Cohomogeneity one Kähler and Kähler-Einstein manifolds with one singular orbit I

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Abstract

Let M be a cohomogeneity one manifold of a compact semisimple Lie group G with one singular orbit $S_0 = G/H$. Then M is G-diffeomorphic to the total space $G \times_H V$ of the homogeneous vector bundle over S_0 defined by a sphere transitive representation of G in a vector space V. We describe all such manifolds M which admit an invariant Kähler structure of standard type. This means that the restriction $\mu : S = Gx = G/L \to F = G/K$ of the moment map of M to a regular orbit $S = G/L$ is a holomorphic map of S with the induced CR structure onto a flag manifold $F = G/K$, where $K = N_G(L)$, endowed with an invariant complex structure J^F. We describe all such standard Kähler cohomogeneity one manifolds in terms of the painted Dynkin diagram associated with $(F = G/K, J^F)$ and a parametrized interval in some T-Weyl chamber.

We determine which of these manifolds admit invariant Kähler-Einstein metrics.

Contents

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1 Introduction and statement of the results

We will study cohomogeneity one Kähler G-manifolds of a compact semisimple Lie group G. By a cohomogeneity one manifold we understand an n-dimensional manifold M together with a proper action of a connected Lie group G which has a (real) codimension one orbit.
It is called a Riemannian (respectively, complex; Kähler) cohomogeneity one manifold if an invariant Riemannian metric g (respectively, an invariant complex structure J; an invariant Kähler structure (J, ω)) is given, where J is a complex structure and ω is a symplectic form such

that $g := -\omega \circ J = \omega(\cdot, J\cdot)$ is a Kähler metric.

Following [15] we will consider Kähler cohomogeneity one G-manifolds (M, J, ω) of the *standard type*, that is manifolds which satisfy the following conditions:

(i) Any regular orbit $S = Gx = G/L$ is an *ordinary* manifold. This means that the normalizer $K = N_G(L)$ of the stability subgroup is the centralizer of a torus in G and $\dim K/L = 1$.

(ii) the CR structure $(\mathcal{H}, J^{\mathcal{H}})$ induced by the complex structure J of M on a (codimension one) regular orbit $S = G/L$ is *projectable*, that is the projection $\pi : S = G/L \to F = G/K$ is a holomorphic map of a CR manifold onto the flag manifold F equipped with a fixed invariant complex structure J^F which does not depend on S.

Condition (ii) depends on the complex structure J on M and shows that the CR structure on a regular orbit G/L is determined by a fixed invariant complex structure J_F on the flag manifold F. In particular, all regular orbits are isomorphic as homogeneous CR manifolds.

The conditions (i),(ii) imply that the moment map $\mu : M \to \mathfrak{g}^* \simeq \mathfrak{g}$ of the symplectic G-manifold (M, ω) maps any regular orbit $(S = Gx = G/L, J^{\mathcal{H}})$ holomorphically to the same flag manifold $(F = G/K, J^F)$. Note that $\pi := \mu|_S : S = G/L \to F = G/K$ is the natural equivariant projection.

A homogeneous CR manifold $(S = G/L, \mathcal{H}, J^{\mathcal{H}})$ which satisfies conditions (i),(ii) is called a *standard homogeneous CR manifold*.

So, we can equivalently say that a complex (in particular, Kähler) cohomogeneity one manifold is of the *standard type* or, shortly, *standard* if all regular orbits are standard CR manifolds associated with a fixed flag manifold $(F = G/K, J^F)$ with a complex structure J^F.

The condition (i) is a weak condition. It is equivalent to the condition that the Lie algebra $\mathfrak{l} = \operatorname{Lie} L$ is not the centralizer of a regular 3-dimensional subalgebra of the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$.

The classification of all non standard homogeneous CR manifolds with non degenerate Levi form is given in [2].

Investigation of invariant Einstein metrics on cohomogeneity one manifolds had been started by D. Page and C.Pope in [14], where they construct the first example of such metrics. L.Berard- Bergery in his famous paper ([6]) developed a systematic approach for construction of invariant Einstein metrics on cohomogeneity one manifolds. A deep investigation of singular ODE for invariant Einstein metrics on cohomogeneity one manifolds had been done by Eschenburg and Wang [9] .

In the Kähler case, Y. Sakane gives in [16] conditions for the existence of Kähler-Einstein metrics on $\mathbb{CP}^1$-bundles P over Hermitian symmetric spaces of compact type which are of cohomogenity one with respect to a maximal compact subgroup of the automorphism group

of P. In [12] and [13] more examples are obtained from these $\mathbb{CP}^1$-bundles by blowing down. We refer the reader also to [11], (see also [5] and the references therein).

Invariant Kähler-Einstein metrics on cohomogeneity one manifolds M of a compact semisimple Lie group G had been studied in two important papers by F. Podestá and A. Spiro [15] and A. Dancer and M. Y. Wang [8]. In both papers the authors reduced the Kähler-Einstein equation for an invariant metric in the regular open submanifold $M_{reg} = G/L \times (a, b)$ of M to an ODE for one function together with some algebraic conditions, which are described in terms of the reductive decomposition associated with a regular orbit G/L. They solved this equation and find necessary and sufficient conditions for extendibility of the Kähler-Einstein metric in M_{reg} to the whole manifold M. They considered some examples of manifolds which satisfy these conditions, but did not study such manifolds systematically.

The main aim of this paper is to give a description of (non compact) standard cohomogeneity one Kähler and Kähler-Einstein manifolds with one singular orbit in terms of painted Dynkin diagrams. We closely follow the approach by F. Podestà and A. Spiro [15], who give a useful description of standard cohomogeneity one Kähler manifolds M in terms of "abstract models" and get an effective criterion of existence of an invariant Kähler-Einstein metric on a (compact) cohomogeneity one manifold with two singular orbits. We will reformulate the basic results of [15] and simplify the proofs.

The structure of the paper is the following. In the Preliminaries we recall the basic facts about cohomogeneity one Riemannian manifolds, CR manifolds and flag manifolds which we need. In Section 3 we define standard homogeneous CR manifolds and standard cohomogeneity one complex and Kähler manifolds and discuss their properties.
A standard homogeneous CR manifold $S = G/L$ with associated flag manifold $F = G/K = G/N_G(L)$ is defined by the standard (reductive) decomposition
$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_F^0 + \mathfrak{m} = \mathfrak{k} + \mathfrak{m}$$
orthogonal with respect to the Killing form B of $\mathfrak{g}$, where $\mathfrak{l} = \mathrm{Lie}\, L$, $\mathfrak{k} = \mathrm{Lie}\, K = \mathfrak{l} + \mathbb{R}Z_F^0$ and Z_F^0 is an Ad_L-invariant vector (called *fundamental vector*), normalized by $B(Z_F^0, Z_F^0) = -1$. We will identify Z_F^0 with a G-invariant vector field on S which is the fundamental vector field of the principal T^1-bundle $\pi : S = G/L \to S/T^1 = G/K = G/L \cdot T^1$. An Ad_K-invariant (integrable) complex structure $J^{\mathfrak{m}}$ in $\mathfrak{m}$ defines a complex structure J_F in the flag manifold $F = G/K$ with the reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ and an invariant projectable CR structure $(\mathcal{H}, J^{\mathcal{H}})$ on $S = G/L$ where $\mathcal{H}$ is the invariant distribution defined by the subspace $\mathfrak{m}$.

In Section 4 we describe standard invariant Kähler structures on a regular cohomogeneity one manifold $M_{reg} = (0, d) \times G/L$. Following

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[15], we show that such structures are in 1-1 correspondence with the parametrized open intervals $(Z_0 Z_d) \subset C(J^F)$ which are parallel to $Z^0 = -i Z_F^0$. Here $C(J^F)$ denotes the T-Weyl chamber (i.e. roughly speaking, the projection of the Weyl chamber of $\mathfrak{g}$ to the center $Z(\mathfrak{k})$ of $\mathfrak{k}$) associated with the invariant complex structure J^F of the flag manifold $F = \mu(S) = G/K$.

In Section 5 we give a new proof of Podestà-Spiro formula for the Ricci form ρ. The Einstein equation for an invariant Kähler metric in M_{reg} reduces to a second order ODE for the function $f(t)$ which defines a parametrization of the interval $(Z_0 Z_d) \subset C(J^F)$ and a linear relation between the initial vector Z_0, the vector Z^0 and the Koszul vector $Z^{Kos} \in C(J^F)$ which defines the invariant Kähler-Einstein metric on the flag manifold $F = (G/K, J_F)$.

To calculate ρ, we construct holomorphic coordinates $\{z_i\}$ in $M_{reg} = (0, d) \times G/L$ which are an extension of local holomorphic coordinates in the flag manifold $F = G/K = G/L \cdot T^1$, and use the formula $\rho = -i \partial \bar\partial \log \mu$, where $\mu(z_i, \bar z_i)$ is the density associated with the volume form

$$vol = \mu(z_i, \bar z_i) dz^m \wedge d\bar z^m, \ m = \dim_{\mathbb{C}} M_{reg}.$$

The singular orbit S_0 of a standard cohomogeneity one manifold is a complex submanifold, hence a flag manifold $(S_0 = G/H, J_S)$. In Section 6, we describe all standard cohomogeneity one manifolds with fixed singular orbit $(S_0 = G/H, J_S)$. Any such manifold $M = M_\varphi$ is defined by a surjective homomorphism $\varphi : H \to U_m$ and is the total space of the homogeneous vector bundle $M_\varphi = G \times_H \mathbb{C}_\varphi^m \to S_0 = G/H$ defined by φ. The flag manifold (S_0, J_S) is determined by a painted Dynkin diagram (PDD). In terms of PDD, the homomorphism φ is defined by a connected component (a white string) of type A_{m-1} of the white subdiagram of PDD and a character $\chi : Z(H) = T^k \to T^1$ of the center $Z(H)$. The complex structure J on M_φ is the natural extension of the complex structure J_F. If $e \neq 0$ is a vector from $\mathbb{C}^m$, then regular orbits $S_t := G \times_H (te) = G/L$ are parametrized by $t > 0$ where $L = H_{te}$ is the stabilizer. The subgroup $K = H_{[e]}$ is the stabilizer of the line $[e] \in P\mathbb{C}^m$. This shows that φ determines the standard reductive decomposition

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R} Z_F^0 + \mathfrak{m}.$$

An invariant Kähler metric in M_φ is defined by an interval $(Z_0 Z_d) \subset C(J_F)$ of the T-Weyl chamber associated with J_F which starts from a face associated with G/H together with a parametrization which satisfies the Verdiani boundary condition.

In the last chapter we give necessary and sufficient conditions for a manifold M_φ to have an invariant Kähler-Einstein metric. We will use this condition in the second part of this paper for explicit description of such Kähler-Einstein metrics on standard cohomogeneity one manifolds of a classical Lie group G.

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2 Preliminaries

2.1 Riemannian cohomogeneity one manifolds

Let G be a compact Lie group and (M, g) a Riemannian cohomogeneity one G-manifold, that is G is an isometry group of (M, g) with a (real) codimension one regular orbit $S = Gx = G/L$. Denote by $\pi : M \to \Omega = M/G$ the natural projection to the orbit space. There are four cases: the orbit space is diffeomorphic to a) $(0,1)$, b) $[0,1)$ c) $[0,1]$ or d) S^1.

In the case b) there is one singular orbit $\pi^{-1}(0) = G/H_0$ and in the case c) there are two singular orbits $S_\epsilon = \pi^{-1}(\epsilon) = G/H_\epsilon$, $\epsilon = 0, 1$. A naturally parametrized geodesic $\gamma(t)$ normal to an orbit remains orthogonal to any orbit and it is called a *normal geodesic*. If it is complete, it intersects any orbit. In the cases b), c), we will assume that $x = \gamma(0)$ belongs to the singular orbit $S_0 = Gx = G/H_0$. Then the stabilizer H_0 transforms γ to any other normal geodesic through x and the isotropy representation $j(H_0)|V$ restricted to the normal space $V = T_x^\perp(S_0)$ acts transitively on the sphere: in other words, the orbit $j(H)v = H/H_v = H/L$ is the sphere. The cohomogeneity one G-manifold M near S_0 is locally G-diffeomorphic to the total space $G \times_H V$ of a homogeneous vector bundle over the singular orbit S_0.

In the case c), when $M/G \simeq [0, 1]$, rescaling the metric we may assume that $\gamma(\epsilon) \in S_\epsilon$, $\epsilon = 0, 1$. Then the cohomogeneity one manifold M is determined by the triple (H_0, L, H_1) of stability subgroups of $\gamma(0), \gamma(1/2), \gamma(1)$ and is denoted by $M(H_0, L, H_1)$. Note that $L \subset H_0 \cap H_1$ and H_ϵ/L are spheres.

In the case d), $M/G \simeq S^1 = \{exp(2\pi it)\}$, we assume that $\gamma(0), \gamma(1) \in S_0 = \pi^{-1}(0)$. Note that the stabilizer $L = G_{\gamma(t)}$ of any regular point of γ preserves pointwisely the geodesic γ. So we can identify any regular orbit $S_t = G(\gamma(t))$ with the same homogeneous space G/L. Deleting singular points (if they exist) or, in the case d), the regular orbit $S_0 = \pi^{-1}(0)$ we get an open dense submanifold M_{reg} of regular points which is G-diffeomorphic to $M_{reg} = (0, 1) \times G/L$.

Note that the orbit space Ω of a Riemannian cohomogeneity one manifold has a structure of a metric space. The following proposition gives a description of cohomogeneity one Riemannian manifolds (M, g) and their orbit spaces.

