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Infinite Energy Solutions for Dissipative Euler Equations in $\mathbb{R}^2$

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Abstract. We study the system of Euler equations with the so-called Ekman damping in the whole 2D space. The global well-posedness and dissipativity for the weak infinite energy solutions of this problem in the uniformly local spaces is verified based on the further development of the weighted energy theory for the Navier–Stokes and Euler type problems. In addition, the existence of weak locally compact global attractor is proved and some extra compactness of this attractor is obtained.

Contents

1. Introduction

We study the following dissipative Euler system in the whole plane $x \in \mathbb{R}^2$:

$$\begin{cases} \partial_t u + (u, \nabla_x)u + \alpha u + \nabla_x p = g, \\ \operatorname{div} u = 0, \qquad u\big|_{t=0} = u_0, \end{cases} \tag{1.1}$$

which differs from the classical Euler equations by the presence of the so-called Ekman damping term αu with $\alpha > 0$. These equations describe, for instance, a 2-dimensional fluid moving on a rough surface and are used in geophysical models for large-scale processes in atmosphere and ocean. The term αu parameterizes the main dissipation occurring in the planetary boundary layer (see, e.g., [25]; see also [6] for the alternative source of damped Euler equations).

The mathematical features of these and related equations are studied in a number of papers (see, for instance, [3,5,7,15–17]) including the analytic properties (which are very similar to the classical Euler

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equations without dissipative term, see [4,21,22,30] and references therein), stability analysis, vanishing viscosity limit, etc.

The attractors for damped Euler equations (1.1) in the case of bounded underlying domains have been studied in [5,7,8,15–17]. Remind that, in contrast to the Navier–Stokes case, the damped Euler equations remain hyperbolic and we do not have any smoothing property on a finite time interval. Moreover, even the asymptotic smoothness as time tends to infinity is much more delicate here. Indeed, similar to the classical Euler equations, following Yudovich, see [31], we have the global existence of smooth solutions, but the best possible estimates for the smooth norms of these solutions grow faster than exponential in time, so they are not helpful for the attractor theory. Thus, to the best of our knowledge, there is no way to obtain more regularity of the attractor than the $L^\infty(\Omega)$ bounds for the vorticity even in the case of bounded underlying domain Ω or periodic boundary conditions. On the other hand, the L^∞-bounds for the vorticity cannot be essentially relaxed if we want to have the uniqueness of a solution. By this reason, the *weak* attractors are normally used in order to describe the longtime behavior of solutions of the damped Euler equations. Some exception is the paper [8] where the so-called trajectory attractor is constructed for this system in a *strong* topology of $W^{1,2}(\Omega)$ based on the enstrophy equality and the energy method.

The situation becomes more complicated when the underlying domain is unbounded, say, $\Omega = \mathbb{R}^2$ and we are interested in the infinite energy solutions. Indeed, although in this case we have an immediate control of the L^∞-norm of the vorticity from the maximum principle:

$$\|\omega(t)\|_{L^\infty} \le Ce^{-\alpha t}\|\omega(0)\|_{L^\infty} + \frac{1}{\alpha}\|\operatorname{curl} g\|_{L^\infty}, \quad \omega := \operatorname{curl} u, \tag{1.2}$$

this gives only growing in time (faster than exponentially) estimates for the velocity u, see [14,28] even in a more simple case of damped Navier–Stokes equations (see also [19,29] for the analogous results for Euler equations), so in order to get the dissipative bounds for the velocity field, we need to use the energy type estimates. For the case of damped Navier–Stokes equations these estimates have been obtained in [36] for the case where the initial data u_0 belong to the so-called uniformly local Sobolev spaces, see also [33,34] for the analogous results for the case of Navier–Stokes equations in cylindrical domains as well as [10,11,23] and references therein for general theory of dissipative PDEs in unbounded domains.

The aim of the present paper is to build up the attractor theory for the damped Euler equation (1.1) in uniformly local spaces extending the results of [8] and [36]. We assume that $g \in \mathcal{H}_b$, where

$$\mathcal{H}_b := \left\{ u \in [L_b^2(\mathbb{R}^2)]^2, \ \operatorname{div} u = 0, \ \operatorname{curl} u \in L^\infty(\mathbb{R}^2) \right\} \tag{1.3}$$

and $L_b^2(\mathbb{R}^2)$ is the usual uniformly local space determined by the norm

$$\|u\|_{L_b^2} := \sup_{x_0 \in \mathbb{R}^2} \|u\|_{L^2(B_{x_0}^1)} < \infty. \tag{1.4}$$

Here and below, $B_{x_0}^R$ means the unit ball of radius R in $\mathbb{R}^2$ centered at $x_0 \in \mathbb{R}^2$. The norm in the space $\mathcal{H}_b$ is defined by the following natural formula:

$$\|u\|_{\mathcal{H}_b} := \|u\|_{L_b^2} + \|\operatorname{curl} u\|_{L^\infty}, \tag{1.5}$$

see Sect. 2 for more details.

By definition, function $u = u(t,x)$ is a weak solution of (1.1) if $u(t) \in \mathcal{H}_b$ for $t \ge 0$ and satisfies equation (1.1) in the sense of distributions, see Definition 3.2 below.

The main result of the paper is the following theorem.

Theorem 1.1. *Let the external forces $g \in \mathcal{H}_b$. Then, for every $u_0 \in \mathcal{H}_b$, problem (1.1) possesses a unique weak solution $u(t)$ and this solution satisfies the following dissipative estimate:*

$$\|u(t)\|_{\mathcal{H}_b} \le Q(\|u_0\|_{\mathcal{H}_b})e^{-\beta t} + Q(\|g\|_{\mathcal{H}_b}), \tag{1.6}$$

where the positive constant β and monotone increasing function Q are independent of $t \ge 0$ and $u_0 \in \mathcal{H}_b$. The solution semigroup $S(t) : \mathcal{H}_b \to \mathcal{H}_b$ associated with this equation possesses a weak locally compact

global attractor $\mathcal{A}$ in the phase space $\mathcal{H}_b$, see Definition 6.1. Moreover, this attractor is a compact set in $[W_{loc}^{1,p}(\mathbb{R}^2)]^2$, for every $p < \infty$ and attracts bounded sets of $\mathcal{H}_b$ in the topology of $[W_{loc}^{1,p}(\mathbb{R}^2)]^2$.

The paper is organized as follows.

In Sect. 2, we recall the definitions and basic properties of the weighted and uniformly local Sobolev spaces, introduce special classes of weights and recall a number of elementary inequalities which will be used throughout of the paper.

In Sect. 3, we introduce (following [36]) a number of technical tools which allows us to treat the pressure term in the proper weighted and uniformly local spaces as well as to exclude it from the various weighted energy estimates.

Section 4 is devoted to the derivation of the basic dissipative estimate (1.6). In this section, based on the new version of the interpolation inequality (proved in the Appendix), we extend the method initially suggested in [36] for the case of damped Navier–Stokes equations to more difficult case of zero viscosity. Moreover, we also indicate here some improvements of the results concerning the classical Navier-Stokes and Euler equations (which corresponds to the case of $\alpha = 0$). In this case, the solution $u(t)$ may grow as $t \to \infty$ and as proved in [36] the growth rate is restricted by the quintic polynomial in time. As indicated in Remark 4.4, using the approach developed in this paper, we may replace the quintic polynomial by the cubic one:

$$\|u(t)\|_{L_b^2(\mathbb{R}^2)} \leq C(t+1)^3. \tag{1.7}$$

Moreover, in the particular case $g = 0$ this estimate can be further improved:

$$\|u(t)\|_{L_b^2(\mathbb{R}^2)} \leq C(t+1), \tag{1.8}$$

see also the recent work [13] where the analogous linear growth estimate has been established for the infinite energy solutions of the Navier–Stokes equations.

The uniqueness of the weak solution of (1.1) is verified in Sect. 5 by adapting the famous Yudovich proof to the case of weighted and uniformly local spaces. Note that, in contrast to [29] and [19], our approach does not use the so-called Serfati identity and is based on the Yudovich type estimates in weighted L^2-spaces. Moreover, following [9], we establish here the so-called weighted enstrophy equality which plays a crucial role in verifying the strong compactness of the attractor.

Finally, the weak locally compact attractor $\mathcal{A}$ is constructed in Sect. 6. Moreover, using the above mentioned weighted enstrophy equality and the energy method (analogously to [8]), we prove the compactness of this weak attractor in the strong topology of the space $[W_{loc}^{1,p}(\mathbb{R}^2)]^2$ for any $p < \infty$.

Note also that, in contrast to the case of 2D Navier–Stokes equations, in the case of Euler equations, the L^∞-estimate for the vorticity holds not only for $\mathbb{R}^2$, but for more or less general unbounded domains. This allows to extend the results of the paper to the case of unbounded domains different from $\mathbb{R}^2$. To this end, one just needs to modify formula (3.3) for pressure by including the proper boundary terms. We return to this problem somewhere else.

2. Preliminaries I: Weighted and Uniformly Local Spaces

In this section, we briefly discuss the definitions and basic properties of the weighted and uniformly local Sobolev spaces (see [23,33,35] for more detailed exposition). We start with the class of admissible weight functions and associated weighted spaces.

Definition 2.1. A positive function $\phi(x)$, $x \in \mathbb{R}^2$, is a weight function of exponential growth rate $\mu \geq 0$ if

$$\phi(x+y) \leq Ce^{\mu|y|}\phi(x), \quad x, y \in \mathbb{R}^2. \tag{2.1}$$

The associated weighted Lebesgue space $L_\phi^p(\mathbb{R}^2)$, $1 \leq p < \infty$, is defined as a subspace of functions belonging to $L_{loc}^p(\mathbb{R}^2)$ for which the following norm is finite:

$$\|u\|_{L_\phi^p}^p := \int_{\mathbb{R}^2} \phi(x)|u(x)|^p \, dx < \infty \tag{2.2}$$

and the Sobolev space $W_\phi^{l,p}(\mathbb{R}^2)$ is the subspace of distributions $u \in \mathcal{D}'(\mathbb{R}^2)$ whose derivatives up to order l inclusively belong to $L_\phi^p(\mathbb{R}^2)$ (this works for positive integer l only; for fractional and negative l, the space $W_\phi^{l,p}$ is defined using the interpolation and duality arguments, see [11,33] for more details).

