Prolongation of Tanaka structures: an alternative approach

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Abstract: The classical theory of prolongation of G-structures was generalized by N. Tanaka to a wide class of geometric structures (Tanaka structures), which are defined on a non-holonomic distribution. Examples of Tanaka structures include subriemannian, subconformal, CR structures, structures associated to second order differential equations and structures defined by gradings of Lie algebras (in the framework of parabolic geometries). Tanaka's prolongation procedure associates to a Tanaka structure of finite order a manifold with an absolute parallelism. It is a very fruitful method for the description of local invariants, investigation of the automorphism group and equivalence problem. In this paper, we develop an alternative constructive approach for Tanaka's prolongation procedure, based on the theory of quasi-gradations of filtered vector spaces, G-structures and their torsion functions.

Key words: G-structures, Tanaka structures, prolongations, automorphism groups, quasi-gradations, torsion functions.

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1 Introduction

Recall that a G-structure of an n-dimensional manifold M is a principal subbundle $\pi_G : P_G \to M$ of the frame bundle of M with structure group $G \subset GL(V)$, $V = \mathbb{R}^n$. Any tensor field which is infinitesimally homogeneous, i.e. whose value at any point has the same normal form with respect to some "admissible" frame, is identified with a G-structure, whose total space P_G is the set of all such admissible frames.
The prolongation of G-structures (see e.g. [?], Chapter VII) is a powerful method in differential geometry which associates to any G-structure $\pi_G : P_G \to M$ of finite order a new manifold $P = P(\pi_G)$ (the full prolongation), with an absolute parallelism (i.e. an $\{e\}$-structure), with the important property that the group of automorphisms $\mathrm{Aut}(P, \{e\})$ of $(P, \{e\})$ is isomorphic to the group of automorphisms $\mathrm{Aut}(\pi_G)$ of π_G. The absolute paralellism $(P, \{e\})$ provides local invariants for π_G (see [?], Theorem 4.1 of Chapter VII). Owing to Kobayashi's theorem (see [?], Theorem 3.2 of Chapter 0), $\mathrm{Aut}(\pi_G) \simeq \mathrm{Aut}(P, \{e\})$ are Lie groups of dimension less or equal to the dimension of P.

The full prolongation P of $\pi_G : P_G \to M$ is defined by consecutive applications of the first prolongation. We briefly recall its construction. It is based on the observation that the bundle $j^1(\pi_G) : J^1 P_G = \mathrm{Hor}(P_G) \to P_G$ of 1-jets of sections of π_G (i.e. horizontal subspaces of TP_G) is a G-structure with structure group

$$G^1 = \mathrm{id} + \mathrm{Hom}(V, \mathfrak{g}) = \{ \begin{pmatrix} \mathrm{id} & 0 \\ A_\cdot & \mathrm{id} \end{pmatrix},\ A_\cdot \in \mathrm{Hom}(V, \mathfrak{g}) \},$$

which is a commutative subgroup of $GL(V + \mathfrak{g})$. Using the torsion functions of $j^1(\pi_G)$, one can reduce the G-structure $j^1(\pi_G)$ to a G-structure $\pi_G^{(1)} : P_G^{(1)} \to P_G$ whose structure group $G^{(1)}$ is the Lie subgroup of G^1 generated by the Lie subalgebra $\mathfrak{g}^{(1)} = \mathrm{Hom}(V, \mathfrak{g}) \cap (V \otimes S^2 V^*) \subset \mathfrak{gl}(V + \mathfrak{g})$. The G-structure $\pi_G^{(1)}$ is called the first prolongation of π_G. If the k-th iterated prolongation $\mathfrak{g}^{(k)} := (\mathfrak{g}^{(k-1)})^{(1)}$ of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ vanishes, then G is called of finite order and the k-th iterated $P_G^{(k)}$ first prolongation of P_G defines an absolute parallelism on the full prolongation $P := P_G^{(k-1)}$.

