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Conjectural upper bounds on the smallest size of a complete cap in PG(N, q), $N \geq 3$

Daniele Bartoli[a, 1,3], Alexander A. Davydov[b, 2,4], Giorgio Faina[a, 1,3], Stefano Marcugini[a, 1,3], Fernanda Pambianco[a, 1,3]

[a] *Dipartimento di Matematica e Informatica*
Università degli Studi di Perugia
Perugia, Italy

[b] *Institute for Information Transmission Problems (Kharkevich institute)*
Russian Academy of Sciences
Moscow, Russian Federation

Abstract

In this work we summarize some recent results to be included in a forthcoming paper [2]. In the projective space PG(N, q) over the Galois field of order q, $N \geq 3$, an iterative step-by-step construction of complete caps by adding a new point at every step is considered. It is proved that uncovered points are evenly placed in the space. A natural conjecture on an estimate of the number of new covered points at every step is done. For a part of the iterative process, this estimate is proved rigorously. Under the mentioned conjecture, new upper bounds on the smallest size $t_2(N, q)$ of a complete cap in PG(N, q) are obtained. In particular,

$$t_2(N, q) < \frac{1}{q-1}\sqrt{q^{N+1}(N+1)\ln q} + \frac{1}{q-3}\sqrt{q^{N+1}} \sim q^{\frac{N-1}{2}}\sqrt{(N+1)\ln q}.$$

The effectiveness of the bounds is illustrated by comparison with complete caps sizes obtained by computer searches. The reasonableness of the conjecture is discussed.

Keywords: Projective spaces, small complete caps, upper bounds

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1 Introduction. The main results

Let $\mathrm{PG}(N,q)$ be the N-dimensional projective space over the Galois field of order q. A cap in $\mathrm{PG}(N,q)$ is a set of points no three of which are collinear. A cap is complete if it is not contained in a larger cap. Caps in $\mathrm{PG}(2,q)$ are also called arcs and they have been widely studied, see e.g. [1, 4].

Points of an n-cap in $\mathrm{PG}(N,q)$ form columns of a parity-check matrix of a linear q-ary code of length n, codimension $N+1$, and minimum distance 4 (exceptions are given by the 5-cap in $\mathrm{PG}(3,2)$ and the 11-cap in $\mathrm{PG}(4,3)$). If $N=3$ it is Almost MDS code. Complete caps correspond to non-extendable quasi-perfect codes of covering radius 2.

Let $t_2(N,q)$ be the *smallest size* of a complete cap in $\mathrm{PG}(N,q)$.

This work is devoted to *upper bounds* on $t_2(N,q)$. It is a hard *open problem.*

The trivial lower bound for $t_2(N,q)$ is $\sqrt{2}q^{\frac{N-1}{2}}$. Constructions of complete caps whose size is close to this lower bound are only known for q even. Using a modification of the approach of [4], the probabilistic upper bound $t_2(N,q) < cq^{\frac{N-1}{2}}\log^{300} q$, where c is a constant independent of q, has been obtained in [3].

Throughout the paper, $D \geq 1$ is a *constant independent of* q.

The main result of the paper is as follows.

Theorem 1.1 **(i)** *Under Conjecture* 2.2(i), *in* $\mathrm{PG}(N,q)$, $N \geq 3$, *it holds that*

$$t_2(N,q) < \frac{\sqrt{D}}{q-1}\sqrt{q^{N+1}(N+1)\ln q} + \frac{\sqrt{q^{N+1}}}{q-3} \sim q^{\frac{N-1}{2}}\sqrt{D(N+1)\ln q}. \quad (1)$$

(ii) *Under Conjecture* 2.2(ii), *the bound* (1) *with* $D=1$ *holds.*

Conjecture 1.2 *In* $\mathrm{PG}(N,q)$, $N \geq 3$, *the upper bound* (1) *with* $D=1$ *holds for all* q *without any extra conditions and conjectures.*

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[3] Email: {`daniele.bartoli,giorgio.faina,stefano.marcugini, fernanda.pambianco`}`@unipg.it`

[4] Email: `adav@iitp.ru`

2 An iterative process. A conjecture

In PG(N, q), $N \geq 3$, let a complete cap be constructed by a step-by-step algorithm (*Algorithm*, for short) which adds one new point to the cap at each step; see e.g. a greedy algorithm that at every step adds to the cap a point providing the maximal possible (for the given step) number of new covered points [1]. A *point* of PG(N, q) is *covered by* a *cap* if the point lies on a bisecant of the cap. The space PG(N, q) contains $\theta_{N,q} = \frac{q^{N+1}-1}{q-1}$ points.