Proposition 1 *Let (M, g) be a Riemannian G-manifold with the orbit space Ω. Then up to a homothety, (M, g) is described as follows:*

a)(No singular orbit) $M = \Omega \times G/L$, where $\Omega \approx (0, 1)$. Moreover, if a normal geodesic is complete, then $\Omega = \mathbb{R}$. In the non-complete case, $\Omega = (0, 1)$ or $\mathbb{R}^+$. The metric is given by

$$g = dt^2 + g_t$$

where g_t, $t \in \Omega$ is a 1-parameter family of invariant Riemannian metrics on G/L.

b)(One singular orbit $S_0 = G/H$) The orbit space is $\Omega = [0, d)$, $d = \infty$ or 1. If a normal geodesic is complete, then $d = \infty$ and the manifold $M = M(H, L) = G \times_H V$ is the homogeneous vector bundle over the singular orbit S_0 defined by a sphere transitive orthogonal representation $\nu : H \to O(V)$ of H in an Euclidean vector space V. In the non complete case M is a tubular invariant neighborhood $M = G \times_H B \subset M(H, L)$ of the zero section where B is the unit ball in V. The invariant metric in M is an invariant extension of the $j(H)$-invariant Riemannian metric in V, which is described by L. Verdiani [18].

c)(Two singular orbits $S_\epsilon = G/H_\epsilon$, $\epsilon = 0, 1$), $\Omega = [0, 1]$. The Riemannian manifold M is obtained by gluing together two manifolds $M_- = \pi^{-1}([0, 1/2))$, $M_+ = \pi^{-1}((1/2, 1])$ of type b) along the isomorphic boundary $\partial M_\pm = G/L$. As a cohomogeneity one manifold it is defined by the triple of subgroups H_0, L, H_1 such that $H_\epsilon/L = S^{n_\epsilon}$ and it is denoted by $M = M(H_0, L, H_1)$.

d) (No singular orbit, Ω is not contractible) $\Omega = S^1$ and M is a fibre bundle over the circle S^1 having as universal cover a Riemannian manifold $\mathbb{R} \times G/L$ of type a).

2.2 Flag manifolds and painted Dynkin diagrams

2.2.1 Isotropy decomposition, T-roots, T-Weyl chambers and invariant complex structures

Let $F = G/K = \mathrm{Ad}_G Z$, where $Z \in \mathfrak{g}$, be a flag manifold, i.e. an adjoint orbit of a compact semisimple Lie group G with the B-orthogonal (where B is the Killing form) reductive decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} = C_{\mathfrak{g}}(Z) + \mathfrak{m}.$$

We can decompose $\mathfrak{k}$ as

$$\mathfrak{k} = Z(\mathfrak{k}) \oplus \mathfrak{k}'$$

where $\mathfrak{k}'$ is the semisimple part and $Z(\mathfrak{k})$ is the center. We fix a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{k}$ (hence also of $\mathfrak{g}$) and denote by R the root system of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ w.r.t. the Cartan subalgebra $\mathfrak{c}^{\mathbb{C}}$. We set

$$R_{\mathfrak{k}} := \{\alpha \in R, \ \alpha(Z(\mathfrak{k})) = 0\}, \ R_{\mathfrak{m}} := R \setminus R_{\mathfrak{k}}.$$

Then

$$\mathfrak{k} = \mathfrak{c} + \mathfrak{g}(R_{\mathfrak{k}})^\tau, \ \mathfrak{m} = \mathfrak{g}(R_{\mathfrak{m}})^\tau,$$

where for a subset $P \subset R$, we set

$$\mathfrak{g}(P) = \sum_{\alpha \in P} \mathfrak{g}_\alpha$$

being $\mathfrak{g}_\alpha$ the root space with root α and V^τ means the fix point set in $V \subset \mathfrak{g}^{\mathbb{C}}$ of the complex conjugation τ. Recall that the Killing form induces an Euclidean metric in the real vector space $i\mathfrak{c}$ and roots are identified with real linear forms on $i\mathfrak{c}$. We set $\mathfrak{t} := iZ(\mathfrak{k}) \subset i\mathfrak{c}$ and denote by

$$\kappa : R \to R|_{\mathfrak{t}}, \ \alpha \mapsto \bar\alpha := \alpha|_{\mathfrak{t}}$$

the restriction map.

Definition 2 *The set $R_T = \kappa(R_{\mathfrak{m}}) = R_{\mathfrak{m}}|\mathfrak{t}$ of linear forms on $\mathfrak{t}$ which are restriction of roots from $R_{\mathfrak{m}}$ is called the system of T-**roots** and connected components C of the set $\mathfrak{t} \setminus \{\ker \bar\alpha, \ \bar\alpha \in R_T\}$ are called T-**Weyl chambers**.*

Sets of T-roots ξ bijectively correspond to irreducible $\mathfrak{k}$-submodules $\mathfrak{m}(\xi) := \mathfrak{g}(\kappa^{-1}(\xi))$ of the complexified isotropy module $\mathfrak{m}^{\mathbb{C}}$ of the flag manifold $F = G/K$.

So a decomposition of the $\mathfrak{k}$-modules $\mathfrak{m}^{\mathbb{C}}$ and $\mathfrak{m}$ into irreducible submodules can be written as

$$\mathfrak{m}^{\mathbb{C}} = \sum_{\xi \in R_T} \mathfrak{m}(\xi), \ \mathfrak{m} = \sum_{\xi \in R_T^+} [\mathfrak{m}(\xi) + \mathfrak{m}(-\xi)]^\tau$$

where $R_T^+ := \kappa(R_{\mathfrak{m}}^+)$ is the system of positive T-roots associated with a system of positive roots R^+, see [3], [1].

We fix a system of simple roots Π_W of $R_{\mathfrak{k}}$ and denote by $\Pi = \Pi_W \cup \Pi_B$ its extension to a system of simple roots of R. Let $R^+ = R^+(\Pi)$ be the associated system of positive roots and $R_{\mathfrak{m}}^+ := R^+ \cap R_{\mathfrak{m}}$. The set $R_T^+ := \kappa(R_{\mathfrak{m}}^+)$ is called positive T-root set.

We need the following

Theorem 3 *[3] There exists a one-to-one correspondence between extensions $\Pi = \Pi_W \cup \Pi_B$ of the system Π_W of simple system of $R_{\mathfrak{k}}$, T-Weyl chambers $C \subset \mathfrak{t}$ and invariant complex structures (ICS) J on $F = G/K$. If $\Pi_B = \{\beta_1, \ldots, \beta_k\}$, then the corresponding T-Weyl chamber is defined by $C = \{\bar\beta_1 > 0, \ldots, \bar\beta_k > 0\}$ where $\bar\beta = \kappa(\beta)$ and the complex structure is defined by $\pm i$-eigenspace decomposition*

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ + \mathfrak{m}^- = \mathfrak{g}(R_{\mathfrak{m}}^+) + \mathfrak{g}(-R_{\mathfrak{m}}^+) \tag{1}$$

of the complexified tangent space $\mathfrak{m}^{\mathbb{C}} = T_{eK}(G/K)$.

The extension $\Pi = \Pi_W \cup \Pi_B$ can be graphically described by a painted Dynkin diagram, i.e. the Dynkin diagram which represents the system Π with the nodes representing Π_B painted in black. Such a diagram, which we sometimes identify with the pair (Π_W, Π_B), allows to reconstruct the flag manifold $F = G/K$ with invariant complex structure J^F as follows: the semisimple part $\mathfrak{k}'$ of the (connected) stability subalgebra $\mathfrak{k}$ is defined as the regular semisimple subalgebra associated with the closed subsystem $R_{\mathfrak{k}} = R \cap \text{span}(\Pi_W)$ and the vectors ih_j defined by condition

$$\beta_k(h_j) = \delta_{kj}, \ \alpha_i(h_j) = 0, \ \beta_j \in \Pi_B, \alpha_i \in \Pi_W$$

form a basis of the center $Z(\mathfrak{k})$. The complex structure is defined by (1).

2.2.2 Invariant symplectic forms and Kähler structures

An element $Z \in \mathfrak{t}$ is called to be K-**regular** if its centralizer $C_G(Z) = K$ or, equivalently, any T-root has a non-zero value on Z.

Proposition 4 *([7], [3]) There exists a natural one-to-one correspondence between elements $Z \in \mathfrak{t}$ and closed invariant 2-forms ω_Z on G/K, given by*

$$Z \leftrightarrow \omega_Z|_o = i\, d(B \circ Z),$$

where d is the exterior differential in the Lie algebra $\mathfrak{g}$ defined by $d\alpha(X, Y) = -1/2\alpha([X, Y])$ and $o = eK \in G/K$.
Moreover, regular elements $Z \in C$ from a T-Weyl chamber C correspond to the Kähler forms ω_Z with respect to the complex structure $J(C)$ associated to C, that is they define an invariant Kähler structure $(\omega_Z, J(C))$. The 2-form $\frac{1}{2\pi}\omega_Z$ is integral if the 1-form $B \circ Z$ has integer coordinates with respect to the fundamental weights π_i associated with the system of black simple roots $\beta_i \in \Pi_B$.

Recall that if $\Pi_W = \{\alpha_1, \ldots, \alpha_m\}$ (resp. $\Pi_B = \{\beta_1, \ldots, \beta_k\}$) is the set of white (resp. black) simple roots, then the fundamental weight π_i associated with β_i, $i = 1, \ldots, k$, is the linear form defined by

$$\frac{2\langle \pi_i, \beta_j \rangle}{\|\beta_j\|^2} = \delta_{ij}, \quad \langle \pi_i, \alpha_j \rangle = 0. \tag{2}$$

The B-dual to π_i vectors h_i form a basis of $\mathfrak{t}$.
Let $E_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in R$, be the *Chevalley basis* of $\mathfrak{g}(R)$ such that $B(E_\alpha, E_{-\alpha}) = \frac{2}{\langle \alpha, \alpha \rangle}$ where $< ., . >$ is the scalar product in $i\mathfrak{c}^* = \mathrm{span}(R)$ induced by the Killing form. We denote by $\omega_\alpha = B \circ E_\alpha$ the dual basis of 1-forms. Then for $Z \in \mathfrak{t}$

$$\omega_Z = -i \sum_{\alpha \in R_\mathfrak{m}^+} \frac{2\alpha(Z)}{< \alpha, \alpha >} \omega_\alpha \wedge \omega_{-\alpha} \tag{3}$$

Indeed,

$$
\begin{array}{ll}
i\, d(B \circ Z)(E_\alpha, E_{-\alpha}) = & -\frac{i}{2} B(Z, [E_\alpha, E_{-\alpha}]) \\
= & -\frac{i}{2} B([Z, E_\alpha], E_{-\alpha}) \\
= & -\frac{i}{2}\alpha(Z) B(E_\alpha, E_{-\alpha}) \\
= & -\frac{i\alpha(Z)}{\langle \alpha, \alpha \rangle} \\
= & -2i\frac{\alpha(Z)}{\langle \alpha, \alpha \rangle} \omega_\alpha \wedge \omega_{-\alpha}(E_\alpha, E_{-\alpha}).
\end{array}
$$

Definition 5 *The 1-form*

$$\sigma = \sum_{\beta \in R_\mathfrak{m}^+} \beta \in \mathfrak{t}^* \subset i\mathfrak{c}^*$$

is called the Koszul form and the dual vector $Z^{Kos} := B^{-1} \circ \sigma$ is called the Koszul vector.

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Proposition 6 *[3] The* **Koszul vector** Z^{Kos} *defines the invariant Kähler-Einstein structure* $(\omega_{Z^{Kos}}, J(C))$ *on* $F = G/K$, *where* $J(C)$ *is the invariant complex structure associated with the T-Weyl chamber C which is defined by* Π_B.

Let us conclude this section by recalling the flag manifolds of the classical groups: (see, for example, [3], [4]):

- $SU(n)/S(U(n_1) \times \cdots \times U(n_s) \times U(1)^m)$
  $n = n_1 + \cdots + n_s + m,\ s, m \geq 0$
- $SO(2n+1)/U(n_1) \times \cdots \times U(n_s) \times SO(2\ell+1) \times U(1)^m$
- $Sp(n)/U(n_1) \times \cdots \times U(n_s) \times Sp(\ell) \times U(1)^m$
- $SO(2n)/U(n_1) \times \cdots \times U(n_s) \times SO(2\ell) \times U(1)^m$
  $n = n_1 + \cdots + n_s + m + \ell,\ s, m, \ell \geq 0, \ell \neq 1$

2.3   Homogeneous CR manifolds

We recall that a *CR structure* on a manifold S is a pair $(\mathcal{H}, J^{\mathcal{H}})$ where $\mathcal{H}$ is a codimension one distribution and $J^{\mathcal{H}} \in \Gamma(\mathrm{End}(\mathcal{H}))$ is a field of complex structures in $\mathcal{H}$ such that the $\pm i$-eigendistributions $\mathcal{H}^{\pm} \subset \mathcal{H}^{\mathbb{C}}$ are involutive (i.e. closed w.r.t. the Lie bracket).

The complex structure J on a manifold M induces a CR structure $(\mathcal{H}, J^{\mathcal{H}})$ on any hypersurface $S \subset M$ where $\mathcal{H} \subset TM$ is the maximal J-invariant distribution and $J^{\mathcal{H}} = J|_{\mathcal{H}}$.

2.3.1   Ordinary homogeneous manifolds and projectable CR structures

Definition 7 *A homogeneous manifold $S = G/L$ of a compact semisimple Lie group G is called an ordinary manifold if the normalizer $K := N_G(L)$ is the centralizer of a torus and contains L as a codimension one subgroup.*

Such a manifold is the total space of a homogeneous principal circle bundle $\pi : G/L \to G/K$ over the flag manifold $F = G/K$.

Let $S = G/L$ be an ordinary homogeneous manifold and $K = N_G(L)$. We define the *standard* reductive decomposition as the B-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_F^0 + \mathfrak{m} \tag{4}$$

where $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z_F^0$ and $B(Z_F^0, Z_F^0) = -1$.
We will call Z_F^0 *fundamental vector* and we will identify it with an invariant vector field on S which is the fundamental vector field of the principal bundle π which generates a commuting with G action of the circle group T^1 on S , that is the structure group of the principal

bundle π. The dual 1-form $\theta^0 := B \circ Z_F^0$ is an invariant 1-form on S which defines a canonical invariant connection in π.