While the prolongation procedure works effectively for G-structures of finite order (e.g. conformal or quaternionic structures), there are other important geometric structures (e.g. CR-structures and other structures defined on a non-integrable distribution), which cannot be treated effectively by this method. To overcome this difficulty, in 1970 Tanaka [?] generalized the prolongation of G-structures to a larger class of geometric structures, called Tanaka structures in [?] and infinitesimal flag structures in [?] (see Definition ??). Examples of Tanaka structures include CR-structures, subriemannian and subconformal structures. Tanaka's prolongation procedure received much attention in the mathematical literature. There are many approaches for the Tanaka prolongation under different assumptions, see [?, ?, ?, ?, ?]. Our approach is a developing and a detalization of the approach from [?], where the first step of the Tanaka prolongation was explained in detail, but the other steps were only stated without proofs. To prove the iterative construction, one has to check many extra conditions, and this will be carefully done in this paper. Our approach is close to the approach of I. Zelenko [?]. The main difference is that we develop and systematically use the theory of quasi-gradations of filtered vector spaces. Together with the well-known theory of Tanaka prolongations of non-positively graded Lie algebras and the torsion functions of G-structures, this provides a conceptual and simple description of each step of the prolongation procedure: the principal bundle

$\bar{\pi}^{(n)} : \bar{P}^{(n)} \to \bar{P}^{(n-1)}$ which relates the n and $(n-1)$-prolongations of a given Tanaka structure is canonically isomorphic to a subbundle of the principal bundle of $(n+1)$-quasi-gradations of $T\bar{P}^{(n-1)}$ and is obtained as the quotient of a G-structure of $\bar{P}^{(n-1)}$, with structure group $G^n GL_{n+1}(\mathfrak{m}_{n-1})$, with suitable properties of the torsion function. These statements are explained in detail in Theorem **??**. In Theorem **??** we state the final result of the Tanaka prolongation procedure, which reduces the local classification of Tanaka structures of finite order to the well understood local classification of absolute parallelisms. This requires the construction of a canonical frame on a prolongation of suitable order and a careful analysis of the behaviour of the automorphisms of a Tanaka structure under the prolongation procedure. We do this in Propositions **??** and Proposition **??**. In the remaining part of the introduction we present the structure of the paper.

Structure of the paper. Section **??** is mainly intended to fix notation. Our original contribution in this section is the theory of quasi-gradations of filtred vector spaces, which is developed in Subsections **??** and **??**. Besides, we recall the definition of the Tanaka prolongation of a non-positively graded Lie algebra [?], the basic facts we need from the theory of G-structures (see e.g. [?]) and the definition of Tanaka structure [?].

In Section **??** we state our main results from this paper, namely Theorems **??** and **??**. All notions used in these statements are defined in the previous section.

The remaining sections are devoted to the proofs of Theorems **??** and **??**. Let $(\mathcal{D}_i, \pi_G : P_G \to M)$ be a Tanaka G-structure of type $\mathfrak{m} = \sum_{i=-k}^{-1} \mathfrak{m}^i$. Basically, the proof of Theorem **??** is divided into two main parts: in a first stage, in Section **??** we construct the starting projection $\bar{\pi}^{(1)} : \bar{P}^{(1)} \to P = P_G$ of the sequence of projections from Theorem **??** (also called the first prolongation of the Tanaka structure $(\mathcal{D}_i, \pi_G)$). For this, we remark that P has a canonical Tanaka $\{e\}$-structure of type $\mathfrak{m}_0 = \mathfrak{m} + \mathfrak{g}^0$ (where $\mathfrak{g}^0 = \mathrm{Lie}(G)$) and we define a G-structure $\pi^1 : P^1 \to P$ as the set of all adapted gradations of TP, or, equivalently, the set of all frames of TP which lift the canonical graded frames of the Tanaka $\{e\}$-structure of P (see Proposition **??** and Definition **??**). Using the torsion, we reduce π^1 to a subbundle $\tilde{\pi}^1 : \tilde{P}^1 \to P$, with structure group $G^1 GL_2(\mathfrak{m}_1)$ and we define $\bar{\pi}^{(1)} : \bar{P}^{(1)} \to P = P_G$ to be the quotient of $\tilde{\pi}^1$ by the normal subgroup $GL_2(\mathfrak{m}_1)$ (see Definition **??**). To a large extent (except Subsection **??**) this material is a rewriting of the construction from [?], using frames instead of coframes (which are more suitable for the higher steps of the prolongation). It is also the simplest part of the prolongation procedure. We skip its details in this introduction and we describe directly the higher steps of the prolongation, where our new approach using quasi-gradations plays a crucial role. Therefore, suppose that the projections $\bar{\pi}^{(i)} : \bar{P}^{(i)} \to \bar{P}^{(i-1)}$ $(i \leq n)$ from Theorem **??** are given. We aim to define $\bar{\pi}^{(n+1)} : \bar{P}^{(n+1)} \to \bar{P}^{(n)}$.