Assume that after the w-th step of Algorithm, a w-cap is obtained that does not cover exactly U_w points. Let $\mathbf{S}(U_w)$ be the set of all w-caps in PG(N, q) each of which does not cover exactly U_w points.

Consider the $(w + 1)$-st step of Algorithm. This step starts from a w-cap $\mathcal{K}_w$ with $\mathcal{K}_w \in \mathbf{S}(U_w)$. The choice $\mathcal{K}_w$ from $\mathbf{S}(U_w)$ is random such that for every cap of $\mathbf{S}(U_w)$ the probability to be chosen is equal to $\frac{1}{\#\mathbf{S}(U_w)}$. So, the set $\mathbf{S}(U_w)$ is considered as an *ensemble of random objects* with the uniform probability distribution. In turn, every point H of PG(N, q) can be considered as a random object that, with some probability $p_w(H)$, is not covered by a randomly chosen w-cap $\mathcal{K}_w$.

Lemma 2.1 *The probability $p_w(H)$ does not depend of the point H; it may be considered as p_w. Moreover, $p_w = U_w/\#\mathrm{PG}(N, q) = U_w/\theta_{N,q}$.*

Let the cap $\mathcal{K}_w$ consist of w points $A_1, A_2, \ldots, A_w$. Let A_{w+1} be the point that will be included into the cap at the $(w + 1)$-st step. The point A_{w+1} defines a bundle of w tangents $\overline{A_1A_{w+1}}, \ldots, \overline{A_wA_{w+1}}$ to $\mathcal{K}_w$, where $\overline{A_iA_j}$ is the line through A_i and A_j. Excluding $A_1, \ldots, A_w$, all the points on the tangents of the bundle are **candidates** to be new covered points at the $(w+1)$-st step. There are $w(q-1)+1$ candidates in the bundle. There are U_w distinct bundles.

Assume that for points of PG(N, q), the *events to be not covered by a randomly chosen w-cap $\mathcal{K}_w$ are independent.* Under this condition, let $\mathbf{E}_{w,q}$ be the **expected value** of the number of points not covered by $\mathcal{K}_w$ among $w(q - 1) + 1$ randomly taken points in PG(N, q). By Lemma 2.1,

$$\mathbf{E}_{w,q} = (w(q - 1) + 1)p_w = (w(q - 1) + 1)U_w/\theta_{N,q}. \tag{2}$$

Since all the candidates lie on some bundle, they cannot be considered as randomly taken points for which the events to be uncovered are independent. On the other side, there are *many random factors* affecting the iterative process, e.g. relative positions and intersections of bisecants and tangents, the number of uncovered points on distinct tangents. Therefore, the conjecture below seems to be reasonable and founded, see also Section 4.

Let $\Delta_w(A_{w+1})$ be the number of new covered points at the $(w+1)$-st step.

Conjecture 2.2 **(i)** *(**the generalized conjecture**) In* $\mathrm{PG}(N,q)$*, for* q *large enough, for every* $(w+1)$*-th step of the iterative process, there exists a* w*-cap* $\mathcal{K}_w \in \mathbf{S}(U_w)$ *such that there exists an uncovered point* A_{w+1} *providing*

$$\Delta_w(A_{w+1}) \geq \frac{\mathbf{E}_{w,q}}{D}. \tag{3}$$

(ii) *(**the basic conjecture**) In* (3) *we have* $D = 1$.

3 Upper bounds on $t_2(N,q)$ and their effectiveness

Theorem 3.1 *Let* $Q := \theta_{N,q}/(q-1)$*. Let* ξ *be a constant independent of* w *with* $\xi \geq 1$*. Under Conjecture 2.2, in PG*(N,q) *the following holds:*

- $t_2(N,q) \leq w+1+\xi$*, where the value* w *satisfies* $\theta_{N,q}\prod_{j=1}^{w}\left(1-\frac{j}{DQ}\right) \leq \xi$;
- $t_2(N,q) \leq \sqrt{2DQ}\sqrt{\ln\frac{\theta_{N,q}}{\xi}}+2+\xi$.