We denote by Z^0 the vector in $\mathfrak{t} = iZ(\mathfrak{k})$ such that $Z_F^0 = iZ^0$ and, by a slight abuse of notation, we will call it also fundamental vector.

The following lemma shows that, given a flag manifold $F = G/K$, almost all closed codimension one subgroups L of K define an ordinary manifold $S = G/L$.

Lemma 8 *[2], [15] Let $F = G/K$ be a flag manifold and L a codimension one closed (normal) subgroup of K. If G/L is not an ordinary manifold, then $L = C_G(A_1)$ is the centralizer of the 3-dimensional regular subalgebra A_1 of $\mathfrak{g}^{\mathbb{C}}$, associated with a long root such that $G/N_G(A_1)$ is the Wolf space (symmetric quaternionic Kähler manifold) or $G = G_2$ and $\mathfrak{a}_1$ is the 3-dimensional subalgebra associated with a short root.*

Now we state some elementary properties of an ordinary manifold $S = G/L$.

Lemma 9 *([2], [15]) Let $S = G/L$ an ordinary manifold. Then,*

i) *Any invariant vector field on S is proportional to Z_F^0.*

ii) *The only invariant codimension one distribution in $S = G/L$ is the distribution $\mathcal{H}$ defined by the Ad_K-invariant subspace $\mathfrak{m}$. This distribution is also Z_F^0-invariant.*

iii) *There is a natural one-to-one correspondence between invariant complex structures J^F on the flag manifold $F = G/K$, Ad_K-invariant complex structures $J^{\mathfrak{m}}$ on $\mathfrak{m}$ (which are integrable in the sense that $\mathfrak{h}^{\mathbb{C}} + \mathfrak{m}^{10}$ is a subalgebra, where $\mathfrak{m}^{10} \subset \mathfrak{m}^{\mathbb{C}}$ is the i-eigenspace of $J^{\mathfrak{m}}$) and invariant CR structures $(\mathcal{H}, J^{\mathcal{H}})$ on S which are also Z_F^0-invariant.*

iv) *Any irreducible K-submodule of $\mathfrak{m} = T_oF$ remains irreducible as L-submodule.*

Following [15] we give

Definition 10 *An invariant CR structure $(\mathcal{H}, J^{\mathcal{H}})$ on an ordinary manifold $S = G/L$ is called projectable if it is Z_F^0-invariant or, equivalently if the projection $\pi : S = G/L \to F = G/K$ is a holomorphic map of the CR manifold S onto the flag manifold F with some invariant complex structure J^F.*
An ordinary manifold $S = G/L$ with a projectable CR structure is called a homogeneous CR manifold of standard type or standard CR manifold.

The following lemma gives a sufficient condition for an invariant CR structure on an ordinary manifold $S = G/L$ to be projectable.

Lemma 11 *([15]) If irreducible Ad_K-submodules of $\mathfrak{m}^{\mathbb{C}}$ are non-equivalent as Ad_L-submodules then any invariant CR structure on an ordinary manifold $S = G/L$ is projectable.*

All compact homogeneous Levi non-degenerate CR manifolds with non projectable CR structure have been classified by [2]. More precisely, they prove

Theorem 12 *[2] A compact homogeneous Levi non-degenerate non projectable CR manifold $(S = G/L, \mathcal{H}, J|_{\mathcal{H}})$ is either the sphere bundle of a compact rank one symmetric space (CROSS) or one of the exceptional homogeneous CR manifolds*

$$SU_n/T^1 \cdot SU_{n-2}, \ SU_p \cdot SU_q/T^1 \cdot U_{p-2} \cdot U_{q-2},$$

$$SU_n/T^1 \cdot SU_2 \cdot SU_2 \cdot SU_{n-4}, \ SO_{10}/T^1 \cdot SO_6, \ E_6/T^1 \cdot SO_8$$

which admit a holomorphic fibration over flag manifolds with fibers S^n, $n = 2, 3, 5, 7, 9$ respectively.

3 Cohomogeneity one Kähler manifolds of standard type

Let (M, ω, J, g) be a cohomogeneity one Kähler G-manifold, where G is a connected compact semisimple Lie group. Deleting singular orbits (if they exist) or, in the case $M/G \simeq S^1$, the regular orbit $S_0 = \pi^{-1}(0)$ we get an open dense submanifold M_{reg} of regular points which we identify as a G-manifold with $M_{reg} = (0, d) \times G/L$, $d = 1$ or ∞.

We may assume that the induced metric $g_{reg} := g|_{M_{reg}}$ has the form

$$g_{reg} = dt^2 + g_t, \ t \in (0, d),$$

where $g_t := g|_{S_t}$ is an invariant metric on the regular orbit $S_t := \{t\} \times G/L = G/L$. We denote by $T_t := \partial_t|_{S_t}$ the unit normal vector field to the orbit S_t which is tangent to normal geodesics $\gamma(t) = (t, x_0)$. Then JT_t is a G-invariant tangent vector field on S_t and $\theta_t := g \circ JT_t$ the dual invariant 1-form.

Any regular orbit S_t carries an invariant CR structure $(\mathcal{H}, J^{\mathcal{H}})$ induced by J, where

$$\mathcal{H} := JT_t^\perp = \ker \theta_t \subset TS_t$$

and $J_t := J|_{\mathcal{H}}$.

Definition 13 *We say that a cohomogeneity one Kähler or complex manifold M is of standard type if a regular orbit $S_t = G\gamma(t) = G/L$ with the induced CR structure is a CR manifold of the standard type (Definition 10 above) and singular orbits (if exist) are flag manifolds with induced complex structure.*

Remark 14 *The last condition is automatically satisfied if M is a Kähler manifold.*

3.1 Moment map of a cohomogeneity one Kähler manifold

Let (M, J, ω, g) be a cohomogeneity one Kähler manifold of a compact semisimple Lie group G. Since the group G is semisimple and preserves the Kähler form, the (G-equivariant) moment map

$$\mu : M \to \mathfrak{g}^* \simeq \mathfrak{g}, \ x \to \mu_x, \ \mu_x(X) = h_X(x)$$

where h_X it the hamiltonian of the Killing vector field $\hat{X}$ generated by $X \in \mathfrak{g}$, is defined.

Lemma 15 *[15] The restriction $\mu_t : S_t = G/L \to F_t := \mu(S_t) = G/K$ of the moment map to a regular orbit is a G-equivariant principal bundle with the structure group $T^1 = K/L$ over a flag manifold $F = G/K$ (i.e. an adjoint orbit of G). In particular, L is a codimension one normal subgroup of the group K, which is the centralizer of a torus in G.*

Proof: Since the moment map μ_t is equivariant, it maps $S_t = G/L$ onto an adjoint orbit $F = G/K$.

Now $\ker \omega_t = \ker \omega|_{TS_t} = \mathbb{R}JT_t$ and $d(h_X) = \omega \circ X$ give $\ker d\mu_t = \mathbb{R}JT_t$. So L is a codimension one subgroup of K. In terms of the Lie algebra, we can write a B-orthogonal decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_F^0 + \mathfrak{m}, \ \mathfrak{k} = \mathfrak{l} + \mathbb{R}Z_F^0$$

where Z_F^0 is an Ad_L-invariant vector, such that the associated invariant vector field on S_t is proportional to JT_t. The corresponding 1-parameter group $T^1 = \exp\mathbb{R}JT_t = \exp\mathbb{R}Z_F^0$ (which is closed in K, since the subgroup L is closed, see [2]) commutes with G and defines the structure of T^1-principal bundle $\pi : S_t \to F = G/K_t$. $\square$

Remark 16 *The subgroup $L \subset K = L \cdot T^1 = L^{con} \cdot T^1$ can be non connected.*

Corollary 17 *A cohomogeneity one Kähler manifold M is of the standard type if the moment map $\mu : S_t = G/L \to F = G/K$ maps any regular orbit with the induced CR structure holomorphically onto the associated flag manifold $F = G/K = G/N_G(L)$ with a fixed invariant complex structure J^F.*

4 Standard invariant Kähler structures on cohomogeneity one manifolds

Let $(S = G/L, \mathcal{H}, J^{\mathcal{H}})$ be a standard CR manifold with holomorphic fibration $\pi : S = G/L \to F = G/K$ over a flag manifold with a complex structure J^F. Infinitesimally, it is described by a standard decomposition

$$\mathfrak{g} = (\mathfrak{l} + \mathbb{R}Z_F^0) + \mathfrak{m} = \mathfrak{k} + \mathfrak{m}$$

where Z_F^0 is the fundamental vector, $\mathfrak{k} = N_\mathfrak{g}(\mathfrak{l})$, together with an Ad_K-invariant complex structure $J^\mathfrak{m}$ in $\mathfrak{m}$. Now we describe invariant Hermitian and Kähler structures of standard type in a regular cohomogeneity one manifold $M_{reg} = (0, d) \times G/L$ which induce a given CR structure $(\mathcal{H}, J^\mathcal{H})$ on each regular orbit $S_t = \{t\} \times G/L = G/L$. We identify Z_F^0 with a G-invariant vector field on $S_t = G/L$ and denote by $\theta^0 := B \circ Z_F^0$ the dual invariant 1-form with kernel $\mathcal{H} = \ker \theta^0$. Any invariant vector field (resp. 1-form) is conformal to Z_F^0 (resp. θ^0).

4.1 Invariant Hermitian structures on M_{reg}

We fix an invariant metric g on $M_{reg} = (0, d) \times G/L$ of the form

$$g = dt^2 + g_t$$

where g_t is an invariant metric in $S_t = S = G/L$ such that the CR structure $J^\mathcal{H}$ is orthogonal.
We describe all g-orthogonal projectable invariant complex structures J on M_{reg} which project onto a given invariant complex structure J^F of the flag manifold $F = G/K$, or, equivalently, are extensions of the associated CR structure $J^\mathcal{H}$. Then (g, J) is an invariant Hermitian structure in M_{reg}.

Since the invariant 1-forms dt, θ^0 are orthogonal to each other and the distribution $\mathcal{H} = \ker\{dt, \theta^0\}$, any extension of $J^\mathcal{H}$ to an orthogonal invariant almost complex structure on M_{reg} can be written as

$$
\begin{array}{llll}
J^*dt & := & dt \circ J = a(t)\theta^0, & J^*\theta^0 = & -\frac{1}{a(t)}dt \\
JT_t & = & \frac{1}{a(t)}Z_F^0, & JZ_F^0 = & -a(t)T_t
\end{array}
\tag{5}
$$

where $a : (0, d) \to \mathbb{R}$ is a non vanishing function.

Proposition 18 *Any extension of the CR structure $J^\mathcal{H}$ to an invariant orthogonal (integrable) complex structure J on M_{reg} is given by (5).*

Proof: We have to check that the almost complex structure J is integrable. Since $J|_\mathcal{H} = J^\mathcal{H}$ is an integrable CR structure in S, it is sufficient to check that the differential of the 1-form $dt - iJ^*dt = dt + ia\theta^0 \in \Omega^{1,0}(M)$ belongs to the space $\Omega^{2,0}(M) + \Omega^{1,1}(M)$ of forms of type $(2, 0)$ and $(1, 1)$. Since $d\theta^0 = \omega^0 \in \Omega^{1,1}(M)$, we have

$$
\begin{aligned}
d(dt + ia\theta^0) &= i\dot{a}dt \wedge \theta^0 + iad\theta^0 \\
&= (dt + ia\theta^0) \wedge i\dot{a}\theta^0 + iad\theta^0 \in \Omega^{2,0}(M) + \Omega^{1,1}(M).
\end{aligned}
$$

$\square$

4.2 Invariant Kähler structures on M_{reg}

We describe all standard invariant Kähler structures $(\omega_{reg}, J_{reg}, g_{reg})$ on $M_{reg} = (0, d) \times G/L$ which induce the projectable CR structure $(\mathcal{H}, J^{\mathcal{H}})$ on $S_t = \{t\} \times S$.

Recall that vectors $Z \in \mathfrak{t} = iZ(\mathfrak{k})$ correspond to invariant closed 2-forms ω_Z on $F = G/K$, whose value at $o = eK$ is given by

$$(\omega_Z)_o(X, Y) = i \, d(B \circ Z)(X, Y), \ X, Y \in \mathfrak{m}.$$

We will denote the pull back of ω_Z to M_{reg} by the same letter ω_Z. In particular, the vector $Z^0 = -iZ_F^0$ defines the invariant 2-form $\omega^0 := \omega_{Z^0} = i \, dB \circ Z^0 = dB \circ Z_F^0 = d\theta^0$ on S which is the curvature of the principal connection θ^0 in the principal bundle $\pi : S = G/L \to F = G/K$. We denote by $C(J^F) \subset \mathfrak{t}$ the T-Weyl chamber which corresponds to the complex structure J^F.

The following proposition shows that a standard invariant Kähler structure in M_{reg} is defined by a parametrized open interval $(Z_0 Z_d)$ in the T-Weyl chamber $C(J^F)$, parallel to Z^0.

Proposition 19 *A standard invariant Kähler structure $(\omega_{reg}, J_{reg}, g_{reg})$ on $M_{reg} = (0, d) \times G/L$ which induces an invariant CR structure $(\mathcal{H}, J^{\mathcal{H}})$ on regular orbits is defined by a parametrized open interval $Z_t = Z_0 + f(t)Z^0 \subset C(J^F)$ where $f : (0, d) \to \mathbb{R}$, $\lim_{t \to 0} f(t) = 0$ is a smooth function with $a(t) = \dot{f}(t) > 0$. More precisely, define*

$$\theta_t := iB \circ Z_t = iB \circ Z_0 + f(t)iB \circ Z^0 = \theta_0 + f(t)\theta^0,$$

$$\omega_0 := d\theta_0 = \omega_{Z_0}, \ \omega^0 := d\theta^0 = \omega_{Z^0}.$$

Then the Kähler structure is given by

$$\omega_{reg} = d(\theta_0 + f(t)\theta^0) = \dot{f}dt \wedge \theta^0 + \omega_0 + f(t)\omega^0,$$

$$g_{reg} = dt^2 + (\dot{f}\theta^0)^2 + \pi^* g_0 + f(t)\pi^* g^0,$$

$$J_{reg} : dt \to -\dot{f}\theta^0, \theta^0 \to \frac{1}{f}dt, \ J|_{\mathcal{H}} = J^{\mathcal{H}}.$$

Here $g^0 = -\omega^0 \circ J^F$, $g_0 = -\omega_0 \circ J^F$ are symmetric bilinear forms on F.
The pair $(\omega_t := \omega_0 + f(t)\omega^0, J^F)$ defines an invariant Kähler structure on F for $t \in (0, d)$.