In Section **??** we define $P^{n+1} \subset \mathrm{Gr}(T\bar{P}^{(n)})$ as the set of all adapted gradations of $T_{\bar{H}^n}\bar{P}^{(n)}$ (for any $\bar{H}^n \in \bar{P}^{(n)}$), whose projection to $T_{\bar{H}^{n-1}}\bar{P}^{(n-1)}$ is compatible with the quasi-gradation $\bar{H}^n \in \mathrm{Gr}_{n+1}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)})$ (see Definition **??**) and we show that the natural map $\pi^{n+1} : P^{n+1} \to \bar{P}^{(n)}$ is a G-structure, with structure group $\mathrm{Id} + \mathfrak{gl}_{n+1}(\mathfrak{m}_n) + \mathrm{Hom}(\sum_{i=0}^{n-1} \mathfrak{g}^i, \mathfrak{m}_n)$ (see Proposition **??**).

The definition of $\bar{\pi}^{(n+1)}$ requires a careful analysis of the torsion functions of the G-structure π^{n+1}. This is done in Sections **??** and **??**. In Section **??** we consider an arbitrary connection ρ on the G-structure $\pi^{n+1} : P^{n+1} \to \bar{P}^{(n)}$ and we study the component $t^\rho : P^{n+1} \to \mathrm{Hom}((\mathfrak{m}^{-1} + \mathfrak{g}^n) \wedge \mathfrak{m}_n, \mathfrak{m}_n)$ of its torsion function (see Theorem **??**). The proof of Theorem **??** is divided into three parts, according to the decomposition of

$\mathrm{Hom}((\mathfrak{m}^{-1}+\mathfrak{g}^n)\wedge\mathfrak{m}_n,\mathfrak{m}_n)$ into the subspaces $\mathrm{Hom}(\mathfrak{g}^n\wedge\mathfrak{m}_n,\mathfrak{m}_n)$, $\mathrm{Hom}(\mathfrak{m}^{-1}\wedge\mathfrak{m},\mathfrak{m}_n)$ and $\mathrm{Hom}(\mathfrak{m}^{-1}\wedge\sum_{i=0}^{n-1}\mathfrak{g}^i,\mathfrak{m}_n)$. In Section **??** we define an action of $G^nGL_{n+1}(\mathfrak{m}_{n-1})$ on P^{n+1} (see Proposition **??**) which is used to treat the $\mathrm{Hom}(\mathfrak{g}^n\wedge\mathfrak{m}_n,\mathfrak{m}_n)$-valued component of t^ρ (see Proposition **??**). The properties of the $\mathrm{Hom}(\mathfrak{m}^{-1}\wedge\mathfrak{m},\mathfrak{m}_n)$-valued component of t^ρ are consequences of the fact that the canonical graded frames of the Tanaka $\{e\}$-structure on $\bar{P}^{(n)}$ are Lie algebra isomorphisms when restricted to $\mathfrak{m}$ (see Proposition **??**). The properties of the remaining $\mathrm{Hom}(\mathfrak{m}^{-1}\wedge\sum_{i=0}^{n-1}\mathfrak{g}^i,\mathfrak{m}_n)$-valued component of t^ρ are inherited from the properties of the torsion function of the G-structure $\tilde{\pi}^n:\tilde{P}^n\to\bar{P}^{(n-1)}$ (see Proposition **??**).