The typical examples of weight functions of exponential growth rate are

$$\phi(x) := e^{-\varepsilon|x-x_0|} \quad \text{or} \quad \phi(x) := e^{-\sqrt{1+\varepsilon^2|x-x_0|^2}}, \quad \varepsilon \in \mathbb{R}, \quad x_0 \in \mathbb{R}^2. \tag{2.3}$$

Another class of admissible weights of exponential growth rate are the so-called polynomial weights and, in particular, the weight function

$$\theta_{x_0}(x) := \frac{1}{1 + |x - x_0|^3}, \quad x_0 \in \mathbb{R}^2, \tag{2.4}$$

which will be essentially used throughout of the paper.

Next, we define the so-called uniformly local Sobolev spaces.

Definition 2.2. The space $L_b^p(\mathbb{R}^2)$ is defined as the subspace of functions of $L_{loc}^p(\mathbb{R}^2)$ for which the following norm is finite:

$$\|u\|_{L_b^p} := \sup_{x_0 \in \mathbb{R}^2} \|u\|_{L^p(B_{x_0}^1)} < \infty \tag{2.5}$$

(here and below $B_{x_0}^R$ denotes the R-ball in $\mathbb{R}^2$ with center x_0). The spaces $W_b^{l,p}(\mathbb{R}^2)$ are defined as subspaces of distributions $u \in \mathcal{D}'(\mathbb{R}^2)$ whose derivatives up to order l inclusively belong to the space $L_b^p(\mathbb{R}^2)$.

The next proposition gives the useful equivalent norms in the weighted Sobolev spaces.

Proposition 2.3. *Let ϕ be the weight function of exponential growth rate and let $1 \le p < \infty$, $l \in \mathbb{R}$ and $R > 0$. Then,*

$$C_1 \int_{x_0 \in \mathbb{R}^2} \phi(x_0)\|u\|_{W^{l,p}(B_{x_0}^R)}^p \, dx_0 \le \|u\|_{W_\phi^{l,p}}^p$$

$$\le C_2 \int_{x_0 \in \mathbb{R}^2} \phi(x_0)\|u\|_{W^{l,p}(B_{x_0}^R)}^p \, dx_0, \tag{2.6}$$

where the constants C_i depend on R, l and p and the constants C and μ from (2.1), but are independent of u and of the concrete choice of the weight ϕ.

For the proof of these estimates, see e.g., [11].

Thus, the norms $\int_{x_0 \in \mathbb{R}^2} \phi(x_0)\|u\|_{W^{l,p}(B_{x_0}^R)}^p \, dx_0$ computed with different R's are equivalent.

The next Proposition gives relations between the weighted and uniformly local norms.

Proposition 2.4. *Let ϕ be the weight of exponential growth rate such that $\int_{x \in \mathbb{R}^2} \phi \, dx < \infty$. Then, for every $u \in W_b^{l,p}(\mathbb{R}^2)$ and every $\kappa \ge 1$,*

$$\|u\|_{W^{l,p}(B_{x_0}^\kappa)}^p \le C \int_{y \in B_{x_0}^\kappa} \|u\|_{W^{l,p}(B_y^1)}^p \, dy \le C_\kappa \int_{y \in \mathbb{R}^2} \phi(y - x_0)\|u\|_{W^{l,p}(B_y^1)}^p \, dy \tag{2.7}$$

and, in particular, fixing $\kappa = 1$ in (2.7) and taking the supremum with respect to $x_0 \in \mathbb{R}^2$, we have

$$\|u\|_{W_b^{l,p}} \le C \sup_{x_0 \in \mathbb{R}^2} \|u\|_{W_{\phi(\cdot-x_0)}^{l,p}}, \tag{2.8}$$

where C is independent of u and the concrete choice of the weight ϕ. In addition,

$$\|u\|_{W_\phi^{l,p}}^p \le C\|\phi\|_{L^1}\|u\|_{W_b^{l,p}}^p, \tag{2.9}$$

where C is also independent of u and the concrete choice of ϕ.

For the proof of these results, see e.g., [33,35].

The next lemma gives a simple, but important estimate for the weights $\theta_{x_0}(x)$ which will allow us to handle the convolution operators in weighted spaces.

Lemma 2.5. *Let $\theta_{x_0}(x)$ be the weight defined via* (2.4). *Then, the following estimate holds:*

$$\int_{x\in\mathbb{R}^2} \theta_{x_0}(x)\theta_{y_0}(x)\,dx \leq C\theta_{x_0}(y_0), \tag{2.10}$$

where C is independent of $x_0, y_0 \in \mathbb{R}^2$.

For the proof of this lemma see, e.g., [36].

Corollary 2.6. *Let $\theta_{x_0}(x)$ be defined via* (2.4). *Then, for every $u \in L^p_{\theta_{x_0}}(\mathbb{R}^2)$, we have*

$$\|u\|^p_{L^p_{\theta_{y_0}}} \leq C\int_{x_0\in\mathbb{R}^2} \theta_{y_0}(x_0)\|u\|^p_{L^p_{\theta_{x_0}}}\,dx_0, \tag{2.11}$$

where C is independent of $y_0 \in \mathbb{R}^2$.

The proof of this corollary can also be found in [36].

We conclude this section by introducing some weights and norms depending on a big parameter R which will be crucial for what follows. First, we introduce the following equivalent norm in the space $W_b^{l,p}(\mathbb{R}^2)$:

$$\|u\|_{W_{b,R}^{l,p}} := \sup_{x_0\in\mathbb{R}^2} \|u\|_{W^{l,p}(B_{x_0}^R)}. \tag{2.12}$$

Then, according to (2.7),

$$\|u\|_{W_b^{l,p}} \leq \|u\|_{W_{b,R}^{l,p}} \leq CR^{2/p}\|u\|_{W_b^{l,p}}, \tag{2.13}$$

where the constant C is independent of $R \geq 1$. We also introduce the scaled weight function

$$\theta_{R,x_0}(x) := \frac{1}{R^3 + |x-x_0|^3} = R^{-3}\theta_{x_0/R}(x/R). \tag{2.14}$$

Then, the scaled analogue of (2.7) reads

$$\|u\|^p_{W^{l,p}(B_{x_0}^{\kappa R})} \leq CR^{-2}\int_{y\in B_{x_0}^{\kappa R}} \|u\|^p_{W^{l,p}(B_y^R)}\,dy$$

$$\leq C_\kappa R\int_{y\in\mathbb{R}^2} \theta_{R,x_0}(y)\|u\|^p_{W^{l,p}(B_y^R)}\,dy, \tag{2.15}$$

where the constants C and C_κ are independent of R and the scaled analogue of (2.10) is

$$\int_{x\in\mathbb{R}^2} \theta_{R,x_0}(x)\theta_{R,y_0}(x)\,dx \leq CR^{-1}\theta_{R,x_0}(y_0), \tag{2.16}$$

where C is independent of $x_0, y_0 \in \mathbb{R}^2$ and $R > 0$. Moreover, multiplying inequality (2.15) by $\theta_{R,y_0}(x_0)$, integrating over x_0 and using (2.16), we see that

$$\int_{x\in\mathbb{R}^2} \theta_{R,x_0}(x)\|u\|^p_{W^{l,p}(B_x^{\kappa R})}\,dx \leq C_\kappa \int_{x\in\mathbb{R}^2} \theta_{R,x_0}(x)\|u\|^p_{W^{l,p}(B_x^R)}\,dx, \tag{2.17}$$

where C_κ is independent of R. We also note that, analogously to (2.9) and using (2.13),

$$\int_{x\in\mathbb{R}^2} \theta_{R,x_0}(x)\|u\|^2_{W^{l,2}(B_x^R)}\,dx \leq CR^{-1}\|u\|^2_{W_{b,R}^{l,2}} \leq C_1R\|u\|^2_{W_b^{l,2}}, \tag{2.18}$$

where the constants C and C_1 are independent of $R \gg 1$.

3. Preliminaries II: Estimating the Pressure

In this section, we introduce the key estimates which allow us to work with the pressure term ∇p in the uniformly local spaces. Note that the Helmholtz decomposition does not work for the general vector fields belonging to $L_b^2(\mathbb{R}^2)$, so the standard (for the bounded domains) approach does not work at least directly and we need to proceed in a bit more accurate way.

As usual, we assume that (1.1) is satisfied in the sense of distributions. Then, taking the divergence from both sides of (1.1) and assuming that the external forces g are divergence free:

$$\text{div } g = 0, \tag{3.1}$$

we have

$$-\Delta_x p = \text{div}((u, \nabla_x)u) = \sum_{i,j=1}^{2} \partial_{x_i} \partial_{x_j} (u_i u_j). \tag{3.2}$$

Thus, formally, p can be expressed through u by the following singular integral operator:

$$p(y) := \int_{\mathbb{R}^2} \sum_{ij} K_{ij}(x - y) u_i(x) u_j(x) \, dx, \quad K_{ij}(x) := \frac{1}{2\pi} \frac{|x|^2 \delta_{ij} - 2x_i x_j}{|x|^4} \tag{3.3}$$

that we present in the form

$$p = \mathbb{K}w := K * w, \;\; w := u \otimes u, \quad K * w = \sum_{ij} K_{ij} * w_{ij}. \tag{3.4}$$

It is well-known that the convolution operator $\mathbb{K}$ is well-defined as a bounded linear operator from $w \in [L^q(\mathbb{R}^2)]^4$ to $p \in L^q(\mathbb{R}^2)$, $1 < q < \infty$, but it is not true neither for $q = \infty$ nor for the uniformly local space $L_b^q(\mathbb{R}^2)$. However, as the following lemma shows, the *gradient* of p [which is sufficient in order to define a solution of (1.1)] is well-defined in uniformly local spaces and has natural regularity properties.

Lemma 3.1. *The operator* $w \to \nabla_x p$, *where* p *is defined via* (3.4) *can be extended by continuity (in* L_{loc}^q) *in a unique way to the bounded operator from* $[W_b^{s,q}(\mathbb{R}^2)]^4$ *to* $[W_b^{s-1,q}(\mathbb{R}^2)]^2$, $1 < q < \infty$ *and* $s \in \mathbb{R}$.