Proof: Using (5), we have

$$\omega_{reg}(T_t, JT_t) = 1 = \omega_{reg}(T_t, \frac{1}{a}Z_F^0).$$

This shows that the Kähler form ω_{reg} on M_{reg} can be written as

$$\omega_{reg} = dt \wedge a(t)\theta^0 + \omega_t^S$$

where $\omega_t^S := \omega|_{S_t}$ is a closed invariant 2-form on $S_t = S$ with kernel $\mathbb{R}Z_F^0$.

The form ω_t^S is the pull back of an invariant symplectic form $\omega_{Z_t} = id(B \circ Z_t)$ associated with a vector $Z_t \in C(J^F)$ (where $C(J^F)$ is the Weyl chamber which corresponds to the complex structure J^F). It is sufficient to check that ω_t^S is invariant with respect to the fundamental vector field Z_F^0. We have

$$\mathcal{L}_{Z_F^0}\omega_t^S = i_{Z_F^0}d\omega_t^S + di_{Z_F^0}\omega_t^S = 0$$

since ω_t^S is closed and has kernel $\mathbb{R}Z_F^0$.

Now the condition that ω_{reg} is closed can be written as

$$dt \wedge a\omega^0 + dt \wedge \dot{\omega}_t^S = 0$$

or

$$-a(t)\omega^0 + \dot{\omega}_t^S = 0 = \omega_{aZ^0 + \dot{Z}_t} = 0$$

or $\dot{Z}_t + aZ^0 = 0$. This implies that the curve Z_t is an (open) interval (maybe, a ray) $(Z_0 Z_d) \subset \bar{C}(J^F)$ with parametrization $Z_t = Z_0 + f(t)Z^0$, $t \in (0, d)$, with $a(t) = \dot{f}(t) > 0$. We may assume that $f(0) = 0$. We have proved that

$$\omega_{reg} = \dot{f}(t)dt \wedge \theta^0 + \omega_{Z_t} = \dot{f}(t)dt \wedge \theta^0 + \omega_0 + f(t)\omega^0.$$

Now we easily calculate the metric g_{reg} as a composition of ω_{reg} and J. □

Since an interval in $C(J^F)$ is not diffeomorphic to a circle, we get

Corollary 20 *There is no cohomogeneity one Kähler manifold of standard type with the orbit space S^1.*

Corollary 21   *i) $a(t) := \dot{f}(t) = \omega_{reg}(T_t, Z_F^0) = g_{reg}(JT_t, Z_F^0)$, $a(t)^2 = g_{reg}((Z_F^0)_{\gamma(t)}, (Z_F^0)_{\gamma(t)})$.*

ii) For any $X \in \mathfrak{g}$ the square norm of the Killing field $\hat{X}$ along a normal geodesic $\gamma(t)$ is given by

$$b_X(t) = |\hat{X}|_{\gamma(t)}^2 = \omega_0(X, JX) + f(t)\omega^0(X, JX) = g_0(X, X) + f(t)g^0(X, X).$$

iii) If $\omega^0(X, JX) \neq 0$, then $\dot{b}_X(t) = a(t)\omega^0(X, JX) = a(t)g^0(X, X) \neq 0$ and the function $b_X(t)$ has no critical points for $t \in (0, d)$. It is true if $0 \neq X \in \mathfrak{m}$.

4.3   Basic properties of standard Kähler cohomogeneity one manifolds with singular orbits

As an application of previous results, we prove two basic properties of a standard Kähler cohomogeneity one manifold with one or two singular orbits (see also [15]).

Proposition 22 *Let M be a standard cohomogeneity one Kähler manifold with the orbit space $M/G = [0,1)$. Then the singular orbit $S_0 = G\gamma(o) = G/H_0$ is a complex submanifold, hence a Kähler flag manifold and $H \supset K = N_G(L)$.*

Proof: The value $\hat{Z}_p^0$ of the Killing vector field $\hat{Z}^0$ generated by the fundamental vector $Z_F^0 \in \mathfrak{g}$ at the point $p := \gamma(0) \in S_0$ is zero, since in the opposite case we get two $\mathrm{Ad}\,_L$-invariant elements $\hat{Z}_p^0, J\hat{Z}_p^0$ in the B-orthogonal complement $\mathfrak{l}^\perp = \mathbb{R}Z_F^0 + \mathfrak{m} \subset \mathfrak{g}$. This proves that $H \supset K$ since $K = N_G(L)$ is a connected subgroup.

For a subspace $\mathfrak{n} \subset \mathfrak{g}$ we denote by $\hat{\mathfrak{n}}_{\gamma(t)}$ the subspace of $T_{\gamma(t)}M$ spanned by the values of the Killing vectors $\hat{X}_{\gamma(t)}$, $X \in \mathfrak{n}$. Since $\omega(\dot{\gamma}(t), \hat{\mathfrak{m}}_{\gamma(t)}) = 0$ for $t \neq 0$ it is true also for $t = 0$. But

$$T_\gamma S_0 = \hat{\mathfrak{g}}_0 = \hat{\mathfrak{m}}_{\gamma(0)}$$

since $(\hat{\mathfrak{l}} + \mathbb{R}\hat{Z}_0)_0 = 0$. So for any normal geodesic $\gamma(t)$ with $\gamma(0) = p$, we get

$$\omega(\dot{\gamma}(0), T_p S_0) = g(\dot{\gamma}(0), JT_p S_0) = \omega(\dot{\gamma}(0), \hat{\mathfrak{m}}_{\gamma(0)}) = g(\dot{\gamma}(0), J\hat{\mathfrak{m}}_{\gamma(0)}) = 0.$$

Since vectors $\dot{\gamma}(0)$ span $T_p^\perp S_0$, we get

$$g(T_p^\perp S_0, JT_p S_0) = 0$$

which shows that S_0 is a complex submanifold. $\qquad\square$

As a corollary, we get the

Proposition 23 *If (M, ω, J, g) is a standard (compact) cohomogeneity one Kähler manifold with two singular orbits $S_\epsilon = G\gamma(\epsilon) = G/H_\epsilon$, $\epsilon = 0, 1$, then the singular orbits are complex submanifolds and $K := N_G(L) = H_0 \cap H_1$.*

Proof: It remains only to check that $H_0 \cap H_1 = K$. Since the metric g is complete, for any normal geodesic $\gamma(t)$, $t \in \mathbb{R}$ through a point $p \in M$ we have
$$\omega(\dot{\gamma}(t), \hat{\mathfrak{m}}_t) = 0.$$
Also, since $\hat{Z}_p^0 = 0$, we get

$$\omega(\dot{\gamma}(0), \hat{Z}_p^0) = 0.$$

Since $T_p S_0 = \hat{\mathfrak{m}}_0$, this means that $\omega(T_p^\perp S_0, T_p S_0) = g(T_p^\perp S_0, JT_p S_0)) = 0$ that is S_0 is a complex submanifold and H_0 and similarly H_1 are the centralizers of some torus in G. Then $H_0 \cap H_1 \supset K$ is also the centralizer of a torus and hence, a connected subgroup. If $K \neq H_0 \cap H_1$, then there is non-zero vector $X \in \mathfrak{m} \cap \mathfrak{h_o} \cap \mathfrak{h}_1$ and the associated function $b_X(t) := |\hat{X}_{\gamma(t)}|^2$ (the square norm of the Killing field $\hat{X}$ along a normal geodesic) vanishes at the points $t = 0, 1$. Hence it has a critical point in $(0, d)$, which contradicts Corollary 21. $\qquad\square$

5 Einstein equation and Kähler-Einstein structure on M_{reg}

To calculate the Ricci form of an invariant Kähler metric on $M_{reg} = (0,d) \times G/L$ we construct local holomorphic coordinates $z^0, z^1, \ldots, z^n$ in M_{reg} which are extension of local holomorphic coordinates $z^1, \ldots, z^n$ of the associated flag manifold $F = G/K$.

5.1 Holomorphic coordinates and Kähler potential of M_{reg}

Let $z_1, \ldots, z_n, t, \phi$ be local coordinates on M_{reg}, where ϕ is a local coordinate on the torus $T^1 = \{e^{i\phi}\}$ such that $K = L \cdot T^1$. In these coordinates, let

$$\theta^0 = c\, d\phi + \Phi(z, \bar{z}) = c\, d\phi + i \sum_{j=1}^{n} (F_j dz_j - \bar{F}_j d\bar{z}_j) \qquad (6)$$

and let Ψ be a solution to the system of partial differential equations

$$\frac{\partial \Psi}{\partial z_j} = F_j, \quad j = 1, \ldots, n \qquad (7)$$

(it is easily checked that $d\theta^0 = \omega^0$ implies $\frac{\partial F_k}{\partial z_j} = \frac{\partial F_j}{\partial z_k}$).
Then, we claim that

$$z_0 = ic\phi + \int \frac{1}{f'} dt + \Psi \qquad (8)$$

is a new holomorphic coordinate on M_{reg}. Indeed, this is equivalent to say that $\bar{\partial} z_0 = (d - id^c)(z_0) = 0$, where $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial) = J^{-1} dJ$. We have

$$dz_0 = ic\, d\phi + \frac{1}{f'} dt + \sum_{j=1}^{n} \frac{\partial \Psi}{\partial z_j} dz_j + \frac{\partial \Psi}{\partial \bar{z}_j} d\bar{z}_j =$$

$$= ic\, d\phi + \frac{1}{f'} dt + \sum_{j=1}^{n} F_j dz_j + \bar{F}_j d\bar{z}_j$$

and

$$d^c z_0 = J^{-1}(ic\, d\phi + \frac{1}{f'} dt + \sum_{j=1}^{n} F_j dz_j + \bar{F}_j d\bar{z}_j) =$$

$$= -icJd\phi - \frac{1}{f'} Jdt - i(\sum_{j=1}^{n} F_j dz_j - \bar{F}_j d\bar{z}_j) =$$

(by $Jdt = -f'\theta^0$ and $J\theta^0 = \frac{1}{f'} dt$)

$$= -i\frac{1}{f'} dt - i(\sum_{j=1}^{n} F_j dz_j + \bar{F}_j d\bar{z}_j) + \theta^0 - i(\sum_{j=1}^{n} F_j dz_j - \bar{F}_j d\bar{z}_j) =$$

$$= -i\frac{1}{f'}dt + c\ d\phi - i(\sum_{j=1}^{n} F_j dz_j + \bar{F}_j d\bar{z}_j)$$

so $dz_0 = id^c z_0$ as required.

5.2 The Ricci form and the Einstein equation for an invariant Kähler metric on M_{reg}

Now, we calculate the Ricci form of an invariant Kähler structure $(\omega_{reg}, J^{reg}, g^{reg})$ on $M_{reg} = (0,d) \times G/L$ associated with a parametrized interval $Z_t = Z_0 + f(t)Z^0$ in the T-Weyl chamber $C(J^F)$ corresponding to the complex structure J^F of the flag manifold $F = G/K$.

By Proposition 19, the Kähler form is given by

$$\omega_{reg} = \dot{f}(t)dt \wedge \theta^0 + (\omega_0 + f(t)\omega^0) = \dot{f}(t)dt \wedge \theta^0 + \omega_{Z_t} \qquad (9)$$

where $\omega_{Z_t} = d(B \circ Z_t)$, so that

$$\omega_{reg}^{n+1} = (n+1)df \wedge \theta^0 \wedge (\omega_{Z_t})^n,$$

$$(n+1)(\omega_{Z_t})^n = (n+1)(-2)^n(\sum_{\alpha \in R_{\mathfrak{m}}^+} \frac{\alpha(Z_t)}{(\alpha,\alpha)}\omega_\alpha \wedge \omega_{-\alpha})^n = h(f)\text{vol}^F,$$

where $R_{\mathfrak{m}}^+$ is the set of positive black roots,

$$\text{vol}^F := (n+1)!(-2)^n \prod_{\alpha \in R_{\mathfrak{m}}^+} \frac{1}{(\alpha,\alpha)}\omega_\alpha \wedge \omega_{-\alpha}$$

is a volume form on $F = G/K$ and

$$h(f) = \prod_{\alpha \in R_{\mathfrak{m}}^+} \alpha(Z_t) = \prod_{\alpha \in R_{\mathfrak{m}}^+} (\alpha_0 + f\alpha^0), \ \ \alpha_0 := \alpha(Z_0), \ \alpha^0 := \alpha(Z^0).$$

So we can finally write

$$\omega_{reg}^{n+1} = \dot{f}h(f)dt \wedge \theta^0 \wedge \text{vol}^F. \qquad (10)$$

Lemma 24 *Let $z^1, \ldots, z^n$ be holomorphic coordinates in $F = G/K$ and $z^0, z^1, \cdots, z^n$ their extension to holomorphic coordinates in $(0,d) \times G/L$, being z_0 defined by above formula (8). Then the density μ associated with the volume form*

$$\omega_{reg}^{n+1} = \mu dz^0 \wedge \cdots \wedge dz^n \wedge d\bar{z}^0 \wedge \cdots \wedge d\bar{z}^n$$

is given by

$$\mu = \frac{i}{2}\dot{f}^2 h(f)\mu^F$$

where μ^F is the density associated with the volume form vol^F.

Proof: By substituting (6) in (10) we have

$$\omega_{reg}^{n+1} = \dot{f}h(f)\mu^F dt \wedge d\phi \wedge dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n \quad (11)$$

By (8) we have $dz_0 = ic \, d\phi + \frac{1}{f'}dt+$ (terms containing dz_j, $d\bar{z}_j$), $d\bar{z}_0 = -ic \, d\phi + \frac{1}{f'}dt+$ (terms containing dz_j, $d\bar{z}_j$) and $-2ic\frac{1}{f'}dt \wedge d\phi = dz_0 \wedge d\bar{z}_0+$ (terms containing dz_i, $d\bar{z}_i$). By replacing this in (11) we conclude the proof.