In Section **??** we determine the homogeneous components of t^ρ which are independent of the connection ρ and we define and study the $(n+1)$-torsion $\bar{t}^{(n+1)}:P^{n+1}\to\mathrm{Tor}^{n+1}(\mathfrak{m}_n)$ of the Tanaka structure $(\mathcal{D}_i,\pi_G)$ (see Definition **??** and Theorem **??**).

With the material from the previous sections, in Section **??** we finally define the G-structure $\tilde{\pi}^{n+1}:\tilde{P}^{n+1}\to\bar{P}^{(n)}$ and the principal bundle $\bar{\pi}^{(n+1)}:\bar{P}^{(n+1)}\to\bar{P}^{(n)}$ we are looking for. Let W^{n+1} be a complement of $\mathrm{Im}(\partial^{(n+1)})$ in the space of torsions $\mathrm{Tor}^{n+1}(\mathfrak{m}_n)$ (see Theorem **??** for the definition of the map $\partial^{(n+1)}$). The G-structure $\tilde{\pi}^{n+1}$ is the restriction of π^{n+1} to $\tilde{P}^{n+1}=(\bar{t}^{(n+1)})^{-1}(W^{n+1})$ and has structure group $G^{n+1}GL_{n+2}(\mathfrak{m}_n)$ (see Proposition **??**). The bundle $\bar{\pi}^{(n+1)}:\bar{P}^{(n+1)}\to\bar{P}^{(n)}$ is defined as the quotient of $\tilde{\pi}^{n+1}$ by the normal subgroup $GL_{n+2}(\mathfrak{m}_n)\subset G^{n+1}GL_{n+2}(\mathfrak{m}_n)$ and satisfies the properties from Theorem **??** (see Proposition **??**). This concludes the proof of Theorem **??**.

In Section **??** we prove Theorem **??**. The construction of the canonical frame F^{can} on $\bar{P}^{(\bar{l})}$ (or on any $\bar{P}^{(\bar{l}')}$, for $\bar{l}'\geq\bar{l}$), required by Theorem **??**, is done in Proposition **??**. In Proposition **??** we show that the automorphism group $\mathrm{Aut}(\mathcal{D}_i,\pi_G)$ of a Tanaka structure $(\mathcal{D}_i,\pi_G)$ (not necessarily of finite order) is isomorphic to the automorphism group of any of the associated G-structures $\tilde{\pi}^n:\tilde{P}^n\to\bar{P}^{(n-1)}$ $(n\geq 1)$. When $(\mathcal{D}_i,\pi_G)$ is of finite order $\bar{l}$, the G-structure $\tilde{\pi}^{\bar{l}'+1}:\tilde{P}^{\bar{l}'+1}\to\bar{P}^{(\bar{l}')}$ is an absolute parallelism for large enough $\bar{l}'$, which coincides with the canonical frame of $\bar{P}^{(\bar{l}')}$ (see Proposition **??**). This fact, combined with Proposition **??** and Kobayashi's theorem mentioned above, completes the proof of Theorem **??**.

2 Preliminary material

2.1 Quasi-gradations of filtred vector spaces

Let $V=V_{-k}\supset V_{-k+1}\supset\cdots\supset V_l$ be a decreasing filtration of a finite dimensional vector space V by subspaces V_i. We define $V_j=\{0\}$ for $j>l$ and $V_j=V$ for $j<-k$.

Definition 1. *i) A gradation $H=\{H^i,\ -k\leq i\leq l\}$ of V is called* **adapted** *(to the filtration $\{V_i\}$) if $V_i=H^i+H^{i+1}+\cdots+H^l$, for any $-k\leq i\leq l$.*

ii) A **quasi-gradation of degree** $m\geq 1$ *(or shortly, m-***quasi-gradation***) of V is a system of subspaces $\bar{H}=\{\bar{H}^i,\ -k\leq i\leq l\}$ such that, for any $-k\leq i\leq l$,*

$$a)V_i=\bar{H}^i+V_{i+1},\quad b)\bar{H}^i\cap V_{i+1}=V_{i+m}.$$

We denote by $\mathrm{Gr}(V)$ and $\mathrm{Gr}_m(V)$ the set of all adapted gradations, respectively the set of all m-quasi-gradations of V. Remark that $\mathrm{Gr}_m(V)=\mathrm{Gr}(V)$ for any $m\geq k+l+1$.