Taking $\xi = \frac{1}{q-1}\sqrt{q^{N+1}}$ in Theorem 3.1, we obtain Theorem 1.1.

To illustrate the *effectiveness of the new upper bounds* we obtained by computer search small complete caps in the following wide regions of q: $\{\text{prime } q \leq 4673\} \cup \{5003, 6007, 7001, 8009\}$ for $\mathrm{PG}(3,q)$ and $\{\text{prime } q \leq 1361\} \cup \{1409\}$ for $\mathrm{PG}(4,q)$.[5] All obtained complete caps satisfy bound (1) with $D = 1$.

4 Reasonableness of the conjecture

For a cap $\mathcal{K}_w$, denote by $\Delta_w^{\text{aver}}(\mathcal{K}_w)$ the average value of $\Delta_w(A_{w+1})$ over all U_w uncovered points A_{w+1}, i.e. $\Delta_w^{\text{aver}}(\mathcal{K}_w) = \frac{1}{U_w}\sum_{A_{w+1}}\Delta_w(A_{w+1}) \geq 1$.

Lemma 4.1 *For any* w*-cap* $\mathcal{K}_w \in \mathbf{S}(U_w)$*, the following inequalities hold.*

$$\max_{A_{w+1}}\Delta_w(A_{w+1}) \geq \Delta_w^{\text{aver}}(\mathcal{K}_w) \geq \max\left\{1, \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1\right\}. \tag{4}$$

The equalities $\max\limits_{A_{w+1}}\Delta_w(A_{w+1}) = \Delta_w^{\text{aver}}(\mathcal{K}_w) = \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1$ *hold if and only if each tangent contains the same number of uncovered points. The equal-*

[5] Calculations were performed using computational resources of Multipurpose Computing Complex of National Research Centre “Kurchatov Institute”, http://computing.kiae.ru

ities $\max_{A_{w+1}} \Delta_w(A_{w+1}) = \Delta_w^{\text{aver}}(\mathcal{K}_w) = 1$ *hold if and only if each tangent contains at most one uncovered point.*

For a **part** of the **iterative process**, we **rigorously prove** Conjecture 2.2.

Theorem 4.2 *Let* $\Phi_{w,q}(D) := D(w-1)\theta_{N,q}(\theta_{N-1,q}+1-w)/(Dw\theta_{N,q}-(1+\theta_{N-1,q}-w)(w(q-1)+1))$, $\Upsilon_{w,q}(D) := D\theta_{N,q}/(w(q-1)+1)$. *Let one of the following conditions hold:* $\Upsilon_{w,q}(D) \geq U_w$, $U_w \geq \Phi_{w,q}(D)$. *Then for any cap* $\mathcal{K}_w$ *of* $\mathbf{S}(U_w)$, *there exists an uncovered point* A_{w+1} *providing the inequality* (3).

Remark 4.3 To illustrate Conjecture 2.2, the values $\Delta_w(A_{w+1})$ were calculated for numerous concrete iterative processes. *For all the calculations done* it holds that $\max_{A_{w+1}} \Delta_w(A_{w+1}) > \mathbf{E}_{w,q}$. The ratio $\max_{A_{w+1}} \Delta_w(A_{w+1})/\mathbf{E}_{w,q}$ has an increasing trend when w grows. In Fig. 1 for a complete k-cap in PG$(3,101)$, $k = 415$, the following values are shown (see (2)–(4)): $\delta_w^{\max} = \frac{1}{\mathbf{E}_{w,q}} \cdot \max_{A_{w+1}} \Delta_w(A_{w+1})$ (*the top solid red curve*), $\delta_w^{\text{aver}} = \frac{1}{\mathbf{E}_{w,q}} \cdot \Delta_w^{\text{aver}}(\mathcal{K}_w)$ (*the 2-nd dashed-dotted blue curve*), $\delta_w^{\min} = \frac{1}{\mathbf{E}_{w,q}} \cdot \min_{A_{w+1}} \Delta_w(A_{w+1})$ (*the 3-rd solid red curve*), $\delta_w^{\text{rigor}} = \frac{1}{\mathbf{E}_{w,q}} \cdot \max\{1, \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1\}$ (*the bottom dotted black curve*). The horizontal axis shows the values of $\frac{w}{k}$. The green lines $y = 1$ and $y = \frac{1}{5}$ correspond to Conjecture 2.2(ii) where $D = 1$ and to Conjecture 2.2(i) with $D = 5$. The signs $\bullet$ correspond to values $\Phi_{w,q}(D)$ and $\Upsilon_{w,q}(D)$ with $D = 1$ and $D = 5$. In Fig. 1, the region where we rigorously prove Conjec-

