) into a new one, with a fundamental matrix $Y^1(z,t) = \Gamma_1(z,t)Y(z,t)$, which is holomorphically invertible at infinity. Since the columns of the matrix $Y(z,t)$ form a basis of horizontal (with respect to the restriction of the connection ∇ to $\overline{\mathbb{C}} \times \{t\}$) sections of the bundle $E_{\overline{\mathbb{C}}\times\{t\}}$ over $\mathbb{C}$, the last relation implies the holomorphic triviality of this bundle.

If $u_1(t) = 0$, then the matrix

$$V_\infty(z) = z^{-K}U(z,t)z^K = z^{-K}\left(I + \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}\frac{1}{z} + \dots\right)z^K$$

is holomorphically invertible at infinity; hence $Y(z,t) = z^K V_\infty(z)$ and $E|_{\overline{\mathbb{C}}\times\{t\}} \cong \mathcal{O}(-1)\oplus\mathcal{O}(1)$. $\quad\square$

Proof of Theorem 3. First we explain that $du_1(t^*) \not\equiv 0$. Indeed, in the opposite case the following equalities should be true:

$$b_{i1}(t^*) = 0, \quad i = 1, \ldots, n, \qquad b_{12}(t^*) + b_{22}(t^*) = 0.$$

Then similarly to the proof of Lemma 1 one gets the relations $b_{12}(t^*) = b_{22}(t^*) = 0$, which contradict the irreducibility of the monodromy representation of the family (14). Thus the Θ-divisor of the Malgrange isomonodromic deformation of system (10) is an analytic submanifold in a neighbourhood $D(t^*)$ of the point t^*.

Now let us estimate the pole orders of the matrices $A_{i1}(t)$, $A_{12}(t)$ and $A_{22}(t)$ along $\Theta \cap D(t^*)$. Return to the proof of Lemma 2. The family obtained from (14) via the gauge transformation (19) coincides with the Malgrange isomonodromic deformation (for $t \in D(t^*) \setminus \Theta$) of the initial system (10). (Indeed, this transformation does not change the connection matrices at the Fuchsian singular points, and nor does it change the holomorphic equivalence classes of the family at the irregular singularities.) Therefore, the coefficient matrix of the Malgrange isomonodromic deformation of the initial system (10) has the form

$$\frac{\partial \Gamma_1}{\partial z} \Gamma_1^{-1} + \Gamma_1 \left(\frac{B_{12}(t)}{(z - a_1(t))^2} + \frac{B_{22}(t)}{(z - a_2(t))^2} + \sum_{i=1}^{n} \frac{B_{i1}(t)}{z - a_i(t)} \right) \Gamma_1^{-1}.$$

As the matrix $\Gamma_1(z, t)$ is holomorphically invertible (with respect to z) in $\mathbb{C}$, one has

$$A_{i1}(t) = \Gamma_1(a_i(t), t) B_{i1}(t) \Gamma_1^{-1}(a_i(t), t), \qquad i = 3, \ldots, n,$$

and for $i = 1, 2$ one has

$$A_{i2}(t) = \Gamma_1(a_i(t), t) B_{i2}(t) \Gamma_1^{-1}(a_i(t), t),$$

$$A_{i1}(t) = \frac{\partial \Gamma_1}{\partial z}(a_i(t), t) B_{i2}(t) \Gamma_1^{-1}(a_i(t), t) + \Gamma_1(a_i(t), t) B_{i1}(t) \Gamma_1^{-1}(a_i(t), t)$$

$$+ \Gamma_1(a_i(t), t) B_{i2}(t) \frac{\partial \Gamma_1^{-1}}{\partial z}(a_i(t), t).$$

Since

$$\Gamma_1(z, t) = U_0'(t)^{-1} \Gamma_1'(z, t) = \begin{pmatrix} \frac{f(t)}{u_1(t)} & -u_1(t) \\ \frac{1}{u_1(t)} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{z}{u_1(t)} & 1 \end{pmatrix} = \begin{pmatrix} z + \frac{f(t)}{u_1(t)} & -u_1(t) \\ \frac{1}{u_1(t)} & 0 \end{pmatrix}$$

and the matrices $B_{ij}(t)$ are holomorphic near the point $t = t^*$, one sees that the same holds for all the matrices $(u_1(t))^2 A_{ij}(t)$. $\square$

Remark. Recall that the Painlevé III and V equations can be described in terms of isomonodromic deformations satisfying Theorem 3 (see details in [10, Ch. 5, §4, 5]): for P_{III} one has $m = n = 2$, and for P_V one has $m = 1$ and $n = 3$. If $t^* \in \Theta$ and $E|_{\overline{\mathbb{C}} \times \{t^*\}} \cong \mathcal{O}(-k) \oplus \mathcal{O}(k)$, then the estimate $2k \leq m + n - 2$ holds [5] (when the monodromy of a connection is irreducible). Thus $2k \leq 2$ and hence $k = 1$ in both cases.

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