For any $1 \leq m \leq p$, we define

$$\Pi_p^m : \mathrm{Gr}_p(V) \to \mathrm{Gr}_m(V),\ \Pi_p^m(\{\bar{H}^i\}) := \{\bar{H}^i + V_{i+m}\}.$$

In particular, there is a natural map

$$\Pi^m : \mathrm{Gr}(V) \to \mathrm{Gr}_m(V),\ \Pi^m(\{H^i\}) := \{H^i + V_{i+m}\}. \tag{1}$$

Definition 2. *Any adapted gradation of V which belongs to $\mathrm{Gr}_{\bar{H}}(V) := (\Pi^m)^{-1}(\bar{H})$ is called* **compatible with the quasi-gradation** $\bar{H} \in \mathrm{Gr}_m(V)$.

Let $\mathrm{gr}(V) := \sum_{i=-k}^{l} \mathrm{gr}^i(V)$, where $\mathrm{gr}^i(V) := V_i/V_{i+1}$, be the graded vector space associated to V. More generally, for any $m \geq 1$, let $\mathrm{gr}_{(m)}(V) := \sum_{i=-k}^{l} \mathrm{gr}^i_{(m)}(V)$, where $\mathrm{gr}^i_{(m)}(V) := V_i/V_{i+m}$. We denote by

$$\mathrm{gr}^i : V_i \to \mathrm{gr}^i(V),\ \mathrm{pr}^i_{(m)} : V_i \to \mathrm{gr}^i_{(m)}(V),\ \mathrm{gr}^i_{(m)} : \mathrm{gr}^i_{(m)}(V) \to \mathrm{gr}^i(V)$$

the natural projections. Remark that $\mathrm{gr}^i = \mathrm{pr}^i_{(1)}$ and $\mathrm{pr}^i_{(1)} = \mathrm{gr}^i_{(m)}$ for $m \geq k+l+1$. Any adapted gradation $H = \{H^i\}$ defines injective maps $\widehat{H}^i : \mathrm{gr}^i(V) \to V_i$, with image $H^i \subset V_i$ (from the direct sum decompositions $V_i = V_{i+1} + H^i$). The next proposition generalizes this statement to quasi-gradations.

Proposition 3. *i) There is a one to one correspondence between the space $\mathrm{Gr}_m(V)$ of m-quasi-gradations $\bar{H} = \{\bar{H}^i\}$ and the space of maps $f = (f^i) : \mathrm{gr}(V) \to \mathrm{gr}_{(m)}(V)$ where*

$$f^i : \mathrm{gr}^i(V) \to \mathrm{gr}^i_{(m)}(V),\ \mathrm{gr}^i_{(m)} \circ f^i = \mathrm{Id}_{\mathrm{gr}^i(V)}, \quad -k \leq i \leq l. \tag{2}$$

*More precisely, any $\bar{H} \in \mathrm{Gr}_m(V)$ defines a map $\widehat{\bar{H}} = (\widehat{\bar{H}}^i) : \mathrm{gr}(V) \to \mathrm{gr}_{(m)}(V)$ which satisfies (**??**) and $\widehat{\bar{H}}^i : \mathrm{gr}^i(V) \to \mathrm{gr}^i_{(m)}(V)$ has image $\bar{H}^i/V_{i+m} \subset \mathrm{gr}^i_{(m)}(V)$. Conversely, any map $f = (f^i) : \mathrm{gr}(V) \to \mathrm{gr}_{(m)}(V)$ as in (**??**) defines $\bar{H} = \{\bar{H}^i\} \in \mathrm{Gr}_m(V)$ by*

$$\bar{H}^i := (\mathrm{pr}^i_{(m)})^{-1}\mathrm{Im}(f^i), \quad -k \leq i \leq l \tag{3}$$

and $f = \widehat{\bar{H}}$.