$\square$

Now we calculate the Ricci form ρ of the invariant Kähler structure using the very well-known formula

$$\rho = -i\partial\bar{\partial}\log\mu = -\frac{1}{2}dd^c \log\mu = -\frac{1}{2}dd^c \log(\dot{f}^2 h(f)) - \frac{1}{2}dd^c \log\mu^F.$$

The second term is the Ricci form $\rho^F = d\sigma$ of (any) invariant Kähler (with respect to the complex structure J^F) metric. We calculate the first term. By Proposition 19

$$d^c \log(\dot{f}^2 h) = J^{-1}d\log(\dot{f}^2 h) = \frac{d}{dt}\log(\dot{f}^2 h)J^{-1}dt = \dot{f}\frac{d}{dt}\log(\dot{f}^2 h)\theta^0,$$

$$dd^c \log(\dot{f}^2 h) = \frac{d}{dt}(\dot{f}\frac{d}{dt}\log(\dot{f}^2 h))dt \wedge \theta^0 + \dot{f}\frac{d}{dt}\log(\dot{f}^2 h)\omega^0$$

and

$$\rho = -\frac{1}{2}\frac{d}{dt}(\dot{f}\frac{d}{dt}\log(\dot{f}^2 h))dt \wedge \theta^0 - \frac{1}{2}\dot{f}\frac{d}{dt}\log(\dot{f}^2 h)\omega^0 + \rho^F.$$

The Einstein equation $\rho = \lambda\omega_{reg} \equiv \lambda\dot{f}dt \wedge \theta^0 + \lambda\omega_0 + \lambda f\omega^0$ can be rewritten as

$$\lambda f + \frac{1}{2}\dot{f}\frac{d}{dt}\log(\dot{f}^2 h) = c, \ c = const \quad (12)$$

$$\rho^F = \lambda\omega_0 + c\omega^0 \quad (13)$$

Since by Proposition 6 we have $\rho^F = d(B \circ Z^{Kos})$, where Z^{Kos} is the Koszul vector, (13) can be rewritten as

$$Z^{Kos} = \lambda Z_0 + cZ^0. \quad (14)$$

Now we write (12) in more explicit form. Since $\dot{f}\frac{d}{dt}\log(\dot{f}^2) = 2\ddot{f}$ and

$$\frac{d}{dt}\log(h(f)) = \frac{d}{dt}\sum_{\alpha \in R_{\mathfrak{m}}^+}\log(\alpha_0 + f\alpha^0) = \sum_{\alpha \in R_{\mathfrak{m}}^+}\frac{\dot{f}\alpha^0}{\alpha_0 + f\alpha^0} =: \dot{f}A(f)$$

where $A(f) = \sum_{\alpha \in R_{\mathfrak{m}}^+}\frac{\alpha^0}{\alpha_0 + f\alpha^0}$, we get

$$\ddot{f} + \frac{1}{2}A(f)\dot{f}^2 + \lambda f = c \quad (15)$$

We proved the following

Proposition 25 *[15] The Kähler metric on $(0,d) \times G/L$ defined by a parametrized interval $Z_t = Z_0 + f(t)Z^0 \subset C$, $t \in (0,d)$ of a Weyl chamber C is a Kähler-Einstein metric if and only if the function $f(t)$ satisfies the equation (15) and the relation (14) holds.*

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5.3 An example: the n-dimensional projective space

Let M be the projective space $\mathbb{CP}^n = \{[x_0, x_1, \ldots x_n]\}$ with affine coordinates $w_k = \frac{x_k}{x_0}$, $k = 1, \ldots, n$, endowed with the Fubini-Study form $\omega_{\mathbb{CP}^n} = i\partial\bar{\partial}\log(1 + |w_1|^2 + \cdots + |w_n|^2)$. Let us denote $x = (x_1, \ldots, x_n)$ and let us consider the action of $G = SU(n)$ on M given by $A[x_0, x] = [x_0, Ax]$.

It is easily checked that this is a cohomogeneity one action, with singular orbits $S_0 = G([x_0, 0]) = [x_0, 0]$, $S_1 = G([0, x]) = \mathbb{CP}^{n-1}$, and regular orbits diffeomorphic to S^{2n-1} (obtained when $x_0 \neq 0$ and $x \neq 0$).

For every regular point $[1, w_1, \ldots, w_n]$, its orbit can be identified with the sphere of radius $r = \sqrt{|w_1|^2 + \cdots + |w_n|^2}$ in $\mathbb{C}^n$, so that the singular orbit S_0 (resp. S_1) is obtained when the radius tends to 0 (resp. to infinity).

The flag $F = G/K$ associated with the regular orbits is $SU(n)/U(n-1) = S^{2n-1}/\{e^{i\phi}\} = \mathbb{CP}^{n-1}$. Let $z_k = \frac{w_k}{w_1}$, $k = 2, \ldots, n$, be the affine coordinate on F. Then we can take $r, \phi, z_2, \ldots, z_n$ as local coordinates on a dense open subset of the union of regular orbits M_{reg}. More precisely, set

$$(w_1, \ldots, w_n) = \left(r\frac{w_1}{\sqrt{|w_1|^2 + \cdots + |w_n|^2}}, \ldots, r\frac{w_n}{\sqrt{|w_1|^2 + \cdots + |w_n|^2}} \right) =$$

$$= \left(\frac{re^{i\phi}}{\sqrt{1 + |z|^2}}, \frac{re^{i\phi}z_2}{\sqrt{1 + |z|^2}}, \ldots, \frac{re^{i\phi}z_n}{\sqrt{1 + |z|^2}} \right). \tag{16}$$

where we are setting $|z|^2 := |z_2|^2 + \cdots + |z_n|^2$.

Using the change of coordinates (16), after a long but straight computation we can see that the restriction ω_{reg} of the Fubini-Study form $\omega_{\mathbb{CP}^n}$ to M_{reg} is

$$\frac{2r}{(1 + r^2)^2}dr \wedge \left(d\phi + \frac{i}{2(1 + |z|^2)}\sum_{j=2}^{n}(z_j d\bar{z}_j - \bar{z}_j dz_j) \right) + \frac{r^2}{1 + r^2}\tilde{\omega} \tag{17}$$

where $\tilde{\omega} = i\partial\bar{\partial}\log(1 + |z|^2)$ is the Fubini-Study form on $\mathbb{CP}^{n-1}$.

In order to find the relation between the parameter r and the parameter t used in the above sections, let us recall that $\frac{\partial}{\partial t}$ is a unit field on M_{reg} normal to each regular orbit. On the one hand, $\frac{\partial}{\partial r}$ is normal to the regular orbits. On the other hand, a straight calculation shows that the coefficient of dr^2 in the espression of the metric g_{reg} is $\frac{2}{(1+r^2)^2}$, so we must have $dt = \frac{\sqrt{2}}{1+r^2}dr$ and, integrating, $r = \tan\left(\frac{t}{\sqrt{2}}\right)$. By replacing this into (17) we find

$$\frac{\sqrt{2}}{2}\sin\left(\frac{t}{\sqrt{2}}\right)dt \wedge \left(d\phi + \frac{i}{2(1 + |z|^2)}\sum_{j=2}^{n}(z_j d\bar{z}_j - \bar{z}_j dz_j) \right) + \sin^2\left(\frac{t}{\sqrt{2}}\right)\tilde{\omega}$$

$$\tag{18}$$

This is exactly formula (9) with $f(t) = \frac{\sqrt{n-1}}{n\sqrt{2}} \sin^2\left(\frac{t}{\sqrt{2}}\right)$, $\omega^0 = n\sqrt{\frac{2}{n-1}}\tilde{\omega}$ and $\theta^0 = n\sqrt{\frac{2}{(n-1)}}\left(d\phi + \frac{i}{2(1+|z|^2)}\sum_{j=2}^{n}(z_j d\bar{z}_j - \bar{z}_j dz_j)\right)$. The correct normalization for θ^0 is obtained as follows. On the Lie algebra $su(n)$ of $G = SU(n)$, θ^0 is defined as $B(Z_F^0, \cdot)$, where $B(X,Y) = 2n\,\mathrm{tr}\,(XY)$ is the Killing-Cartan form and $Z_F^0 \in su(n)$ is such that $B(Z_F^0, Z_F^0) = -1$. This last condition easily implies that $Z_F^0 = \frac{i}{n\sqrt{2(n-1)}}\mathrm{diag}\,((n-1), -1, \ldots, -1)$. Then, for every $X_I \in T_I(SU(n))$ having $i\alpha_1, \ldots, i\alpha_n$ ($\alpha_k \in \mathbb{R}$) on the diagonal, we have

$$\theta^0(X_I) = B(Z_F^0, X_I) = -\frac{2n}{n\sqrt{2(n-1)}}((n-1)\alpha_1 - \alpha_2 - \cdots - \alpha_n) =$$

$$= -\sqrt{\frac{2}{(n-1)}}n\alpha_1$$

and for general $X_g \in T_g(SU(n))$, $g = (a_{ij})$,

$$\theta^0(X_g) = \theta^0(dg^{-1}X_g) = -n\sqrt{\frac{2}{(n-1)}}(\bar{a}_{11}a'_{11} + \cdots + \bar{a}_{n1}a'_{n1}).$$

Hence $\theta^0 = -\sqrt{\frac{2}{(n-1)}}(\bar{a}_{11}da_{11} + \cdots + \bar{a}_{n1}da_{n1})$. Now, if we take into account that $\mathbb{CP}^{n-1} = SU(n)/U(n-1)$ via the action $\mathbb{CP}^{n-1} = \{A[1,0,\ldots,0]\} = \{[a_{11},\ldots,a_{n1}]\}$ and accordingly to (16) we replace $a_{11} = \frac{e^{i\phi}}{\sqrt{1+|z|^2}}, a_{21} = \frac{e^{i\phi}z_2}{\sqrt{1+|z|^2}}, \ldots, a_{n1} = \frac{e^{i\phi}z_n}{\sqrt{1+|z|^2}}$, a straight calculation yields exactly the above expression for θ^0.

Finally, let us verify that $f(t) = \frac{\sqrt{n-1}}{n\sqrt{2}} \sin^2\left(\frac{t}{\sqrt{2}}\right)$ satisfies the Einstein equation (15). In this case, since $F = \mathbb{CP}^{n-1}$, the set $R_{\mathfrak{m}}^+$ contains the roots $\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_1 - \varepsilon_n$. Moreover, since the singular orbit S^0 is a fixed point, we must have $Z_0 = 0$ and then $\alpha_0 = \alpha(Z_0) = 0$. Hence $A(f) = \sum_{\alpha \in R_{\mathfrak{m}}^+} \frac{\alpha^0}{\alpha_0 + f\alpha^0} = \frac{n-1}{f}$ and (15) reduces to

$$\ddot{f} + \frac{n-1}{2}\frac{\dot{f}^2}{f} + \lambda f = c.$$

One immediately sees that $f(t) = \frac{\sqrt{n-1}}{n\sqrt{2}} \sin^2\left(\frac{t}{\sqrt{2}}\right)$ is a solution to this equation for $c = \frac{\sqrt{n-1}}{\sqrt{2}}$ and $\lambda = n+1$, which is exactly the value of the Einstein constant for $\omega_{\mathbb{CP}^n} = i\log(1 + |w_1|^2 + \cdots + |w_n|^2)$.

6 Standard cohomogeneity one Kähler manifolds with one singular orbit

In the sequel we will consider only standard cohomogeneity one G-manifolds M (Definition 13) with one (complex) singular orbit ($S_0 =$

$G/H, J^S$). For brevity, we will call these manifolds just *standard C1 manifolds*.

First of all, we prove that any such manifold is the total space of the homogeneous vector bundle $M_\varphi = G \times_H V_\varphi \to S_0 = G/H$ over a flag manifold $S_0 = G/H$ defined by a representation $\varphi : H \to GL(V_\varphi)$ with $\varphi(H) \simeq U_m$. Then we give a description of these manifolds in terms of painted Dynkin diagrams and determine the invariant Kähler metrics on them.

6.1 Reduction to admissible vector bundles

Let $(S_0 = G/H, J^S)$ be a flag manifold with invariant complex structure.

Definition 26 *A complex linear representation $\varphi : H \to GL(V_\varphi) = GL_m(\mathbb{C})$ and the associated homogeneous vector bundle $M_\varphi = G \times_H V_\varphi$ are called admissible if $\varphi(H) \simeq U_m$.*

Note that an admissible representation is defined by a normal subgroup $N = \ker(\varphi)$ such that $H/N \simeq U_m$. Such a subgroup N is also called an *admissible subgroup*.

The main result of this section is the following

Theorem 27 *Let M be a standard complex C1 manifold. Then M is the total space M_φ of an admissible vector bundle $M_\varphi = G \times_H V_\varphi \to S_0$ over the singular orbit $(S_0 = G/H, J^S)$.*

Proof: Any cohomogeneity one manifold M with one singular orbit can be identified with the homogeneous vector bundle $M = G \times_H V$ over the singular orbit $S_0 = G/H$ associated to some sphere transitive representation $\varphi : H \to GL(V_\varphi)$, where V_φ is the normal space of the singular orbit S (Proposition 1). Since, by assumption, the orbit S_0 is a complex submanifold, the linear group $\varphi(H)$ preserves a complex structure in V. Checking the Borel list of sphere transitive linear groups, we conclude that $\varphi(H) \simeq H/\ker(\varphi) \simeq SU_m, U_m, Sp_{m/2}$ or $T^1 \cdot Sp_{m/2}$.
It remains to check that the only possible case is $\varphi(H) = U_m$ which is clear for $m = 1$. For $m > 1$, the result follows from the following two lemmas.