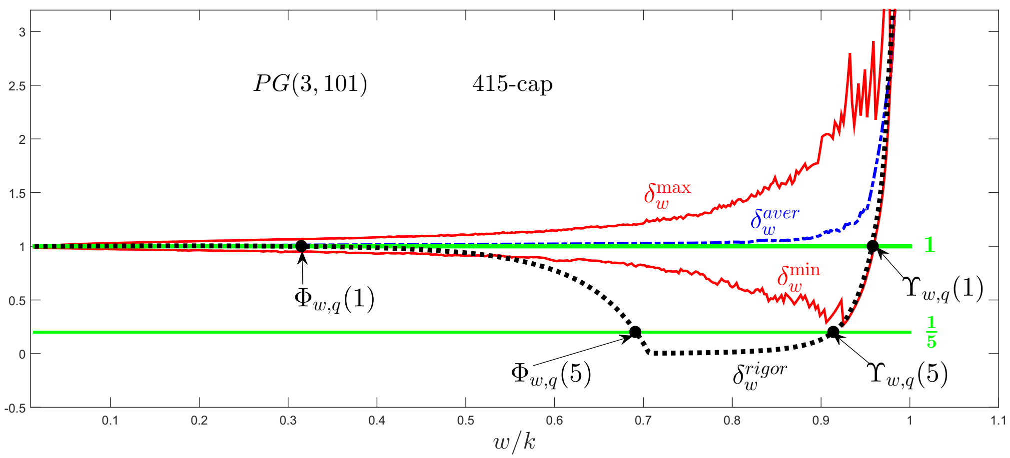

Fig. 1. **Illustration of reasonableness of Conjectures 2.2(i) and 2.2(ii)**

ture 2.2 lies on the left of $\Phi_{w,q}(D)$ and on the right of $\Upsilon_{w,q}(D)$. This region takes $\sim 35\%$ of the whole iterative process for $D = 1$ and $\sim 75\%$ for $D = 5$.

Remark 4.4 Let $\gamma_{w,j}$ be the number of uncovered points on the j-th tangent after the w-th step of Algorithm. The lower estimate in (4) is attained in two cases: either *every tangent contains the same number of uncovered points* (i.e. $\gamma_{w,j} = \gamma_{w,i}$ for all pairs i, j) or *each tangent contains at most one uncovered point*. The 1-st situation holds in the first steps of the iterative process only. Then while $U_w(D) \geq \Phi_{w,q}(D)$ holds, the differences $\gamma_{w,j} - \gamma_{w,i}$ are relatively small and estimate (4) works "well". As U_w decreases, the differences relatively increase, and the estimate becomes worse in the sense that actually the value of $\Delta_w^{\text{aver}}(\mathcal{K}_w)$ is considerably greater than $\max\left\{1, \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1\right\}$.

The 2-nd situation is possible, in principle, when $U_w \leq \theta_{N-1,q} + 1 - w$ and the average number γ_w^{aver} of uncovered points on a tangent is smaller than 1. But on this stage of the iterative process variations in the values $\gamma_{w,j}$ are relatively big; and again the value of $\Delta_w^{\text{aver}}(\mathcal{K}_w)$ is considerably greater than $\max\left\{1, \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1\right\}$. In the final region of the process, where $U_w \leq \Upsilon_{w,q}(D)$ and $\frac{\mathbf{E}_{w,q}}{D} \leq 1$, estimate (4) becomes reasonable once more.

Thus, in the region $\Phi_{w,q}(D) > U_w > \Upsilon_{w,q}(D)$ the estimate (4) does not reflect the real situation effectively. In fact, in this region *the value of* $\Delta_w^{aver}(\mathcal{K}_w)$ (presented by curve δ_w^{aver} in Fig. 1) *is considerably greater than* $\max\left\{1, \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1\right\}$ (presented by curve δ_w^{rigor} in Fig. 1).

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