ii) A gradation H is compatible with an m-quasi-gradation $\bar{H}$ if and only if

$$\mathrm{pr}^i_{(m)} \circ \widehat{H}^i = \widehat{\bar{H}}^i, \quad -k \leq i \leq l. \tag{4}$$

Proof. The proof is straightforward and we omit details. We only define the map $\widehat{\bar{H}}$ associated to the quasi-gradation $\bar{H} \in \mathrm{Gr}_m(V)$, and this is done as for gradations. Namely, from Definition **??**, $V_i/V_{i+m} = \bar{H}^i/V_{i+m} + V_{i+1}/V_{i+m}$ (direct sum decomposition). This induces an isomorphism between $\mathrm{gr}^i(V) = (V_i/V_{i+m})/(V_{i+1}/V_{i+m})$ and $\bar{H}^i/V_{i+m} \subset \mathrm{gr}^i_{(m)}(V) = V_i/V_{i+m}$, which gives the required map $\widehat{\bar{H}}^i$. Alternatively, $\widehat{\bar{H}}^i$ associates to $[y] \in \mathrm{gr}^i(V)$ the unique $[z] \in \mathrm{pr}^i_{(m)}(\bar{H}^i) \subset \mathrm{gr}^i_{(m)}(V)$, such that $\mathrm{gr}^i_{(m)}([z]) = [y]$. $\square$

2.2 Lifts and quasi-gradations

Let $\mathfrak{m} = \sum_i \mathfrak{m}^i$ be a graded vector space, V a filtered vector space and $u : \mathfrak{m} \to \mathrm{gr}(V)$ a graded vector space isomorphism. Since $\mathfrak{m}$ is graded, it is filtered in a natural way by the subspaces $\mathfrak{m}_i := \sum_{j\geq i} \mathfrak{m}^j$.

Definition 4. *A* **lift** *of u is a filtration preserving isomorphism $F : \mathfrak{m} \to V$ which satisfies $\mathrm{gr}^i \circ F|_{\mathfrak{m}^i} = u|_{\mathfrak{m}^i}$, for any i. More generally, an m-***lift** *($m \geq 1$) is a map $F = (F^i) : \mathfrak{m} \to \mathrm{gr}_{(m)}(V)$, where $F^i : \mathfrak{m}^i \to \mathrm{gr}^i_{(m)}(V)$ are such that $\mathrm{gr}^i_{(m)} \circ F^i = u|_{\mathfrak{m}^i}$, for any i.*

We remark that F is a lift of u if and only if it is filtration preserving and $\mathrm{gr}^i \circ F|_{\mathfrak{m}_i} = u \circ \pi_{\mathfrak{m}^i}|_{\mathfrak{m}_i}$, for any i. (We always denote by $\pi_{\mathfrak{m}^i} : \mathfrak{m} \to \mathfrak{m}^i$ the natural projection onto the degree i-component $\mathfrak{m}^i$ of a graded vector space $\mathfrak{m}$). The next theorem generalizes Lemma 7.1 of [?].

Theorem 5. *There is a one to one correspondence between the space of m-quasi-gradations of V and the space of m-lifts of u. More precisely, any m-quasi-gradation $\bar{H}$ defines an m-lift, by $F^i_{\bar{H}} := \widehat{\bar{H}}^i \circ u|_{\mathfrak{m}^i}$. Conversely, any m-lift $F = (F^i)$ defines an m-quasi-gradation $\bar{H}^i := (\mathrm{pr}^i_{(m)})^{-1} F^i(\mathfrak{m}^i)$ and $F = F_{\bar{H}}$.*