Lemma 28 *Let H be the stability subgroup of a flag manifold $F = G/H$. Then there is no normal subgroup $N \subset H$ with $H/N \simeq SU_m$, $m > 1$.*

Proof: If, by contradiction, a normal subgroup $N \subset H$ such that $H/N \simeq SU_m$ exists, then there is an ideal $\mathfrak{su}_m$ of $\mathfrak{h}$ corresponding to a connected component of type A_{m-1} in the white subdiagram Π_W of the painted Dynkin diagram $\Pi = \Pi_W \cup \Pi_B$ of the flag manifold G/H.

For the classical groups, we can always assume that the inclusion of A_{m-1} into Π has the form

$$\underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_{m-1}}{\circ} - \bullet - \cdots$$

Then the ideal $A_{m-1} = \mathfrak{su}_m$ is embedded into a subalgebra $A_m = \mathfrak{su}_{m+1}$ and the subgroup $H = N_G(\mathfrak{h}) \supset N_{SU_{m+1}}(\mathfrak{su}_m) = U_m$ contains U_m as a normal subgroup. Since $U_m = SU_m \times T^1/\mathbb{Z}_m$, we have $H/N \simeq SU_m/\mathbb{Z}_m$. This implies that there is no normal subgroup N with $H/N \simeq SU_m$, for $m > 1$, as required.

Consider now exceptional groups. There are only two flag manifolds of type F_4, where the inclusion of the subalgebra $A_{m-1} = \mathfrak{su}_m$ is not as above:

$$\bullet - \bullet \Rightarrow \circ - \circ$$

$$\circ - \circ \Rightarrow \bullet - \bullet$$

Indeed, in these cases the (white) subalgebra $\mathfrak{h}' = A_2 = \mathfrak{su}_3$ is embedded into $\tilde{\mathfrak{h}} = C_3 = \mathfrak{sp}_6$ for the first diagram and into $\tilde{\mathfrak{h}} = B_3 = \mathfrak{so}_7$ for the second diagram. In both cases we have $N_{\tilde{H}}(SU_3) = U_3 = SU_3 \times T^1/\mathbb{Z}_3$. Repeating the above argument, we conclude that $H/N = SU_3/\mathbb{Z}_3$.

In the case when G has type E_ℓ, $\ell = 6, 7, 8$, we can always embed A_{m-1} into A_m or D_m and use similar arguments with the exception of the cases

$$
\begin{array}{c}
\bullet \\
| \\
\circ - \circ - \cdots - \circ - \circ - \circ
\end{array}
$$

which correspond to the flag manifold $G/H = E_\ell/U_\ell$, $\ell = m = 6, 7, 8$. In this case we have $H = SU_\ell \cdot T^1$, that is, at the Lie algebra level, $\mathfrak{h} = \mathfrak{su}_l + i\mathbb{R}d$ where ad_d defines a gradation of depth 2 or 3 of the complex Lie algebra $\mathfrak{e}_\ell^{\mathbb{C}}$. If we denote by $V = \mathbb{C}^\ell$ the standard U_ℓ-module, then the gradation is given by (see [10], section 3.5)

$$\mathfrak{e}_\ell^{\mathbb{C}} = \mathfrak{u}_\ell + \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_1 + \mathfrak{g}_2$$

for $\ell = 6, 7$, where $\mathfrak{g}_{-1} = \Lambda^3 V$, $\mathfrak{g}_{-2} = \Lambda^6 V$, $\mathfrak{g}_1 = \Lambda^3 V^*$, $\mathfrak{g}_{-2} = \Lambda^4 V^*$, and by

$$\mathfrak{e}_8^{\mathbb{C}} = \mathfrak{u}_8 + \mathfrak{g}_{-3} + \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3$$

where $\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 2}$ are defined as above and $\mathfrak{g}_{-3} = V \otimes \Lambda^8 V$, $\mathfrak{g}_3 = V^* \otimes \Lambda^8 V^*$. The isotropy action of the semisimple part SU_ℓ of the stability subgroup $H = SU_\ell \cdot T^1$ is the standard action on forms.

Since the isotropy action of $i \cdot d$ on the space $\mathfrak{g}_k$ is $ik \cdot \mathrm{id}$ and $\mathfrak{h} = \mathfrak{u}_\ell = \mathfrak{g}_0$, we have $[d, \mathfrak{h}] = 0$, that is $i \cdot d \in \mathfrak{z}(\mathfrak{h})$, so that $N = \exp(\mathbb{R} i \cdot d) = Z(H)$. By $Z(H) \cap SU_m = Z(U_m) \cap SU_m = \mathbb{Z}_m$ we have then

$$H/N = H/\exp(\mathbb{R} i \cdot d) = H/Z(H) = (H \cap SU_m)/(Z(H) \cap SU_m) = SU_m/\mathbb{Z}_m.$$

The case of the group G_2 is similar. $\qquad\square$

Lemma 29 *There is no normal subgroup $N \subset H$ with the quotient $H/N \simeq Sp_m$ or $T^1 \cdot Sp_m$ such that the associated cohomogeneity one manifold $M = G \times_H V_\varphi$ has ordinary regular orbits.*

Proof: The only flag manifold G/H of a classical Lie group which admits a normal subgroup $N \subset H$ with indicated quotient has the form $G/H = Sp_n/(U_{n_1} \times \cdots \times U_{n_s} \times T^k \times Sp_m) = Sp_n/(N \times Sp_m)$. The associated cohomogeneity one manifold $Sp_n \times_H V_\varphi$ has regular orbits given, for non-zero $v \in V_\varphi$, by

$$Sp_n/L = Sp_n/N \times (Sp_m)_v = Sp_n/N \times Sp_{m-1}$$

which are not ordinary since $N_G(L) \setminus L \supset N_{Sp_m}(Sp_{m-1}) \supset Sp_1$ and Sp_1 is not one-dimensional.

For the exceptional case, we have to consider the cases of $F_4/Sp_3 \cdot T^1$ with Dynkin diagram

$$\bullet - \circ \Rightarrow \circ - \circ$$

and $F_4/Sp_2 \cdot T^2$ with Dynkin diagram

$$\bullet - \circ \Rightarrow \circ - \bullet$$

In both cases, the associated cohomogeneity one manifold has non ordinary regular orbits. For example, in the first case the corresponding regular orbit is $F_4/(T^1 \cdot Sp_3)_v = F_4/Sp_2$, so that $N_G(L) \setminus L \supset N_{F_4}(Sp_2) \supset Sp_1$ and we conclude as in the classical case above. The case of $F_4/Sp_2 \cdot T^2$ is similar. This finishes the proof of the lemma. $\square$

6.1.1 Description of admissible homogeneous vector bundles in terms of Dynkin diagrams

The following proposition describes the admissible homogeneous vector bundles $\pi : M_\varphi = G \times_H V_\varphi \to S_0 = G/H$, over a flag manifold $(S_0 = G/H, J^S)$ in terms of painted Dynkin diagrams and characters $\chi : T^k \to T^1$ of the connected center T^k of the stabilizer $H = H' \cdot T^k$.

Proposition 30 *Let $(S_0 = G/H = G/H' \cdot T^k, J^S)$ be a flag manifold associated with painted Dynkin diagram $\Pi = \Pi_B \cup \Pi_W$. Then an admissible homogeneous vector bundle $M_\varphi = G \times_H V_\varphi$ is defined by a pair (A_{m-1}, χ) where A_{m-1} is a connected component of Π_W of type A_{m-1} (i.e. a string of length $m - 1$), and $\chi : T^k \to T^1$ a character.*

Proof: The stability subalgebra of S_0 admits a direct sum decomposition

$$\mathfrak{h} = \mathfrak{su}_m \oplus \mathfrak{n}' \oplus Z(\mathfrak{h})$$

where $\mathfrak{su}_m$ is the ideal associated with the string A_{m-1} and the corresponding decomposition of the stability subgroup is $H = SU_m \cdot N' \cdot T^k$. Then the extension of the tautological representation of SU_m in a vector space $V_\varphi = \mathbb{C}^m$ by a character $\chi : T^k \to T^1 \cdot \mathrm{id}_{V_\varphi}$ is an admissible representation $\varphi : H \to \varphi(H) = U(V_\varphi) \simeq U_m$. The converse statement is also clear. $\qquad\qquad\square$

6.2 Regular submanifold M_{reg} of M_φ

6.2.1 Invariant complex structures in the projective space $PV_\varphi = H[e_0] = U_m/U_{m-1} \times U(1)_0$

Let $\pi : M_\varphi = G \times_H V_\varphi \to S_0 = G/H$ be an admissible vector bundle over a flag manifold $(S_0 = G/H, J^S)$ with reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$.

We identify V_φ with the arithmetic complex vector space $\mathbb{C}^m$ with the standard Hermitian form $< .,. >$ and the standard basis $e_0 = (1, \cdots, 0)^T, \cdots, e_{m-1} = (0, \cdots, 0, 1)^T$. Then the stability subalgebra may be written as
$\mathfrak{h} = \mathfrak{n} \oplus \mathfrak{u}_m$ where $\mathfrak{u}_m$ is the Lie algebra of skew-Hermitian matrices. The orbit $\varphi(H)e_0 = H/H_{e_0}$ is the sphere with the reductive decomposition of $\mathfrak{h} = \mathfrak{n} \oplus \mathfrak{u}_m$ given by

$$\mathfrak{h} = (\mathfrak{n} \oplus \mathfrak{u}_{m-1}) + (\mathbb{R}I_0 + \mathfrak{q})$$

where $I_0 = \mathrm{diag}(i, 0_{m-1})$ and

$$\mathfrak{q} = \{C_X := \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}, X \in \mathbb{C}^{m-1}\} \simeq \mathbb{C}^{m-1}.$$

The diagonal Lie algebra $\mathfrak{c}_{\mathfrak{u}_m}$ is a Cartan subalgebra of $\mathfrak{u}_\mathfrak{m}$ and a basic vector e_j is a weight vector for $\mathfrak{c}_{\mathfrak{u}_m}$ with weight ϵ_j where

$$\epsilon_j(\mathrm{diag}(x_0, x_1, \cdots, x_{m-1})) = x_j.$$

The elementary matrices $E_{ij} \in \mathfrak{u}_m^\mathbb{C} = \mathfrak{gl}_m(\mathbb{C})$ are root vectors with roots $\alpha_{ij} = \epsilon_i - \epsilon_j$. The reductive decomposition $\mathfrak{u}_m = (\mathfrak{u}_{m-1} + \mathbb{R}I_0) + \mathfrak{q}$ of the projective space $\mathbb{C}P^{m-1} = U_m/U_{m-1} \times U(1)_0$ defines a decomposition of the root system $R_{\mathfrak{u}_m}$ of $\mathfrak{u}_m^\mathbb{C}$ into the union of white roots $R_{\mathfrak{u}_m}^0 = \{\epsilon_i - \epsilon_j, i, j > 0\}$ (which are roots of the stability subalgebra) and the complementary black roots $R_{\mathfrak{u}_m}' = \{\pm\alpha_{0j} = \pm(\epsilon_0 - \epsilon_j), j > 0\}$. The multiplication by $\pm i$ in $T_o\mathbb{C}P^{m-1} = \mathfrak{q} = \mathbb{C}^{m-1}$ defines two (opposite) invariant complex structures $\pm J_\mathfrak{q}$ which define two invariant complex structures $\pm J^{\mathbb{C}P^{m-1}}$ on $\mathbb{C}P^{m-1}$. They correspond to the following painted Dynkin diagrams:

$$\overset{\alpha_{01}}{\bullet} - \overset{\alpha_{12}}{\circ} - \cdots - \overset{\alpha_{m-1m}}{\circ} \qquad \overset{\alpha_{10}}{\bullet} - \overset{\alpha_{21}}{\circ} - \cdots - \overset{\alpha_{mm-1}}{\circ} \ .$$

Note that $\alpha_{01}(\mathrm{id}_0) = 1$ where $\mathrm{id}_0 = -iI_0 = \mathrm{diag}(1, 0, \cdots, 0)$. So the T-Weyl chamber associated with the complex structure $\pm J^{\mathbb{C}P^{m-1}}$ is $\pm \mathbb{R}^+ \mathrm{id}_0$.

Deleting the singular orbit $S_0 = G/H$ which is the zero section of π, we get a regular open submanifold which is the union of the codimension one regular orbits parametrized by $t > 0$
$S_t = G(te_0) = G \times_H (te_0) = G/L$ where $L = H_{e_0} = \ker(\varphi) \cdot U_{m-1}$ is the stabilizer of the point e_0. So we identify M_{reg} with

$$M_{reg} = \mathbb{R}^+ \times G/L.$$

The projectivization $G \times_H PV_\varphi$ of the vector bundle π is a homogeneous manifold
$F = G\{te_0\} = G \times_H [e_0] = G/K$ where the stabilizer $K = H_{[e_0]} = L \times U(1)_0$ with $U(1)_0 = \mathrm{diag}(U(1), \mathrm{id}_{m-1})$.
We will assume that the regular orbit $S = G/L$ is an ordinary manifold. Then $N_G(L) = K = L \times U(1)_0$ and the natural projection $G/L \to F = G/K$ is a principal $U(1)_0$-bundle over the flag manifold F. The restriction of this projection to a fibre $V(x) = \pi^{-1}(x)$ is the standard projection $S^{2m-1} = S_t \cap V(x) \to PV(x)$ of the sphere onto the projective space. To be specific, we assume that $J_F = J_F^+$ corresponds to $\beta = \alpha_{12}$ such that the system of black roots

$$\Pi_B^F = \{\beta = \alpha_1, \beta_1, \cdots, \beta_k\}$$

and the T-Weyl chamber is defined by conditions

$$C(J_F) = \{\beta > 0, \beta_1 > 0, \cdots, \beta_k > 0\} \subset \mathfrak{t} = iZ(\mathfrak{k})$$

We have the following standard (B-orthogonal) decomposition

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_F^0 + \mathfrak{m} = \mathfrak{l} + \mathbb{R}Z_F^0 + \mathfrak{q} + \mathfrak{p}$$

where $Z_F^0 \in \mathbb{R}I_0$ is the fundamental vector, i.e. the vector of $Z(\mathfrak{k}) = Z(\mathfrak{l}) \oplus \mathbb{R}I_0$ orthogonal to $\mathfrak{l}$ and normalized by the condition $B(Z_F^0, Z_F^0) = -1$. Note that by assumption, its centralizer is $\mathfrak{k}$.