For the proof of this lemma see [36]. We denote the operator constructed in Lemma 3.1 by $\nabla_x P$ and the corresponding term in the Euler equation is denoted by $\nabla_x P(u \otimes u)$. Then, in particular

$$\nabla_x P(u \otimes u) : [W_b^{1,2q}(\mathbb{R}^2)]^2 \to [L_b^q(\mathbb{R}^2)]^2, \quad 1 < q < \infty. \tag{3.5}$$

Here the operator $\nabla_x P(u \otimes u)$ is considered as a nonlinear (quadratic) operator $u \to \nabla_x P(u \otimes u)$.

Now we are ready to define a weak solution of problem (1.1).

Definition 3.2. Let the external forces $g \in \mathcal{H}_b$, where the space $\mathcal{H}_b$ is defined as follows:

$$\mathcal{H}_b := \{u \in [L_b^2(\mathbb{R}^2)]^2, \quad \text{div } u = 0, \quad \text{curl } u \in L^\infty(\mathbb{R}^2)\}. \tag{3.6}$$

Here and below $\text{curl } u := \partial_{x_2} u_1 - \partial_{x_1} u_2$ and the norm in this space is given by (1.5). A vector field $u(t, x)$ is a weak solution of the damped Euler problem (1.1) if

$$u \in L^\infty(\mathbb{R}_+, \mathcal{H}_b) \tag{3.7}$$

and the Euler equation is satisfied in the sense of distributions with $\nabla_x p = \nabla_x P(u \otimes u)$ defined in Lemma 3.1. Note that, according to the interpolation and embedding theorems,

$$u \in L^\infty(\mathbb{R}_+, W_b^{1,q}(\mathbb{R}^2)) \subset L^\infty(\mathbb{R}_+ \times \mathbb{R}^2), \quad 2 < q < \infty, \tag{3.8}$$

see e.g., [20]. Thus, $u \otimes u \in L^\infty(\mathbb{R}_+, W_b^{1,q}(\mathbb{R}^2))$, so, due to the previous lemma, the pressure term $\nabla_x p \in L^\infty([0, T], L_b^q(\mathbb{R}^2))$ and the equation (1.1) can be understood as equality in this space. Moreover, from Eq. (1.1), we then conclude that $\partial_t u \in L^\infty(\mathbb{R}_+, L_b^q(\mathbb{R}^2))$ and, therefore, $u \in C([0, T], L_b^q(\mathbb{R}^2))$. Thus, the initial data is also well-defined.

Remark 3.3. We emphasize once more that only the gradient of pressure ∇p is well-defined as an element of $L^\infty([0,T], L_b^q)$, but the pressure itself may be unbounded as $|x| \to \infty$. To be more precise, the operator ∇P defined above satisfies

$$- \operatorname{div} \nabla P(w) = \sum_{ij} \partial_{x_i} \partial_{x_j} w_{ij}, \quad \operatorname{curl} \nabla P(w) = 0 \tag{3.9}$$

in the sense of distributions. These relations can be justified by approximating w by finite functions and passing to the limit analogously to the proof of Lemma 3.1 given in [36]. Therefore, there is a function

$$p \in L^\infty([0,T], W_{loc}^{1,q}(\mathbb{R}^2))$$

such that

$$\nabla p = \nabla P(w),$$

see [30], but this function may grow as $|x| \to \infty$ (in fact, one can only guarantee that $p \in BMO(\mathbb{R}^2)$, however, functions with bounded mean oscillation may grow as $|x| \to \infty$, say, as a polynomial of $\log |x|$, see [20]). Thus, in general $p \notin L^\infty(\mathbb{R}^2)$.

Note also that the choice of $\nabla p = \nabla P(u \otimes u)$ is not unique. If p_1 and p_2 both satisfy (3.2) (for the same velocity field u), then the difference $p_1 - p_2$ solves $\Delta(p_1 - p_2) = 0$ (in the sense of distributions) and, consequently, $p_1 - p_2$ is a *harmonic* function. Moreover, every harmonic function with bounded gradient is linear, so, if we want to look for a velocity field u in the proper uniformly local space, the most general choice of the pressure is

$$\nabla p = \nabla P(u \otimes u) + \vec{C}(t), \tag{3.10}$$

where the constant vector $\vec{C}(t)$ depends on time only (it is independent of x) and ∇P is defined in Lemma 3.1. In the present paper, we consider the choice $\vec{C}(t) \equiv 0$. In fact, the vector $\vec{C}(t)$ should be treated as one more external data and can be chosen arbitrarily, but this does not lead to more general theory since everything can be reduced to the case of $\vec{C} \equiv 0$ by replacing the external force g by $g - \vec{C}(t)$.

We conclude this preliminary section by reminding the key estimate which allows us handle the pressure term in weighted energy estimates, see [36] for more details. To this end, we introduce for every $x_0 \in \mathbb{R}^2$ and $R > 1$ the cut-off function φ_{R,x_0} which satisfies

$$\varphi_{R,x_0}(x) \equiv 1, \quad \text{for } x \in B_{x_0}^R, \quad \varphi_{R,x_0}(x) \equiv 0, \quad \text{for } x \notin B_{x_0}^{2R}, \tag{3.11}$$

and

$$|\nabla_x \varphi_{R,x_0}(x)| \leq CR^{-1} \varphi_{R,x_0}^{1/2}(x), \tag{3.12}$$

where C is independent of R (obviously such family of cut-off functions exist). Then, the following result holds.

Lemma 3.4. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $w \in [L_b^p(\mathbb{R}^2)]^4$ and $v \in [W^{1,q}(\mathbb{R}^2)]^2$ be divergence free. Then the following estimate holds:*

$$|(\nabla_x P(w), \varphi_{R,x_0} v)| \leq C \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \|w\|_{L^p(B_x^R)} \, dx \cdot \|\varphi_{R,x_0}^{1/2} v\|_{L^q}, \tag{3.13}$$

where C is independent of R and x_0, while $\theta_{R,x_0}(x)$ is defined by (2.14).

For the proof of this lemma, see [36].

4. Dissipative Estimates for the Velocity Field

In this section, we prove the dissipative estimate for the solutions of (1.1) in the phase space $\mathcal{H}_b$. We start with recalling the L^∞-estimate for the vorticity $\omega = \operatorname{curl} u$ which satisfies the scalar transport equation

$$\partial_t \omega + \alpha\omega + (u, \nabla_x)\omega = \operatorname{curl} g, \quad \omega\big|_{t=0} = \omega_0 := \operatorname{curl} u_0. \tag{4.1}$$

Lemma 4.1. *Let u be a weak solution of the Euler problem* (1.1). *Then, the vorticity ω satisfies the following estimate:*

$$\|\omega(t)\|_{L^\infty} \leq C\|w(0)\|_{L^\infty} e^{-\alpha t} + \frac{\|\operatorname{curl} g\|_{L^\infty}}{\alpha}, \tag{4.2}$$

where the constant C is independent of t and u.

Indeed, the desired estimate (4.2) is an immediate corollary of the maximum principle applied to the transport equation (4.1). The validity of the maximum principle can be easily justified using the fact that the weak solution of the damped Euler equation is unique (which will be proved in the next section) and approximating the solution u by the smooth ones by the vanishing viscosity method. Thus, we only need to estimate the L_b^2-norm of u. To do that, we will extend the approach developed in [36] to the case of Euler equations.

Theorem 4.2. *Let the above assumptions hold. Then the Euler equation* (1.1) *possesses at least one weak solution which satisfies the following dissipative estimate:*

$$\|u(t)\|_{\mathcal{H}_b} \leq Q(\|u_0\|_{\mathcal{H}_b})e^{-\beta t} + Q(\|g\|_{\mathcal{H}_b}), \tag{4.3}$$

where $\beta > 0$ and Q is a monotone function.

Proof. For simplicity, we first derive the desired estimate (4.3) in the non-dissipative case $\beta = 0$ and then indicate the changes to be made in order to verify the dissipation. The existence of a solution can be obtained after that in a standard way using e.g., the vanishing viscosity method which we leave to the reader.

We will systematically use the weight functions

$$\theta_{R,x_0}(x) := \frac{1}{R^3 + |x - x_0|^3} \tag{4.4}$$

and the family of cut-off functions $\varphi_{R,x_0}(x)$ which equal to one if $x \in B_{x_0}^R$ and zero outside $B_{x_0}^{2R}$ such that

$$|\nabla_x \varphi_{R,x_0}(x)| \leq CR^{-1}\varphi_{R,x_0}(x)^{1/2}. \tag{4.5}$$

introduced in Sects. 2 and 3.