6.2.2 Extension of the complex structure J^S to an invariant complex structure in F

Let $\Pi = \Pi_W \cup \Pi_B$ be the PDD associated with (S_0, J^S) and $\Pi_B = \{\beta_1, \cdots, \beta_k\}$. We may assume that the T-Weyl chamber $C(J^S)$ associated with the complex structure J^S is defined by

$$C(J^S) = \{\beta_1 > 0, \cdots, \beta_k > 0\} \subset \mathfrak{t}_S = iZ(\mathfrak{h}) = iZ(\mathfrak{n}) + \mathbb{R}\mathrm{id}_m.$$

We extend the complex structure $J_\mathfrak{p} = J^S|\mathfrak{p}$ to a Ad_K-invariant complex structure $J_\mathfrak{m}^\pm$ on $\mathfrak{m} = \mathfrak{q} + \mathfrak{p}$ choosing one of the complex structures $\pm J_\mathfrak{q}$, described above. This defines two invariant complex structures

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$J_F^\pm$ in the flag manifold $F = G/K$ which are consistent with the complex structure J^S in S_0 such that the natural projection $F \to S_0$ is holomorphic. The PDD of the flag manifold $(F, J_F^\pm)$ is obtained from the PDD of (S_0, J^S) by painting in black one of the end roots $\beta = \alpha_1$ or α_{m-1} of the white string $A_{m-1} = \{\alpha_1, \cdots, \alpha_{m-1}\} \subset \mathfrak{t}_F = iZ(\mathfrak{k}) = \mathfrak{t}_S + \mathbb{R}Z^0$, $Z_0 = -iZ_F^0$.

Since $\beta(Z^0) \neq 0$, changing Z^0 to $-Z^0$ if necessary, we may assume that $\beta(Z^0) > 0$.

6.2.3 Decomposition $TM_\varphi = T^h M_\varphi + T^v M_\varphi$ of the tangent bundle

The decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ defines a G-invariant principal connection in the principal bundle $G \to S_0 = G/H$ such that the decomposition of the tangent bundle TG into horizontal subbundle $T^h G$ and vertical subbundle is given by

$$T_a G = T_a^h G + T_a^v G = a\mathfrak{p} + a\mathfrak{h} = (L_a)_* \mathfrak{p} + (L_a)_* \mathfrak{h}.$$

It defines a similar decomposition of the tangent bundle TM_φ given by

$$T_{[a,v]} M_\varphi = T_{[a,v]}^h M_\varphi + T_{[a,v]}^v M_\varphi = T_a^h G + T_v V_\varphi = a\mathfrak{p} + V_\varphi.$$

In particular, along the radial line $\mathbb{R}e_0 \in \pi^{-1}(eH) \subset M_\varphi$ the tangent space $T_{te_0} M_{reg}$ can be written as

$$T_{te_0} M_\varphi = T_{te_0}^h M_{reg} + T_{te_0}^v M_\varphi = \mathfrak{p} + V_\varphi.$$

Proposition 31 *Let g_V be a $\varphi(H) = U_m$-invariant metric in V_φ and $g_\mathfrak{p}^t$, $t \in \mathbb{R}$ a 1-parameter family of Ad_K-invariant metrics in $\mathfrak{p}$.*
Then the metric $g_{te_0} = g_\mathfrak{p}^t \oplus g_V$ in $\mathfrak{p} + V_\varphi = T_{te_0} M_{reg}$ is extended by left translations from G to a smooth invariant metric in M_φ.

Proof: Indeed, the metric $g_\mathfrak{p}$ defines an invariant metric in $S_0 = G/H$ which induces the G-invariant metric g^h in the horizontal distribution $T^h M_\varphi$ via the isomorphism $\pi_* : T_x^h M_\varphi \to T_{\pi(x)} S_0$. The metric g_V in the fibre $V_\varphi = \pi^{-1}(eH)$ induces an invariant metric g^v in the vertical distribution $T^v M_\varphi$. Then $g = g^h \oplus g^v$ is an invariant Riemannian metric in M_φ. $\qquad\square$

6.2.4 Description of the character

For simplicity, we will assume that G acts effectively on S_0. Then G has no center and we may identify G with the adjoint group Ad_G and H with the adjoint subgroup $\mathrm{Ad}_H \subset \mathrm{Ad}_G$. Then the group of characters $\mathcal{X}(T^k)$ is identified with the lattice $Q_T = \mathrm{span}_{\mathbb{Z}} R_T \subset \mathfrak{t}_H^*$ of T-roots as follows.

We denote by

$$Q_T^* := \{h \in \mathfrak{t}, \ \bar\beta(h) \in \mathbb{Z}, \bar\beta \in R_T\} = \mathrm{span}(h_1, \cdots, h_k) \subset \mathfrak{t}_H := iZ(\mathfrak{h})$$

the lattice in $\mathfrak{t}_H = iZ(\mathfrak{h})$ dual to Q_T with basis $(h_i, i = 1, \cdots, k)$ dual to the basis $\{\bar{\beta}_j = \kappa(\beta_j), \beta_j \in \Pi_B\}$ of Q_T, $\bar{\beta}_j(h_k) = \delta_{jk}$.
Then $2\pi Q_T^*$ is the kernel of the exponential map

$$\exp \circ i : \mathfrak{t} \to T^k,\ h \mapsto \exp ih$$

and we may identify T^k with $\mathfrak{t}_H/2\pi(Q_T)^*$.
A weight

$$\Lambda = \sum_{j=1}^{k} p_j \bar{\beta}_j \in Q_T = \mathcal{X}(T^k) : h = \sum_{j=1}^{k} x_j h_j \mapsto \Lambda(h) = \sum_{j=1}^{k} p_j x_j$$

defines a character

$$\chi = \chi_\Lambda : T^k \to \mathbb{C}^*,\ \exp ih \mapsto e^{2\pi i \Lambda(h)} = e^{2\pi i p_j x_j}.$$

(The reader is referred for example to [17] for the correspondence between homogeneous line bundles and characters).

6.3   Invariant Kähler structures on M_φ

Let $(S_0 = G/H, J^S)$ be a flag manifold with reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ associated with painted Dynkin diagram $\Pi = \Pi_W \cup \Pi_B$, $\Pi_B = \{\beta_1, \cdots, \beta_k\}$ and the complex structure J^S associated with the T-Weyl chamber $C(J^S) = \{\beta_1 > 0, \cdots, \beta_k > 0\} \subset \mathfrak{t}_S = iZ(\mathfrak{h})$. Let $M_\varphi = G \times_H V_\varphi$, $V_\varphi = \mathbb{C}^{m-1}$ be the admissible homogeneous vector bundle associated with a pair (A_{m-1}, χ) where A_{m-1} is a connected component of Π_W of type A_{m-1} (i.e. a white string of length $m-1$ and $\chi : T^\ell = Z^{con}(H) \to T^1$ a character.
We assume that the regular orbit $S_t = G(te_0) = G/L$ is ordinary. Then the projectivization $F = G \times_H PV_\varphi = G/K = G/L \times U(1)$ is a flag manifold. Let J^F be an extension of the complex structure J^S to an invariant complex structure of F defined by extension of the complex structure $J_\mathfrak{p} = J^S|_\mathfrak{p}$ to a complex structure $J_\mathfrak{m} = J_\mathfrak{q} \oplus J_\mathfrak{p}$ on $\mathfrak{m} = \mathfrak{q} + \mathfrak{p} = T_{eK}F$ such that the PDD of $(F = G/K, J^F)$ is obtained by painting in black the end root β of the string A_{m-1}.
The standard decomposition associated with the C1 manifold M_{reg} may be written as

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_F^0 + \mathfrak{m} = (\mathfrak{n} + \mathfrak{su}_{m-1} + \mathbb{R}I_{m-1}) + (\mathfrak{q} + \mathfrak{p})$$

where $\mathfrak{n} = \ker(\varphi)$, $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z_F^0$ and Z_F^0 is the fundamental vector which is identified with the fundamental vector field on M_{reg} whose restriction to a regular orbit S_t is the fundamental vector field of the principal $U(1)$ -bundle $S_t = G/L \to F = G/K = G/L \times U(1)$.

The main result of this section is the following theorem.

Theorem 32 *Let $M_\varphi = G \times_H V_\varphi$ be as above a standard C1 manifold having as singular orbit the flag manifold $(S_0 = G/H, J^S)$, where the*

complex structure J^S is associated with the T-Weyl chamber $C(J^S) = \{\beta_1 > 0, \cdots, \beta_k > 0\} \subset iZ(\mathfrak{h})$ and let J^F be an extension of the complex structure J^S to an invariant complex structure of $F = G \times_H PV_\varphi = G/K$ associated with the T-Weyl chamber $C(J^F) = \{\beta > 0, \beta_1 > 0, \cdots, \beta_k > 0\}$.

Let $Z_F^0 = \kappa I_0$ be the fundamental vector, where $\kappa^{-1} = (-B(I_0, I_0))^{\frac{1}{2}}$ and I_0 is defined in Section 6.2.1, and let θ^0 be the dual to Z_F^0 invariant 1-form on the regular part M_{reg} of M_φ. We may assume that $\beta(Z^0) > 0$ where $Z^0 = -iZ_F^0 \in iZ(\mathfrak{k})$.

Let $Z_0 \in C(J^S)$ be a vector from the T-Weyl chamber $C(J^S)$ which is the face of the Weyl chamber $C(J^F) = \{\beta = 0\}$. Then a segment in $C(J^F)$ with a parametrization $Z_0 + f(t)Z^0$, $\dot{f}(t) > 0$ defines a Kähler metric in M_{reg} given by (see Proposition 19)

$$g_{reg} = dt^2 + (\dot{f}\theta^0)^2 + \pi_F^* g_0 + f(t)\pi_F^* g^0,$$

where $\pi_F : M_{reg} \to F$ is the natural projection, $g_0 = \omega_{Z_0} \circ J^F$ is a symmetric bilinear form on F (which is the pull back of an the invariant Kähler metric on S_0) and $g^0 = -\omega_{Z^0} \circ J^F$ is a symmetric bilinear form on F.

The Kähler structure smoothly extends to a geodesically complete invariant Kähler structure on M_φ if and only if the function $f(t)$ is extended to a smooth even function on $\mathbb{R}$ such that $Z_0 + f(t)Z^0 \in C(J^F)$ and satisfies the following Verdiani conditions :

$$f(0) = \dot{f}(0) = 0, \ \ddot{f}(0) = \kappa$$

Proof: In the notations used before the statement, let $X \in \mathfrak{q}$, $Y \in \mathfrak{m}$, then

$$g_o(X, Y) = d(B \circ Z_0)(X, J_\mathfrak{m}Y) = -B(Z_0, [X, J_\mathfrak{m}Y]) =$$

$$= -B([Z_0, X], J_\mathfrak{m}Y]) = 0.$$

This shows that $\pi_S^* g_0$ is a metric in the horizontal subbundle $T^h M_\varphi$. Similarly, for $X \in \mathfrak{q}$, $Y \in \mathfrak{p}$, we get

$$g^0(X, Y) = -B(Z_F^0, [X, J_\mathfrak{p}Y]) = -B([Z_F^0, X], J_\mathfrak{p}Y) = 0$$

since $[Z_F^0, X] \in \mathfrak{q}$ and $J_\mathfrak{p}Y \in \mathfrak{p}$. This shows that $g_{reg}(T^h M_{reg}, T^v M_{reg}) = 0$. The horizontal part g_{reg}^h of the metric g_{reg} at a point te_0 can be written as

$$g_{reg}^h = g_{reg}|_\mathfrak{p} = \pi_F^* g_0 + f\pi_F^* g^0.$$

Under the assumptions of the theorem, it is extended to a smooth metric in $T^h M_{reg}$. The vertical part is $g_{reg}^v = g_{reg}|_V = dt^2 + (\dot{f}\theta^0)^2 + f\pi_F^* g^0|_\mathfrak{q}$. Since Z_F^0 and $\mathfrak{q}$ belong to $\mathfrak{u}_m \subset \mathfrak{h}$, using calculation in $\mathfrak{u}_m$ we get for $X, Y \in \mathfrak{q}$ $g_{reg}^v(X, Y) = -d(B \circ Z_F^0)(X, iY) = B(Z_F^0, [X, iY]) = B(Z_F^0, X^*iY + iY^*X) = -2 < X, Y > B(Z_F^0, I_0) = \frac{2}{\kappa} < X, Y >$ where $< X, Y >= Im(X^*Y) = \frac{1}{2}(X^*Y + Y^*X)$ is the standard metric in $\mathfrak{q} = \mathbb{C}^{m-1}$.

Now, let $g_{eucl} = dt^2 + t^2\eta^2 + g_{eucl}|_\mathfrak{q}$ be the flat euclidean metric on V, where η denotes the 1-form dual to the vector field generated by I_0.

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Since $\theta^0 = \frac{1}{\kappa}\eta$ we can rewrite $g_{reg}|_V = dt^2 + (\frac{\dot{f}}{\kappa t})^2 t^2 \eta^2 + f\pi_F^* g^0|_{\mathfrak{q}}$ and apply Verdiani criterion (Theorem 1 in [18]) which says that $g_{reg}^v|_{V_\varphi \setminus \{0\}}$ is extended to V_φ if and only if $f(t)$ satisfies the stated conditions, i.e. is a smooth even function on $\mathbb{R}$ with $\dot{f}(0) = 0$, $\ddot{f}(0) = \kappa$. Then by Proposition 31 the metric g_{reg} is extended to a Riemannian metric g_M on M_φ. Since this metric is Kähler on M_{reg}, the corresponding complex structure J_{reg} is parallel. It is clear that J_{reg} is extended to a parallel complex structure on M_φ. Hence the metric g_M is Kähler.