Then, multiplying equation (1.1) by $u\varphi_{R,x_0}$ (where R is sufficiently large number and $x_0 \in \mathbb{R}^2$) and following [36], after the integration over x, we estimate the nonlinear term as follows:

$$|((u, \nabla_x)u, \varphi_{R,x_0}u)| = \frac{1}{2}|(u.\nabla_x\varphi_{R,x_0}, |u|^2)| \leq CR^{-1}\|u\|^3_{L^3(B_{x_0}^{2R})}.$$

After the standard transformations, we get

$$\frac{d}{dt}\|u(t)\|^2_{L^2_{\varphi_{R,x_0}}} + \alpha\|u\|^2_{L^2_{\varphi_{R,x_0}}} \leq C\|g\|^2_{L^2_{\varphi_{R,x_0}}}$$
$$+ CR^{-1}\|u\|^3_{L^3(B_{x_0}^{2R})} + 2|(\nabla_x P(u \otimes u), \varphi_{R,x_0}u)|. \tag{4.6}$$

Here, the constant C is independent of R. To estimate the term containing pressure, we use Lemma 3.4 with $q = 3$ and $p = 3/2$. Then, due to (3.13) together with the Hölder and Young inequalities,

$$|(\nabla_x P(u \otimes u), \varphi_{R,x_0}u)|$$
$$\leq C\int_{\mathbb{R}^2} \theta_{R,x_0}(x)\|u \otimes u\|_{L^{3/2}(B_x^R)}\, dx \cdot \|u\|_{L^3(B_{x_0}^{2R})}$$

$$\leq C \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \|u\|^2_{L^3(B^R_x)} \, dx \cdot \|u\|_{L^3(B^{2R}_{x_0})}$$

$$\leq C \left(\int_{\mathbb{R}^2} \theta_{R,x_0}(x) \, dx \right)^{1/3} \left(\int_{\mathbb{R}^2} \theta_{R,x_0}(x) \|u\|^3_{L^3(B^R_x)} \, dx \right)^{2/3} \cdot \|u\|_{L^3(B^{2R}_{x_0})}$$

$$\leq C R^{-1/3} \left(\int_{\mathbb{R}^2} \theta_{R,x_0}(x) \|u\|^3_{L^3(B^R_x)} \, dx \right)^{2/3} \cdot \|u\|_{L^3(B^{2R}_{x_0})}$$

$$\leq C \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \|u\|^3_{L^3(B^R_x)} \, dx + C R^{-1} \|u\|^3_{L^3(B^{2R}_{x_0})}, \tag{4.7}$$

where all constants are independent of $R \gg 1$. Thus, (4.6) now reads

$$\frac{d}{dt} \|u(t)\|^2_{L^2_{\varphi_{R,x_0}}} + \alpha \|u\|^2_{L^2_{\varphi_{R,x_0}}} \leq C \|g\|^2_{L^2_{\varphi_{R,x_0}}}$$

$$+ C R^{-1} \|u\|^3_{L^3(B^{2R}_{x_0})} + C \int_{\mathbb{R}^2} \theta_{R,x_0}(x) \|u\|^3_{L^3(B^R_x)} \, dx. \tag{4.8}$$

We set

$$Z_{R,y_0}(u) := \int_{x_0 \in \mathbb{R}^2} \theta_{R,y_0}(x_0) \|u\|^2_{L^2_{\varphi_{R,x_0}}} \, dx_0. \tag{4.9}$$

Then, using (2.17), we have

$$C_2 \int_{y \in \mathbb{R}^2} \theta_{R,y_0}(y) \|u\|^2_{L^2(B^R_y)} \, dy \leq Z_{R,y_0}(u) \leq C_1 \int_{y \in \mathbb{R}^2} \theta_{R,y_0}(y) \|u\|^2_{L^2(B^R_y)} \, dy, \tag{4.10}$$

where C_i are independent of R. Multiplying inequality (4.8) by $\theta_{R,y_0}(x_0)$, integrating over $x_0 \in \mathbb{R}^2$ and using (2.16), we see that, for sufficiently large R,

$$\frac{d}{dt} Z_{R,x_0}(u(t)) + \alpha Z_{R,x_0}(u(t)) \leq C Z_{R,x_0}(g)$$

$$+ C R^{-1} \int_{x \in \mathbb{R}^2} \theta_{R,x_0}(x) \|u\|^3_{L^3(B^R_x)} \, dx, \tag{4.11}$$

where the positive constant C is independent of R.

Thus, we only need to estimate the integral in the RHS of (4.11) which, however, a bit more delicate than in [36] since now we have no the control of the H^1-norm. We use instead the interpolation inequality proved in Appendix

$$\|u\|^3_{L^3(B_R)} \leq C \left(\frac{1}{R} \|u\|^3_{L^2(B_{2R})} + \|u\|^{5/2}_{L^2(B_{2R})} \|\omega\|^{1/2}_{L^\infty(B_{2R})} \right). \tag{4.12}$$

which holds for every $R > 0$ and every $u \in \mathcal{H}_b$. Using this inequality, we have

$$\int_{x \in \mathbb{R}^2} \theta_{R,x_0}(x) \|u\|^3_{L^3(B^R_x)} \, dx \tag{4.13}$$

$$\leq C \int_{x \in \mathbb{R}^2} \theta_{R,x_0}(x) \left(\frac{1}{R} \|u\|^3_{L^2(B^{2R}_x)} + \|u\|^{5/2}_{L^2(B^{2R}_x)} \|\omega\|^{1/2}_{L^\infty(B^{2R}_x)} \right) \, dx$$

$$\leq C \int_{x \in \mathbb{R}^2} \theta_{R,x_0}(x) \|u\|^2_{L^2(B^{2R}_x)} \left(\frac{1}{R} \|u\|_{L^2(B^{2R}_x)} + \|u\|^{1/2}_{L^2(B^{2R}_x)} \|\omega\|^{1/2}_{L^\infty(B^{2R}_x)} \right) \, dx$$

$$\leq C \left(R^{-1} \|u\|_{L^2_{b,2R}} + \|u\|^{1/2}_{L^2_{b,2R}} \|\omega\|^{1/2}_{L^\infty} \right) \int_{x \in \mathbb{R}^2} \theta_{R,x_0}(x) \|u\|^2_{L^2(B^{2R}_x)} \, dx$$

$$\leq C R^{1/2} (\|\omega\|_{L^\infty} + R^{-1} \|u\|_{L^2_{b,R}}) Z_{R,x_0}(u).$$

Inserting this estimate into (4.11), we finally get

$$\frac{d}{dt}Z_{R,x_0}(u(t)) + \frac{\alpha}{2}Z_{R,x_0}(u(t)$$
$$+ \left(\frac{\alpha}{2} - CR^{-1/2}(\|\omega(t)\|_{L^\infty} + R^{-1}\|u(t)\|_{L^2_{b,R}})\right)Z_{R,x_0}(u(t)) \le CZ_{R,x_0}(g). \tag{4.14}$$

This estimate is an analogue of the estimate (5.18) in [36], so arguing exactly as in the proof of estimate (5.23) there, we end up with the inequality

$$\|u(t)\|_{L^2_b} \le C(\|u_0\|_{\mathcal{H}_b} + \|g\|_{\mathcal{H}_b} + 1)^3. \tag{4.15}$$

For the convenience of the reader, we give below a schematic derivation of (4.15) from the key estimate (4.14). The details can be found in [36]. Indeed, under the additional assumption that the constant $R = R(u_0, g)$ satisfies

$$KR^{-1/2}(R^{-1}\|u\|_{L^2_{b,R}} + \|\omega\|_{L^\infty}) \le \frac{\alpha}{2}, \quad t \ge 0, \tag{4.16}$$

we apply the Gronwall inequality to (4.14) and we obtain

$$Z_{R,x_0}(u(t)) \le Z_{R,x_0}(u_0)e^{-\alpha t/2} + CZ_{R,x_0}(g) \le CR(\|u_0\|^2_{L^2_b} + \|g\|^2_{L^2_b}). \tag{4.17}$$

Therefore, taking into the account (4.2) and (2.15), we have the desired control

$$R^{-1}\|u(t)\|_{L^2_{b,R}} + \|w(t)\|_{L^\infty} \le C_1(\|u_0\|_{\mathcal{H}_b} + \|g\|_{\mathcal{H}_b} + 1). \tag{4.18}$$

Thus, to finish the proof of (4.15), we only need to fix the parameter R in such way that our extra assumption (4.16) is satisfied. Inserting the obtained estimate (4.18) into the left-hand side of (4.16), we see that it will be formally satisfied if

$$R^{-1/2} = \frac{\alpha}{2KC_1}(\|u_0\|_{L^2_b} + \|\operatorname{curl} u_0\|_{L^\infty} + 1 + \|g\|_{L^2_b} + \|\operatorname{curl} g\|_{L^\infty})^{-1} \tag{4.19}$$

and this estimate together with (4.18) gives the desired estimate (4.15). Of course, the above arguments are formal but they can be made rigorous exactly as in [36].

Remind that estimate is still not dissipative in time. In order to obtain its dissipative analogue, we just need to take $R = R(t)$ depending on time and argue exactly as in Section 6 of [36] (see also [26] for the analogous estimate in the case of the Cahn–Hilliard equation in $\mathbb{R}^3$). Indeed, as shown there, if we replace (4.19) by

$$R(t)^{-1/2} = \frac{\alpha}{2KC_1}\left(\|u_0\|_{\mathcal{H}_b}e^{-\beta t} + 1 + \|g\|_{\mathcal{H}_b}\right)^{-1}, \tag{4.20}$$

where $0 < \beta \le \alpha$ is small enough then, arguing exactly as above we obtain that

$$\|u(t)\|_{L^2_b} \le \|u\|_{L^2_{b,R}} \le C\|u_0\|^3_{\mathcal{H}_b}e^{-\beta t} + C(1 + \|g\|_{\mathcal{H}_b})^3. \tag{4.21}$$

Thus, the desired estimate (4.3) for the L^2_b-norm of the velocity field is obtained. Since the control of the L^∞-norm of the vorticity has been already obtained in (4.2), the theorem is proved. $\square$

Remark 4.3. The analogue of the dissipative estimate (4.3) for the case of damped 2D Navier–Stokes system

$$\partial_t u + (u, \nabla_x)u + \alpha u + \nabla_x p = \nu\Delta_x u + g, \quad \operatorname{div} u = 0 \tag{4.22}$$

has been previously obtained in [36]. However, the proof given there used essentially the viscous term $\nu\Delta_x u$ and the obtained estimate was not uniform with respect to $\nu \to 0$. In contrast to this, based on the new version of the interpolation inequality, see (4.12), we have checked that the above estimate holds for the limit case $\nu = 0$. Moreover, as not difficult to see, the dissipative estimate (4.3) is now *uniform* with respect to $\nu \to 0$.

Remark 4.4. The method described above works in the case of classical Euler equations (which corresponds to $\alpha = 0$) as well. However, in this case we cannot expect any dissipative estimates and the analogue of (4.3) will be growing in time:

$$\|u(t)\|_{\mathcal{H}_b} \leq C(1 + \|g\|_{\mathcal{H}_b} + \|u_0\|_{\mathcal{H}_b})^3 (t+1)^5. \tag{4.23}$$

The proof of this estimate repeats word by word the one given in [36] for the case of damped Navier–Stokes equations (if we use the new version of the key interpolation inequality). In addition, the following estimate stated in [36] remains true:

$$\frac{1}{(t+1)^4} \|u(t)\|_{L^2_{b,(t+1)^4}} \leq C(t+1), \tag{4.24}$$

where the constant C depends on u_0 and g, but is independent of t. As elementary examples with $g = const$, $u = tg$ show, in contrast to (4.23), estimate (4.24) on the mean value of the energy with respect to the expanding balls of radii $R(t) = (t+1)^4$ is sharp.

Moreover, in the important particular case $g = 0$ this estimate can be essentially improved arguing exactly as in [36]:

$$\frac{1}{(t+1)^2} \|u(t)\|_{L^2_{b,(t+1)^2}} \leq C, \tag{4.25}$$

so the L^2_b-norm of the velocity field in this case can grow at most as a quadratic polynomial in time. The usage of the following L^∞ analogue of the interpolation inequality (4.12):

$$\|u\|_{L^\infty(B^R_0)} \leq C \left(\|\operatorname{curl} u\|^{1/2}_{L^\infty(B^{2R}_0)} \|u\|^{1/2}_{L^2(B^{2R}_0)} + \frac{1}{R} \|u\|_{L^2(B^{2R}_0)} \right), \tag{4.26}$$

which can be proved analogously to (4.12), allows us to improve essentially the inequality (4.23). Indeed, applying (4.26) for the vector field $u(t)$, fixing $R = (t+1)^4$ and using that

$$\|\operatorname{curl} u(t)\|_{L^\infty} \leq C(\|\operatorname{curl} u_0\|_{L^\infty} + t\|\operatorname{curl} g\|_{L^\infty}) \tag{4.27}$$

(which is the analogue of (4.2) for the case of $\alpha = 0$) together with estimate (4.24), we see that

$$\|u(t)\|_{L^2_b} \leq \|u(t)\|_{L^\infty} \leq C(t+1)^3, \tag{4.28}$$

where C depends on g and u_0 but is independent of t. Finally, in the particular case where $g = 0$, taking $R = (t+1)^2$ and using estimate (4.25), we see that

$$\|u(t)\|_{L^2_b} \leq C(t+1), \tag{4.29}$$

where C depends on u_0, but is independent of t. Actually, we do not know whether or not the $\|u(t)\|_{L^2_b}$ norm can grow as $t \to \infty$. However, it has been recently established in [12] that in the case of damped Navier–Stokes equation in an infinite cylinder (with the periodicity assumption with respect to one variable, say, x_1), the corresponding solution remains bounded as $t \to \infty$. We also mention that estimate (4.29) has been recently obtained in [13] based on a slightly different representation of the non-linearity and pressure term in Navier–Stokes equation which allows to avoid the usage of rather delicate interpolation inequality (4.26).

5. Uniqueness and Enstrophy Equality

The aim of this section is to adapt the Yudovich proof of uniqueness for the Euler equations (see [31]) to the case of uniformly local spaces. The key technical tool for this proof is the following lemma.

Lemma 5.1. *Let the vector field $u \in \mathcal{H}_b$. Then, the following estimate holds*

$$\|u\|_{W^{1,p}_b} \leq Cp\|u\|_{\mathcal{H}_b}, \tag{5.1}$$

where the constant C is independent of $p > 2$ and $u \in \mathcal{H}_b$.

Proof. Estimate (5.1) follows from the analogous estimate in the case of bounded domains established by Yudovich. □

Proposition 5.2. *Let* $\Omega \subset \mathbb{R}^2$ *be a smooth simply connected bounded domain. Then, for any vector field* $v \in [W^{1,p}(\Omega)]^2$ *such that* $v.n\big|_{\partial\Omega} = 0$ *and any* $1 < p < \infty$*, the following estimate holds:*

$$\|v\|_{W^{1,p}(\Omega)} \le C\left(p + \frac{1}{p-1}\right)\left(\|\operatorname{div} v\|_{L^p(\Omega)} + \|\operatorname{curl} v\|_{L^p(\Omega)}\right), \tag{5.2}$$

where the constant C *is independent of* p *and* v*.*

Indeed, due to Leray-Helmholtz decomposition, the vector field v can be expressed in terms of inverse Laplacians as follows:

$$v = -\nabla_x(-\Delta_x)_N^{-1}\operatorname{div} v - \nabla_x^\perp(-\Delta_x)_D^{-1}\operatorname{curl} v, \tag{5.3}$$

where $(-\Delta_x)_D^{-1}$ and $(-\Delta_x)_N^{-1}$ are inverse Laplacians with Dirichlet and Neumann boundary conditions respectively and $\nabla_x^\perp := (-\partial_{x_2}, \partial_{x_1})$ is the rotated gradient. Thus, to verify (5.2), it is enough to know that

$$\|Av\|_{W^{2,p}(\Omega)} \le C\left(p + \frac{1}{p-1}\right)\|v\|_{L^p(\Omega)}, \tag{5.4}$$

for the case $A = (-\Delta_x)_D^{-1}$ and $A = (-\Delta_x)_N^{-1}$ (of course, for the case of Neumann boundary conditions we need the extra zero mean assumption). The proof of (5.4) can be found in [32].

Thus, estimate (5.2) is verified and we may return to the proof of the desired estimate (5.1). Let φ_{x_0} be the smooth cut-off function which equals one identically if $x \in B_{x_0}^1$ and zero outside of the ball $B_{x_0}^2$. Then applying (5.2) to the function $v := \varphi_{x_0}u$ and $\Omega = B_{x_0}^2$, we get

$$\|u\|_{W^{1,p}(B_{x_0}^1)} \le Cp(\|\operatorname{curl} u\|_{L^p(B_{x_0}^2)} + \|u\|_{L^\infty(B_{x_0}^2)})$$
$$\le Cp(\|\operatorname{curl} u\|_{L^\infty(B_{x_0}^3)} + \|u\|_{L^2(B_{x_0}^3)}), \tag{5.5}$$

where we have used the obvious estimate $\|u\|_{L^\infty(B_{x_0}^2)} \le C(\|\operatorname{curl} u\|_{L^\infty(B_{x_0}^3)} + \|u\|_{L^2(B_{x_0}^3)})$ and the fact that $p > 2$ is separated from the singularity at $p = 1$. Taking the supremum over $x_0 \in \mathbb{R}^2$, we obtain the desired estimate (5.1) and finish the proof of the lemma. □

The main result of the section is the following theorem.

Theorem 5.3. *Let* $u_1(t)$ *and* $u_2(t)$ *be two weak solutions of the damped Euler equation* (1.1)*. Then, the following estimate holds:*

$$\|u_1(t) - u_2(t)\|_{L_b^2}^2 \le Ke\left(\frac{\|u_1(0) - u_2(0)\|_{L_b^2}^2}{K}\right)^{e^{-Lt}}, \tag{5.6}$$

where the positive constants K *and* L *depend on the* $\mathcal{H}_b$*-norms of* $u_1(0)$ *and* $u_2(0)$*, but are independent of* t*. In particular, the weak solution of the damped Euler solution is unique.*

Proof. Let u_1 and u_2 be two solutions of (1.1) and $w = u_1 - u_2$. Then, this function solves

$$\partial_t w + (w, \nabla_x)u_1 + (u_2, \nabla_x)w + \alpha w + \nabla_x p = 0. \tag{5.7}$$

Multiplying (5.7) by $w\varphi_{R,x_0}$, where φ_{R,x_0} is the same as in (4.5) and $R > 1$ and $x_0 \in \mathbb{R}^2$ are arbitrary, after the straightforward calculations, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L_{\varphi_{R,x_0}}^2}^2 + \alpha\|w\|_{L_{\varphi_{R,x_0}}^2}^2 \le C(|\nabla_x u_1| + |\nabla_x u_2|, w^2\varphi_{R,x_0})$$
$$+ CR^{-1}(|u_2|, w^2)_{L^2(B_{x_0}^{2R})} + |(\nabla_x P(u_1 \otimes u_1 - u_2 \otimes u_2), w\varphi_{R,x_0})|.$$

To estimate the term with pressure, we use (3.13) with $p = 2$ which gives

$$|(\nabla_x P(u_1 \otimes u_1 - u_2 \otimes u_2), w\varphi_{R,x_0})|$$
$$\leq C\|w\|_{L^2(B_{x_0}^{2R})} \int_{y \in \mathbb{R}^2} \theta_{R,x_0}(y)\|u_1 \otimes u_1 - u_2 \otimes u_2\|_{L^2(B_y^R)} \, dy.$$

Now using $\|u_i\|_{L^\infty} \leq C$, $i = 1, 2$, where the constant C depends on the $\mathcal{H}_b$-norms of the initial data (thanks to the dissipative estimate (4.3) and the obvious embedding $\mathcal{H}_b \subset L^\infty$), together with the Cauchy-Schwartz inequality and the straightforward inequality

$$\|u\|_{L^2(B_{x_0}^{2R})}^2 \leq C_R \int_{y \in \mathbb{R}^2} \theta_{R,x_0}(y)\|u\|_{L^2(B_y^R)}^2 \, dy,$$

we end up with

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2_{\varphi_{R,x_0}}}^2 + \alpha\|w\|_{L^2_{\varphi_{R,x_0}}}^2 \leq C_R \int_{y \in \mathbb{R}^2} \theta_{R,x_0}(y)\|w\|_{L^2(B_y^R)}^2 \, dy \tag{5.8}$$
$$+ C(|\nabla_x u_1| + |\nabla_x u_2|, w^2\varphi_{R,x_0}).$$

Thus, we only need to estimate the most complicated last term in the RHS of (5.8). To this end, we will essentially use (5.1) and the fact that $\|u_i\|_{\mathcal{H}_b} \leq C$. Then, due to the interpolation inequality

$$\|w\|_{L^{2p/(p-1)}} \leq C\|w\|_{L^\infty}^\theta \|w\|_{L^2}^{1-\theta}, \quad \theta = \frac{1}{p}$$

which holds for any $p > 2$, we end up with

$$|(|\nabla_x u_1| + |\nabla_x u_2|, w^2\varphi_{R,x_0})| \leq C_R(\|u_1\|_{W_b^{1,p}} + \|u_2\|_{W_b^{1,p}})\|w\|_{L^{2p/(p-1)}(B_{x_0}^{2R})}^2$$
$$\leq Cp\|w\|_{L^\infty}^{2/p}\|w\|_{L^2(B_{x_0}^{2R})}^{2(p-1)/p} \leq Cp\|w\|_{L^2(B_{x_0}^{2R})}^{2(p-1)/p}.$$