The claim about the completeness of the metric on M_φ follows from the following

Lemma 33 *A metric of the form $dt^2 + g_t$ on a manifold $M = \mathbb{R} \times N$, where g_t, $t \in \mathbb{R}$ is a family of metrics on the compact manifold N, is complete.*

Proof of the Lemma: By Hopf-Rinow theorem, a metric is complete if and only if the closed balls are compact. This is true under our assumptions since any closed ball of radius r in M is contained in the compact set $[-r, r] \times N$. $\qquad\square$

Definition 34 *An interval, together with a parametrization as described in Theorem 32, is called admissible.*

Remark 35 *Notice that for any $d > 0$, an admissible parametrization $f : (0, +\infty) \to (0, d)$ exists, take for example the function*

$$f(t) = d(1 - e^{-\frac{\kappa}{2d}t^2}).$$

This shows that, without additional conditions, a bounded segment in the Weyl chamber can define a complete Kähler metric (compare with the case of Kähler-Einstein metrics in the next section).

7 Kähler-Einstein metrics on standard cohomogeneity one manifolds

In this section we give necessary and sufficient conditions for the existence of (complete and non-complete) Kähler-Einstein metrics on standard C1 manifolds.

Our first result is the following

Theorem 36 *Let M be a standard cohomogeneity one manifold, i.e. (see Theorem 27) the total space of an admissible bundle $M_\varphi = G \times_H V_\varphi \to S_0$ over the singular orbit $(S_0 = G/H, J^S)$.*
Let $(F = G \times_H PV_\varphi = G/K, J^F)$ be the flag manifold associated with regular orbits, $Z^{Kos} \in C(J^F)$ be the Koszul vector which defines the invariant Kähler-Einstein metric on (F, J^F) associated with the invariant complex structure J^F and $(Z_0 Z_d) \subseteq C = C(J^F)$ an interval in the T-Weyl chamber $C(J^F)$ which represents a standard invariant Kähler structure on M_φ (Theorem 32).

Then, the interval $(Z_0 Z_d)$ with a parametrization $f(t)$ defines a Kähler-Einstein structure with Einstein constant λ if and only if the vectors Z^{Kos}, Z_0, Z^0 are related by

$$Z^{Kos} = \lambda Z_0 + \kappa m Z^0 \tag{19}$$

where $m = \dim(V_\varphi)$, k is defined by $Z_F^0 = \kappa I_o$ (Theorem 32) and $f(t)$ is the solution of the equation

$$\ddot{f}(t) + \frac{1}{2}A(f)\dot{f}^2 + \lambda f = \kappa m \tag{20}$$

with the initial conditions
$\lim_{t\to 0} f(t) = \lim_{t\to 0} \dot{f}(t) = 0, \lim_{t\to 0} \ddot{f}(t) = \kappa$
where $A(f) = \sum_{\alpha \in R_{\mathfrak{m}}^+} \frac{\alpha(Z^0)}{\alpha(Z_0) + f\alpha(Z^0)}$, being $R_{\mathfrak{m}}^+$ the set of the positive black roots of G/K (see formulas (14) and (15)).
Moreover, the Kähler-Einstein metric can be extended to a complete metric if and only if $\lambda \leq 0$, and in this case the segment extends to a ray $Z_0 + \mathbb{R}^+ Z^0$ in $C(J^F)$.

Proof:
The calculations made in Section 5.2 show that the Kähler metric determined by the interval $(Z_0 Z_d)$ together with a parametrization $Z(t) = Z_0 + f(t)Z^0$ satisfies the Einstein condition if and only if

$$Z^{Kos} = \lambda Z_0 + c Z^0 \tag{21}$$

$$\ddot{f}(t) + \frac{1}{2}A(f)\dot{f}^2 + \lambda f = c \tag{22}$$

for some constant c. So, we must show that $c = \kappa m$.
In order to do that, recall that in Theorem 32 it was shown that the metric extends to the singular orbit if and only if the function f satisfies the initial conditions $\lim_{t\to 0} f(t) = \lim_{t\to 0} \dot{f}(t) = 0, \lim_{t\to 0} \ddot{f}(t) = \kappa$. This implies that $f(t) = \frac{\kappa}{2}t^2 + O(t^3)$, and then

$$\frac{1}{2}A(f)\dot{f}^2 = \sum_{\alpha \in R_{\mathfrak{m}}^+} \frac{\alpha(Z^0)(\kappa^2 t^2 + O(t^3))}{\alpha(Z_0) + \alpha(Z^0)(\frac{\kappa}{2}t^2 + O(t^3))}$$

when $t \to 0$ tends to 0 if $\alpha(Z_0) \neq 0$ and to 2κ if $\alpha(Z_0) = 0$. Since $Z_0 \in C_\beta = \{\beta = 0, \beta_1 > 0, \ldots, \beta_k > 0\}$ where $\{\beta_1, \ldots, \beta_k\}$ (resp. $\{\beta, \beta_1, \ldots, \beta_k\}$) is the set of black roots in the Dynkin diagram of G/H (resp. G/K), then a positive black root $\alpha \in R_{\mathfrak{m}}^+$ is a black root of G/H if and only if $\alpha(Z_0) \neq 0$ (recall that, since $Z_0 \in Z(\mathfrak{k})$ then every white root vanishes on Z_0).
In other words, the number of roots $\alpha \in R_{\mathfrak{m}}^+$ for which $\alpha(Z_0) = 0$ equals the number of positive black roots of G/K minus the number of positive black roots of G/H, i.e. equals $dim_{\mathbb{C}}(G/K) - dim_{\mathbb{C}}(G/H)$ which, by $G/K = G \times_H PV_\varphi$, is equal to $m - 1$. It follows then that

$\frac{1}{2}A(f)\dot{f}^2 \to \kappa(m-1)$ for $t \to 0$, which, combined with the other initial conditions, implies that

$$\ddot{f}(t) + \frac{1}{2}A(f)\dot{f}^2 + \lambda f \to \kappa + \kappa(m-1) = \kappa m.$$

This shows that $c = \kappa m$.

Notice that $f(t)$ extends to a smooth even function. In fact it follows by a straight calculation using equation (22) that, under the given initial conditions, $\lim_{t \to 0} f^{(3)}(t) = 0$, which shows that f extends to a C^3 function invariant by reflection at 0. Then, it gives rise to a C^2-Einstein metric and we can apply a result by DeTurck and Kazdan (see, for example, [5]) to conclude that f is C^∞.

In order to end the proof of the theorem, we need to prove the following

Lemma 37 *If the condition (19) is fulfilled, then the function $f(t)$ parametrizing the segment $(Z_0 Z_d)$ which gives the Kähler-Einstein metric is the inverse to the function*

$$t(f) = \int_0^f \sqrt{\frac{P(s)}{2 \int_0^s (c - \lambda v) P(v) dv}} ds \qquad (23)$$

where P is the polynomial defined by $P(x) = \Pi_{\alpha \in R_{\mathfrak{m}}^+}(\alpha(Z_0) + x\, \alpha(Z^0))$.

Proof: The proof is based on the fact that, if f satisfies the ordinary differential equation $\ddot{f}(t) + \frac{1}{2}A(f)\dot{f}^2 + \lambda f = c$, where $A(f)$ is any function of f, then

$$(f')^2 = e^{-\int_0^f A(v) dv} \left(\int_0^f 2(c - \lambda v) e^{\int_0^v A(s) ds} dv + D \right) \qquad (24)$$

Indeed, by the substitution $p(f) = f'$ we get $f''(t) = p'(f)f' = p'p$, so that the equation can be rewritten

$$p'p + \frac{1}{2}A(f)p^2 + \lambda f = c$$

that is, by setting $u(f) = p^2(f)$,

$$u' = -A(f)u + 2(c - \lambda f).$$

Then (24) follows by using the general formula to solve a linear first order ODE $x' = Px + Q$, that is $x(t) = e^{\int P(t) dt} \left[\int Q(t) e^{-\int P(t) dt} dt + D \right]$.

In particular, in our case $A(f) = \sum_{\alpha \in R_{\mathfrak{m}}^+} \frac{\alpha(Z^0)}{\alpha(Z_0) + f\alpha(Z^0)}$, we have $e^{\int A(f) df} = \Pi_{\alpha \in R_{\mathfrak{m}}^+}(\alpha(Z_0) + f\alpha(Z^0)) = P(f)$ which, replaced into (24) yields

$$(f')^2 = \frac{1}{P(f)} \left(\int_0^f 2(c - \lambda v) P(v) dv + D \right) \qquad (25)$$

from which formula (23) follows by extracting the square root, inverting and integrating, and by taking into account that for $t \to 0$ we have $f(t) \to 0$ and $f'(t) \to 0$. $\qquad \square$

Now, using formula (23) we are ready to end the proof of Theorem 36.

If $\lambda > 0$, then the metric is not complete by Myers' theorem, which implies that a complete Kahler-Einstein manifold with positive Einstein constant is compact (and then in the C1 case should have two singular orbits).

Let then $\lambda \leq 0$. Then, by (19) we have $\kappa m Z^0 = Z^{Kos} - \lambda Z_0$ and since Z^{Kos}, Z_0 belong to $C(J^F)$ then also $Z^0 \in C(J^F)$.

This implies that the polynomial $P(x) = \Pi_{\alpha \in R_{\mathfrak{m}}^+}(\alpha(Z_0) + x\ \alpha(Z^0))$ is strictly positive for $x > 0$: indeed, by definition, a root is black if and only if in its decomposition as sum of the basis roots there is one of those corresponding to the black vertices of the painted Dynkin diagram, that is $\beta, \beta_1, \ldots, \beta_p$. Since for any white root γ one has $\gamma(Z_0) = \gamma(Z^0) = 0$ (since $Z_0, Z^0 \in Z(\mathfrak{k})$), the factors $\alpha(Z_0) + x\ \alpha(Z^0)$ of the polynomial $P(x)$ are positive for $x > 0$ if and only if $\beta(Z_0) + x\ \beta(Z^0) = \beta(Z_0 + x Z^0)$, $\beta_j(Z_0) + x\ \beta_j(Z^0) = \beta_j(Z_0 + x Z^0)$, $j = 1, \ldots, p$, are positive, which is true by definition of $C(J^F)$.

From this and from $\lambda \leq 0$, it follows that the polynomial $Q(s) = 2 \int_0^s (\kappa m - \lambda v) P(v) dv$ appearing in the formula (23) has first derivative $Q'(s) = 2(\kappa m - \lambda s) P(s)$ always positive for $s > 0$, so it is strictly increasing and (being $Q(0) = 0$) we have $Q(s) > 0$ for any s.

Then the integrand $\sqrt{\frac{P(s)}{2 \int_0^s (\kappa m - \lambda v) P(v) dv}}$ in (23) is always well-definite and obviously positive. Moreover, being the square root of the ratio between one polynomial of degree N and a polynomial of degree $N + 2$ (both with no positive real roots), it goes to infinity like $1/s$, and then if $f \to \infty$ then one has $t \to \infty$: this shows that f does not blow in finite time, and then it is defined on $(0, +\infty)$, which, together with the fact that $Z_0 + f(t) Z^0 \in C(J^F)$ (we have already observed above that $Z^0 \in C(J^F)$) proves that the metric is complete.

Now, in order to prove that $Z_0 + f Z^0$ is actually a ray in $C(J^F)$, we observe that the function f does not stay bounded.

Indeed, if so, then either there would be a point $t_0 > 0$ for which if $f'(t_0) = 0$, or $f'(t) \to 0$ for $t \to \infty$. In the first case, let t_0 be the first positive value for which if $f'(t_0) = 0$: by the initial conditions, it should be $f''(t_0) \leq 0$. But, from equation (22), one gets $f''(t_0) = c - \lambda f(t_0)$ and then, since $c = \kappa m > 0$ and $\lambda \leq 0$, we have $f''(t_0) > 0$, a contradiction. In the second case, we can conclude by the same argument since, from equation (22),

$$f'' = c - \lambda f - \frac{1}{2} A(f) f'^2 \to c - \lambda f.$$

$\square$

Using Theorem 36 and Lemma 37, together with the description of standard C1 manifolds in terms of Dynkin diagrams and characters given in Section 6, it is possible to find explicit conditions (at least when G is a classical group) for the existence and completeness of Kähler-Einstein standard C1 manifolds having a given flag manifold $S_0 = G/H$ as (the only) singular orbit, for any sign of the Einstein constant. More precisely, one has

Proposition 38 *Let G a simply connected Lie group with Lie algebra $\mathfrak{g}$ equal to one of the classical Lie algebras $su_n, sp_{2n}, so_{2n}, so_{2n+1}$, and let $(S_0 = G/H, J^S)$ be a flag manifold associated with painted Dynkin diagram $\Pi = \Pi_B^H \cup \Pi_W^H$ (possibly consisting of more connected components) which begins with a white A_{m-1} string.*

Let G/K be the flag manifold obtained by painting in black one of the end roots of the A_{m-1} string, and let $n_1, \ldots, n_p$ be the coefficients of the fundamental weights associated to the roots in Π_B^H in the decomposition of the Koszul form of G/K.

Then,

(i) *there exists a Kähler-Einstein standard C1 manifold M (having S_0 as only singular orbit) with Einstein constant $\lambda = 0$ if and only if $n_1, n_2, \ldots, n_p$ (resp. $n_1 + 1, n_2, \ldots, n_p$) are divisible by m if the A_{m-1} string is a connected component of the diagram (resp. otherwise). In particular, if $m = 1$ this condition is trivially fulfilled.*

(ii) *for any $\lambda \neq 0$, there always exists a Kähler-Einstein standard C1 manifold M (having S_0 as only singular orbit) with Einstein constant λ.*

If the above conditions are satisfied then, in accordance with above Theorem 36, the Kahler-Einstein metric can be chosen to be complete for $\lambda \leq 0$, while for $\lambda > 0$ the metric is never complete. Notice that these results are obviously consistent with Myers' theorem and with [8].

We give a proof of Proposition 38 and illustrate it with some examples in the second part of this paper.

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