Let us take here $p = \ln\left(\frac{K}{\|w\|_{L^2(B_{x_0}^{2R})}^2}\right)$, where K is large enough to guarantee that $p > 2$. Such $K = K(\|u_1(0)\|_{\mathcal{H}_b}, \|u_2(0)\|_{\mathcal{H}_b})$ exists since u_1 and u_2 are globally bounded in the L_b^2-norm and, consequently,

$$\|w\|_{L^2(B_{x_0}^{2R})} \leq \|u_1\|_{L^2_{b,2R}} + \|u_2\|_{L^2_{b,R}} \leq Q_R(\|u_1(0)\|_{\mathcal{H}_b} + \|u_2(0)\|_{\mathcal{H}_b})$$

for some monotone increasing function Q_R. Then, we get

$$|(|\nabla_x u_1| + |\nabla_x u_2|, w^2\varphi_{R,x_0})| \leq C\|w\|_{L^2(B_{x_0}^{2R})}^2 \ln\left(\frac{K}{\|w\|_{L^2(B_{x_0}^{2R})}^2}\right)$$

and (5.8) reads

$$\frac{d}{dt}\|w\|_{L^2_{\varphi_{R,x_0}}}^2 \leq C_R \int_{y \in \mathbb{R}^2} \theta_{R,x_0}(y)\|w\|_{L^2(B_y^R)}^2 \, dy$$
$$+ C\|w\|_{L^2(B_{x_0}^{2R})}^2 \ln\left(\frac{K}{\|w\|_{L^2(B_{x_0}^{2R})}^2}\right). \tag{5.9}$$

We now use the fact that the function $z \to z\log\frac{K}{z}$ is concave. Then, due to Jensen inequality

$$\int_{x_0 \in \mathbb{R}^2} \theta_{R,y}(x_0)\|w\|_{L^2(B_{x_0}^{2R})}^2 \ln\left(\frac{K}{\|w\|_{L^2(B_{x_0}^{2R})}^2}\right) dx_0$$
$$\leq \int_{x_0 \in \mathbb{R}^2} \theta_{R,y}(x_0)\|w\|_{L^2(B_{x_0}^{2R})}^2 \, dx_0 \ln\left(\frac{K \int_{x_0 \in \mathbb{R}^2} \theta_{R,y}(x_0) \, dx_0}{\int_{x_0 \in \mathbb{R}^2} \theta_{R,y}(x_0)\|w\|_{L^2(B_{x_0}^{2R})}^2 \, dx_0}\right).$$

Using also that the function $z \to z \ln \frac{K}{z}$ is monotone increasing if $z \leq Ke^{-1}$, together with (4.10) and (2.17), we get

$$\int_{x_0 \in \mathbb{R}^2} \theta_{R,y}(x_0) \|w\|^2_{L^2(B_{x_0}^{2R})} \ln \left(\frac{K}{\|w\|^2_{L^2(B_{x_0}^{2R})}} \right) dx_0 \leq CZ_{R,y}(w) \ln \frac{K_1}{Z_{R,y}(w)}$$

for some new constant $K_1 > 0$. Multiplying now (5.9) by $\theta_{R,y}(x_0)$, $y \in \mathbb{R}^2$, integrating over $x_0 \in \mathbb{R}^2$ and (2.10), we finally arrive at

$$\frac{d}{dt} Z_{R,y}(w) \leq LZ_{R,y}(w) + CZ_{R,y}(w) \ln \frac{K}{Z_{R,y}(w)} \tag{5.10}$$

for some new constants L and K depending on the $\mathcal{H}_b$-norms of the initial data. Integrating this inequality, we have

$$Z_{R,y}(w(t)) \leq K \left(\frac{Z_{R,y}(w(0))}{K} \right)^{e^{-Lt}} e^{1-e^{-Lt}}. \tag{5.11}$$

Fixing, say, $R = 1$ in this estimate and taking the supremum over $y \in \mathbb{R}^2$, we end up with the desired estimate (5.6) and finish the proof of the theorem. $\qquad \square$

To conclude this section, we recall a weighted enstrophy equality which will be used in the next section to verify the convergence to the attractor in a strong topology. Indeed, multiplying formally equation (4.1) by $\omega\phi_{\varepsilon,x_0}$ where $\phi_{\varepsilon,x_0}(x) := e^{-\varepsilon|x-x_0|}$, integrating over $x \in \mathbb{R}^2$ and using that $\operatorname{div} u = 0$, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2_{L^2_{\phi_{\varepsilon,x_0}}} + \alpha\|\omega\|^2_{L^2_{\phi_{\varepsilon,x_0}}} - \frac{1}{2}(u.\nabla_x\phi_{\varepsilon,x_0}, |\omega|^2) = (\operatorname{curl} g, \omega\phi_{\varepsilon,x_0}). \tag{5.12}$$

However, these arguments require justification since for weak solutions $\omega \in L^\infty([0,T] \times \mathbb{R}^2)$ only and the term $((u,\nabla_x)\omega, \omega\phi_{\varepsilon,x_0})$ is not rigorously defined. We overcome this difficulty using the mollification operators and arguing as in [9].

Theorem 5.4. *Let u be a weak solution of the damped Euler problem* (1.1) *and let $\varepsilon > 0$ and $x_0 \in \mathbb{R}^2$ be arbitrary. Then, the function $t \to \|\omega(t)\|^2_{L^2_{\phi_{\varepsilon,x_0}}}$ is absolutely continuous and the equalitys* (5.12) *holds for almost all t.*

Proof. Indeed, let $S_\mu v := \rho_\mu * v$ where $\rho_\mu(x) = \mu^{-2}\phi(x\mu^{-1})$, $\mu > 0$ and ρ is a standard mollification kernel. Then, applying S_μ to both sides of Eq. (4.1) and denoting $\omega_\mu := S_\mu\omega$, we have

$$\partial_t\omega_\mu + \alpha\omega_\mu + (u,\nabla_x)\omega_\mu = \operatorname{curl} g_\mu + R_\mu, \tag{5.13}$$

where $R_\mu := ((u,\nabla_x)\omega)) * \rho_\mu - (u,\nabla_x)(\omega * \rho_\mu)$. Using the fact that $u(t) \in W_b^{1,p}(\mathbb{R}^2)$ for all $p < \infty$ and arguing exactly as in [9], we see that R_μ is uniformly with respect to $\mu \to 0$ bounded in $L^\infty([0,T], L_b^p(\mathbb{R}^2))$ and

$$R_\mu \to 0 \quad \text{in } L^1([0,T], L_{loc}^p(\mathbb{R}^2)), \tag{5.14}$$

see [9], Lemma II.1, page 516. Multiplying (5.13) by $\omega_\mu\phi_{\varepsilon,x_0}$ (which is now allowed since ω_μ is smooth in x) after the integration over x and t, we get that for every $t \geq s \geq 0$

$$\frac{1}{2}\left(\|\omega_\mu(t)\|^2_{L^2_{\phi_{\varepsilon,x_0}}} - \|\omega_\mu(s)\|^2_{L^2_{\phi_{\varepsilon,x_0}}}\right) = \int_s^t \Bigg((\operatorname{curl} g_\mu, \omega_\mu(\tau)\phi_{\varepsilon,x_0})$$

$$+ (R_\mu(\tau), \omega_\mu(\tau)\phi_{\varepsilon,x_0}) + \frac{1}{2}(u(\tau).\nabla_x\phi_{\varepsilon,x_0}, |\omega_\mu(\tau)|^2) - \alpha\|\omega_\mu(\tau)\|^2_{L^2_{\phi_{\varepsilon,x_0}}}\Bigg) d\tau.$$

Passing to the limit $\mu \to 0$ in this equality and using (5.14) together with the standard convergence properties of the mollification operators as well as that

$$\omega \in L^\infty(R_+ \times \mathbb{R}^2) \cap C_w([0,T], L^2_{\phi_{\varepsilon,x_0}}(\mathbb{R}^2)), \tag{5.15}$$

we end up with the integral equality

$$\frac{1}{2}\left(\|\omega(t)\|^2_{L^2_{\phi_\varepsilon,x_0}} - \|\omega(s)\|^2_{L^2_{\phi_\varepsilon,x_0}}\right) = \int_s^t \bigg((\operatorname{curl} g, \omega(\tau)\phi_{\varepsilon,x_0})$$
$$+ \frac{1}{2}(u(\tau).\nabla_x\phi_{\varepsilon,x_0}, |\omega(\tau)|^2) - \alpha\|\omega(\tau)\|^2_{L^2_{\phi_\varepsilon,x_0}}\bigg)\, d\tau$$

which is equivalent to (5.12) and finish the proof of the theorem. $\qquad\square$

6. The Attractor

The aim of this section is to verify the existence of the attractor for the damped Euler equation in the uniformly local spaces. We first recall that according to Theorems 4.2 and 5.3, Eq. (1.1) generates a solution semigroup $S(t)$ in the phase space $\mathcal{H}_b$:

$$S(t) : \mathcal{H}_b \to \mathcal{H}_b, \quad S(t)u_0 := u(t), \quad t \geq 0, \tag{6.1}$$

where $u(t)$ is a unique solution of (1.1) with the initial data $u_0 \in \mathcal{H}_b$. Moreover, according to estimate (4.3) this semigroup is dissipative in the space $\mathcal{H}_b$, i.e., the following estimate holds:

$$\|S(t)u_0\|_{\mathcal{H}_b} \leq Q(\|u_0\|_{\mathcal{H}_b})e^{-\beta t} + Q(\|g\|_{\mathcal{H}_b}) \tag{6.2}$$

for some positive β and monotone increasing Q which are independent of t and u_0 and, according to estimates (5.6) and (5.11), the maps $S(t)$ are locally Hölder continuous in the space $L^2_b(\mathbb{R}^2)$ and as well as in the space $L^2_{\theta_{R,x_0}}(\mathbb{R}^2)$.

As usual, in the case of unbounded domains and infinite energy solutions, see [23] for more details, we cannot expect the existence of a global attractor in the uniform topology of $\mathcal{H}_b$, but only in the local topology of the space

$$\mathcal{H}_{loc} := \left\{u \in [L^2_{loc}(\mathbb{R}^2)]^2, \quad \operatorname{div} u = 0, \quad \omega \in L^\infty_{loc}(\mathbb{R}^2)\right\}. \tag{6.3}$$

However, we do not know whether or not the above defined semigroup $S(t)$ is asymptotically compact in the strong topology of $\mathcal{H}_{loc}$, so we have to use the weak star topology in $\mathcal{H}_{loc}$ (which we will further denote by $\mathcal{H}^{w^*}_{loc}$) in order to define the convergence to the global attractor. We recall that a sequence $u_n \in \mathcal{H}_{loc}$ converges weakly star in $\mathcal{H}_{loc}$ to some function $u \in \mathcal{H}_{loc}$ iff for any ball $B^R_{x_0}$ the restrictions $u_n|_{B^R_{x_0}}$ converge weakly to $u|_{B^R_{x_0}}$ in $L^2(B^R_{x_0})$ and the restrictions $\omega_n|_{B^R_{x_0}}$ converge weakly star to $\omega|_{B^R_{x_0}}$ in the space $L^\infty(B^R_{x_0})$. Remind also that any closed ball in $\mathcal{H}_b$ is *metrizable* and is *compact* in the topology $\mathcal{H}^{w^*}_{loc}$, see [27]. Thus, we will use the following version of a global attractor.

Definition 6.1. Let $S(t) : \mathcal{H}_b \to \mathcal{H}_b$ be a semigroup. Then, a set $\mathcal{A} \subset \mathcal{H}_b$ is a weak locally compact attractor for this semigroup iff:

1. The set $\mathcal{A}$ is bounded and closed in $\mathcal{H}_b$ and is compact in the topology $\mathcal{H}^{w^*}_{loc}$;
2. It is strictly invariant: $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$;
3. It attracts bounded (in the norm of $\mathcal{H}_b$) sets in the topology of $\mathcal{H}^{w^*}_{loc}$, i.e., for every bounded set $B \subset \mathcal{H}_b$ and every neighbourhood $\mathcal{O}(\mathcal{A})$ of the attractor $\mathcal{A}$ in the topology $\mathcal{H}^{w^*}_{loc}$, there exists $T = T(B, \mathcal{O})$ such that

$$S(t)B \subset \mathcal{O}(\mathcal{A}) \tag{6.4}$$

for all $t \geq T$.

The main result of the section is the following theorem.

Theorem 6.2. *Let the above assumptions hold. Then the solution semigroup $S(t) : \mathcal{H}_b \to \mathcal{H}_b$ associated with the damped Euler equation* (1.1) *possesses a weak locally compact attractor $\mathcal{A}$ in $\mathcal{H}_b$ which is generated by all bounded solutions of the equation:*

$$\mathcal{A} = \mathcal{K}\big|_{t=0}, \tag{6.5}$$

where $\mathcal{K} \subset L^{\infty}(\mathbb{R}, \mathcal{H}_b)$ is the set of all weak solutions $u(t)$ of Eq. (1.1) *which are defined for all $t \in \mathbb{R}$ and are bounded in $\mathcal{H}_b$.*

Proof. Indeed, according to the dissipative estimate (6.2), the ball

$$\mathcal{B}_R := \{u \in \mathcal{H}_b, \ \|u\|_{\mathcal{H}_b} \le R\} \tag{6.6}$$

is absorbing for the semigroup $S(t)$ if R is large enough. This ball is metrizable and compact in the weak star topology of $\mathcal{H}_{loc}^{w^*}$. Thus, the considered semigroup possesses an absorbing ball $\mathcal{B}_R$ which is bounded in $\mathcal{H}_b$ and compact in $\mathcal{H}_{loc}^{w^*}$. Moreover, using the fact that the semigroup is Hölder continuous on $\mathcal{B}_R$ (due to estimate (5.11)) together with compactness of the embedding $\mathcal{H}_{loc} \subset L^2_{loc}$, it is straightforward to show that, for every fixed $t \ge 0$, the operators $S(t)$ are continuous on $\mathcal{B}_R$ in the topology $\mathcal{H}_{loc}^{w^*}$. Thus, all assumptions of the abstract attractor existence theorem (see e.g., [1]) are satisfied and the existence of the attractor $\mathcal{A}$ is proved. Formula (6.5) for the attractor's structure is also an immediate corollary of this theorem. So, Theorem 6.2 is proved. □

Corollary 6.3. *Let the above assumptions hold. Then, for every $\varepsilon > 0$ and every $p < \infty$, the weak locally compact attractor $\mathcal{A}$ is compact in the strong topology of $[W_{loc}^{1-\varepsilon,p}(\mathbb{R}^2)]^2$ and attracts the images of bounded sets in $\mathcal{H}_b$ in this strong topology, i.e., for every bounded set $B \subset \mathcal{H}_b$ and every $R > 0$ and $x_0 \in \mathbb{R}^2$*

$$\lim_{t \to \infty} \mathrm{dist}_{W^{1-\varepsilon,p}(B_{x_0}^R)} \left((S(t)B)\big|_{B_{x_0}^R}, \mathcal{A}\big|_{B_{x_0}^R} \right) = 0, \tag{6.7}$$

where $\mathrm{dist}_V(X, Y)$ is a non-symmetric Hausdorff distance from a set X to a set Y in a metric space V.

Indeed, the convergence (6.7) is an immediate corollary of the definition of the attractor $\mathcal{A}$ and the compactness of the embedding $\mathcal{H}_{loc} \subset W_{loc}^{1-\varepsilon,p}(\mathbb{R}^2)$.

We conclude this section by establishing, analogously to [8], that we may take $\varepsilon = 0$ in (6.7).

Theorem 6.4. *Let the above assumptions hold. Then the attractor $\mathcal{A}$ of the solution semigroup associated with the damped Euler equation* (1.1) *is compact in $W_{loc}^{1,p}(\mathbb{R}^2)$ for any $p < \infty$ and attracts the images of bounded sets in $\mathcal{H}_b$ in the strong topology of this space.*

Proof. Indeed, due to the interpolation, it is sufficient to verify the asymptotic compactness of $S(t)$ in $W_{loc}^{1,2}(\mathbb{R}^2)$ or, which is the same, the asymptotic compactness of the associated vorticity ω in $L^2_{loc}(\mathbb{R}^2)$. To verify this assertion, following [8], we use the so-called energy method, see also [2,24]. Let $u_0^n \in \mathcal{B}_R$ be a sequence of the initial data and $t_n \to \infty$ be a sequence of times. We need to verify that the sequence $S(t_n)u_0^n$ is precompact in $W_{loc}^{1,2}(\mathbb{R}^2)$.

Let $u_n(t)$, $t \ge -t_n$, be the solutions of the following damped Euler problems:

$$\partial_t u_n + (u_n, \nabla_x)u_n + \nabla_x p_n + \alpha u_n = g, \quad u_n\big|_{t=-t_n} = u_0^n \tag{6.8}$$

and the associated vorticities $\omega_n = \mathrm{curl}\, u_n$ solve

$$\partial_t \omega_n + (u_n, \nabla_x)\omega_n + \alpha \omega_n = \mathrm{curl}\, g, \quad \omega_n\big|_{t=0} = \mathrm{curl}\, u_0^n. \tag{6.9}$$

To verify the desired asymptotic compactness (and to finish the proof of the theorem), we only need to verify that the sequence $\omega_n(0)$ is precompact in $L^2_{loc}(\mathbb{R}^2)$. We first note that, due to the dissipative estimate (4.3), the sequence u_n is uniformly bounded in $\mathcal{H}_b$:

$$\|u_n\|_{L^{\infty}(\mathbb{R}, \mathcal{H}_b)} + \|\partial_t u_n\|_{L^{\infty}(\mathbb{R}, L_b^q(\mathbb{R}^2))}, \quad q < \infty, \tag{6.10}$$

where the control over the norm of $\partial_t u_n$ is obtained from Eq. (6.8) analogously to Definition 3.2 (to simplify the notations, we extend u_n and $\partial_t u_n$ by zero for $t \leq t_n$). Thus, without loss of generality, we may assume that

$$\begin{aligned}
u_n &\to u \quad \text{weakly star in} \quad L^\infty_{loc}(\mathbb{R}, \mathcal{H}_{loc}), \\
\partial_t u_n &\to \partial_t u \quad \text{weakly star in } L^\infty_{loc}(\mathbb{R}, L^q_{loc}(\mathbb{R}^2)).
\end{aligned} \tag{6.11}$$

Then, due to the compactness arguments,

$$u_n \to u \quad \text{strongly in } C_{loc}(\mathbb{R} \times \mathbb{R}^3) \tag{6.12}$$

and, in particular,

$$\begin{aligned}
\omega_n &\to \omega \quad \text{weakly star in } L^\infty_{loc}(\mathbb{R} \times \mathbb{R}^2) \quad \text{and} \\
\omega_n &\to \omega \quad \text{strongly in } C_{loc}(\mathbb{R}, W^{-1,2}_{loc}(\mathbb{R}^2)).
\end{aligned} \tag{6.13}$$

Passing to the limit $n \to \infty$ in a straightforward way in Eq. (6.8), we see that the limit function $u(t)$, $t \in \mathbb{R}$, solves the damped Euler equation (1.1) and, therefore, $u \in \mathcal{K}$. Moreover, from (6.13), we conclude that

$$\omega_n(0) \to \omega(0) \quad \text{weakly in } L^2_{loc}(\mathbb{R}^2) \tag{6.14}$$

and using that $\omega_n(0)$ is uniformly bounded in $L^2_b(\mathbb{R}^2)$ the last convergence implies that

$$\omega_n(0) \to \omega(0) \quad \text{weakly in } L^2_{\phi_{\varepsilon,x_0}}(\mathbb{R}^2) \tag{6.15}$$

for all $\varepsilon > 0$ and $x_0 \in \mathbb{R}^2$.

At the second step, we will show that the convergence in (6.15) is actually *strong* which will complete the proof of the theorem. To this end, it is sufficient to prove that

$$\|\omega_n(0)\|_{L^2_{\phi_{\varepsilon,x_0}}} \to \|\omega(0)\|_{L^2_{\phi_{\varepsilon,x_0}}} \tag{6.16}$$

for some $\varepsilon > 0$ and $x_0 \in \mathbb{R}^2$. To this end, we will use the enstrophy equality (5.12) for Eq. (6.9) which we rewrite in the following form:

$$\begin{aligned}
&\|\omega_n(0)\|^2_{L^2_{\phi_{\varepsilon,x_0}}} + \int_{-t_n}^0 e^{\alpha s} \int_{x \in \mathbb{R}^2} \left(\alpha - \frac{1}{2} \phi_{\varepsilon,x_0}(x)^{-1} u_n(x,s).\nabla_x \phi_{\varepsilon,x_0}(x) \right) \\
&\quad \times \phi_{\varepsilon,x_0}(x)|\omega_n(s,x)|^2 \, dx \, ds \\
&= \|\omega_n(-t_n)\|^2_{L^2_{\phi_{\varepsilon,x_0}}} e^{-\alpha t_n} + \int_{-t_n}^0 e^{\alpha s}(\operatorname{curl} g, \omega_n(s)\phi_{\varepsilon,x_0}) \, ds.
\end{aligned} \tag{6.17}$$

Remind that

$$|\nabla_x \phi_{\varepsilon,x_0}(x)| \leq C\varepsilon \phi_{\varepsilon,x_0}(x) \tag{6.18}$$

and therefore we may fix $\varepsilon > 0$ being small enough that

$$\alpha - \frac{1}{2}\phi_{\varepsilon,x_0}(x)^{-1} u_n(x,s).\nabla_x \phi_{\varepsilon,x_0}(x) \geq 0 \tag{6.19}$$

for all $(s,x) \in \mathbb{R}_- \times \mathbb{R}^2$. Then, using the classical result on the weak lower semicontinuity of convex functionals, see e.g., [18], together with the strong convergence (6.12) and weak convergence (6.13), we conclude that

$$\begin{aligned}
&\int_{s \in \mathbb{R}_-} \int_{x \in \mathbb{R}^2} F(s,x,u(s,x),\omega(s,x)) \, ds \, dx \\
&\leq \liminf_{n \to \infty} \int_{-t_n}^0 \int_{x \in \mathbb{R}^2} F(s,x,u_n(s,x),\omega_n(s,x)) \, ds \, dx,
\end{aligned}$$

where

$$F(s, x, u, \omega) := e^{\alpha s} \left(\alpha - \frac{1}{2} \phi_{\varepsilon,x_0}(x)^{-1} u . \nabla_x \phi_{\varepsilon,x_0}(x) \right) \phi_{\varepsilon,x_0}(x) |\omega|^2 \tag{6.20}$$

is a positive function. Passing now to the limit $n \to \infty$ in (6.17), we arrive at

$$\limsup_{n \to \infty} \|\omega_n(0)\|_{L^2_{\phi_{\varepsilon,x_0}}}^2 + \int_{-\infty}^0 \int_{x \in \mathbb{R}^2} F(s, x, u(s,x), \omega(s,x)) dx \, ds$$
$$\leq \int_{-\infty}^0 e^{\alpha s} (\operatorname{curl} g, \omega(s) \phi_{\varepsilon,x_0}) \, ds. \tag{6.21}$$

On the other hand, according to the enstrophy equality for the limit functions u and ω,

$$\|\omega(0)\|_{L^2_{\phi_{\varepsilon,x_0}}}^2 + \int_{-\infty}^0 \int_{x \in \mathbb{R}^2} F(s, x, u(s,x), \omega(s,x)) dx \, ds$$
$$= \int_{-\infty}^0 e^{\alpha s} (\operatorname{curl} g, \omega(s) \phi_{\varepsilon,x_0}) \, ds. \tag{6.22}$$

Thus,

$$\limsup_{n \to \infty} \|\omega_n(0)\|_{L^2_{\phi_{\varepsilon,x_0}}}^2 \leq \|\omega(0)\|_{L^2_{\phi_{\varepsilon,x_0}}}^2 \leq \liminf_{n \to \infty} \|\omega_n(0)\|_{L^2_{\phi_{\varepsilon,x_0}}}^2 . \tag{6.23}$$

Therefore, the convergence (6.16) is verified and the theorem is proved. $\square$

Appendix: The Interpolation Inequality

The aim of Appendix is to verify the following interpolation inequality.

Lemma 6.5. *Let $u \in L^2_b(\mathbb{R}^2)$ be a divergence free vector field such that $\omega := \operatorname{curl} u \in L^\infty(\mathbb{R}^2)$. Then, the following inequality holds:*

$$\|u\|_{L^3(B^R_{x_0})}^3 \leq C \left(\frac{1}{R} \|u\|_{L^2(B^{2R}_{x_0})}^3 + \|u\|_{L^2(B^{2R}_{x_0})}^{5/2} \|\omega\|_{L^\infty(B^{2R}_{x_0})}^{1/2} \right). \tag{6.24}$$

where $R > 0$, $x_0 \in \mathbb{R}^2$ are arbitrary and the constant C is independent of R, x_0, and u.

To verify this inequality, we use the following result proved in [36].

Proposition 6.6. *Let the vector field $u \in [W_0^{1,2}(B^{2R}_{x_0})]^2$ be such that*

$$\operatorname{div} u, \operatorname{curl} u \in L^\infty(B^{2R}_{x_0}).$$

Then,

$$\|u\|_{L^3(B^{2R}_{x_0})} \leq C \|u\|_{L^2(B^{2R}_{x_0})}^{5/6} \left(\|\operatorname{curl} u\|_{L^\infty(B^{2R}_{x_0})} + \|\operatorname{div} u\|_{L^\infty(B^{2R}_{x_0})} \right)^{1/6}, \tag{6.25}$$

where the constant C is independent of R and x_0. Moreover, for $2 < p < \infty$,

$$\|u\|_{L^\infty(B^{2R}_{x_0})} \leq C \|u\|_{L^2(B^{2R}_{x_0})}^\theta \left(\|\operatorname{curl} u\|_{L^p(B^{2R}_{x_0})} + \|\operatorname{div} u\|_{L^p(B^{2R}_{x_0})} \right)^{1-\theta}, \tag{6.26}$$

where $\theta = \frac{1}{2} - \frac{1}{2(p-1)}$, the value C may depend on p, but is independent of R and $x_0 \in \mathbb{R}^2$.

Proof of the lemma. We first note that (6.24) is homogeneous, so (scaling $x \to Rx$ if necessary) it is sufficient to prove the assertion for $R = 1$ and $x_0 = 0$ only. Let now $\varphi \in C_0^\infty(B_0^{4/3})$ be the cut-off function such that $\varphi(x) \equiv 1$ if $x \in B_0^1$. Then applying inequality (6.25) with $R = 2$ to the vector field φu, we get

$$\|u\|_{L^3(B_0^1)}^3 \leq \|\varphi u\|_{L^3(B_0^{4/3})}^3 \leq C\|u\|_{L^2(B_0^2)}^{5/2}\|\omega\|_{L^\infty(B_0^2)}^{1/2}$$
$$+ C\|u\|_{L^2(B_0^2)}^{5/2}\|u\|_{L^\infty(B_0^{4/3})}^{1/2}. \tag{6.27}$$

Applying now inequality (6.26) with $p = 4$ and $\theta = 1/3$ to the vector field $\varphi_1 u$, where $\varphi_1 \in C_0^\infty(B_0^{5/3})$ is a new cut-off function which equals to one if $x \in B_0^{4/3}$, we get

$$\|u\|_{L^\infty(B_0^{4/3})}^{1/2} \leq C\|u\|_{L^2(B_0^2)}^{1/6}\|\omega\|_{L^4(B_0^2)}^{1/3} + \|u\|_{L^2(B_0^2)}^{1/6}\|u\|_{L^4(B_0^{5/3})}^{1/3}. \tag{6.28}$$

In the first term, we replace the L^4-norm of ω by its L^∞-norm, insert the obtained result to (6.27) and use the Young inequality, this gives

$$\|u\|_{L^3(B_0^1)}^3 \leq C\|u\|_{L^2(B_0^2)}^{5/2}\|\omega\|_{L^\infty(B_0^2)}^{1/2} + C\|u\|_{L^2(B_0^2)}^{8/3}\|\omega\|_{L^\infty(B_0^2)}^{1/3}$$
$$+ C\|u\|_{L^2(B_0^2)}^{8/3}\|u\|_{L^4(B_0^{5/3})}^{1/3} \leq C\|u\|_{L^2(B_0^2)}^3 + C\|u\|_{L^2(B_0^2)}^{5/2}\|\omega\|_{L^\infty(B_0^2)}^{1/2}$$
$$+ C\|u\|_{L^2(B_0^2)}^{8/3}\|u\|_{L^4(B_0^{5/3})}^{1/3}. \tag{6.29}$$

Thus, we only need to estimate the L^4-norm in the RHS of (6.29). To this end, we introduce one more cut-off function $\varphi_2 \in C_0^\infty(B_0^2)$ such that $\varphi_2(x) = 1$ for $x \in B_0^{5/3}$ and use the following Ladyzhenskaya type inequality for vector fields $v \in W_0^{1,2}(B_0^2)$:

$$\|v\|_{L^4(B_0^2)}^4 \leq C\|v\|_{L^2(B_0^2)}^2 \left(\|\operatorname{div} v\|_{L^2(B_0^2)}^2 + \|\operatorname{curl} v\|_{L^2(B_0^2)}^2 \right). \tag{6.30}$$

Applying this inequality to the vector field $\varphi_2 u$ and estimating again the L^2-norm of the vorticity by its L^∞-norm, we have

$$\|u\|_{L^4(B_0^{5/3})}^{1/3} \leq C\|u\|_{L^2(B_0^2)}^{1/6}\|\omega\|_{L^\infty(B_0^2)}^{1/6} + C\|u\|_{L^2(B_0^2)}^{1/3}. \tag{6.31}$$

Inserting this estimate to the RHS of (6.29) and using Young inequality again, we derive the desired estimate (6.24). □

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