Proof. Let $\bar{H} \in \mathrm{Gr}_m(V)$. From the definitions of $F^i_{\bar{H}}$ and $\widehat{\bar{H}}^i$, $\mathrm{gr}^i_{(m)} \circ F^i_{\bar{H}} = \mathrm{gr}^i_{(m)} \circ \widehat{\bar{H}}^i \circ u|_{\mathfrak{m}^i} = u|_{\mathfrak{m}^i}$, i.e. $F_{\bar{H}}$ is an m-lift. Conversely, if F is an m-lift, then $F \circ u^{-1} : \mathrm{gr}(V) \to \mathrm{gr}_{(m)}(V)$ satisfies the properties from Proposition **??**. We deduce that $\bar{H} := \{\bar{H}^i\}$ where

$$\bar{H}^i = (\mathrm{pr}^i_{(m)})^{-1} \mathrm{Im}(F \circ u)^i = (\mathrm{pr}^i_{(m)})^{-1} F^i(\mathfrak{m}^i)$$

is an m-quasi-gradation. It remains to prove that $F = F_{\bar{H}}$. For this, let $x \in \mathfrak{m}^i$. Since $\bar{H}^i = (\mathrm{pr}^i_{(m)})^{-1} F^i(\mathfrak{m}^i)$, $F^i(x) \in \mathrm{pr}^i_{(m)}(\bar{H}^i)$. Since $\mathrm{gr}^i_{(m)} \circ F^i(x) = u(x)$, we obtain $F^i_{\bar{H}}(x) = \widehat{\bar{H}^i}(u(x)) = F^i(x)$, as needed (the second equality follows from the proof of Proposition **??**, by taking $[y] = u(x)$ and $[z] = F^i(x)$). □

In view of the above theorem, we identify the space $\mathrm{Gr}_m(V)$ of m-quasi-gradations with the space of m-lifts of u. To avoid confusion, lifts of u will be denoted by F_H and m-lifts by $F_{\bar{H}}$. The map (**??**), in terms of m-lifts, is

$$\Pi^m : \mathrm{Gr}(V) \to \mathrm{Gr}_m(V),\ F_H = (F^i_H) \mapsto F_{\bar{H}} = (F^i_{\bar{H}} := \mathrm{pr}^i_{(m)} \circ F^i_H). \tag{5}$$

We end this subsection by discussing group actions on the space of quasi-gradations. For this, we need to introduce new notation, which will be used also later in the paper. Recall that if $U := \sum_i U^i$ and $W := \sum_j W^j$ are graded vector spaces, then $U \wedge W := \sum_i (U \wedge W)^i$ and $\mathrm{Hom}(U,W) = \sum_i \mathrm{Hom}^i(U,W)$ are graded as well, where $(U \wedge W)^i := \sum_{j+r=i} U^j \wedge W^r$ and $\mathrm{Hom}^i(U,W) := \sum_j \mathrm{Hom}(U^j, W^{j+i})$. For any $A \in \mathrm{Hom}(U,W)$, we denote by $A^i \in \mathrm{Hom}^i(U,W)$ its degree i homogeneous component. In particular, the vector subspaces

$$\mathfrak{gl}^j(\mathfrak{m}) := \{A \in \mathfrak{gl}(\mathfrak{m}),\ A(\mathfrak{m}^i) \subset \mathfrak{m}^{i+j},\ \forall i\}$$

define a gradation of $\mathfrak{gl}(\mathfrak{m})$. This is a Lie algebra gradation: $[\mathfrak{gl}^j(\mathfrak{m}), \mathfrak{gl}^r(\mathfrak{m})] \subset \mathfrak{gl}^{j+r}(\mathfrak{m})$, for any j, r. Consider the subalgebra $\mathfrak{gl}_m(\mathfrak{m}) := \sum_{i\geq m} \mathfrak{gl}^i(\mathfrak{m})$ and

$$\mathrm{GL}_m(\mathfrak{m}) := \{B \in \mathrm{GL}(\mathfrak{m}) :\ B = \mathrm{Id} + A,\ A \in \mathfrak{gl}_m(\mathfrak{m})\}$$

the Lie group with Lie algebra $\mathfrak{gl}_m(\mathfrak{m})$. For $m \geq 2$, $GL_m(\mathfrak{m})$ is a normal subgroup of $GL_1(\mathfrak{m})$. Any class $[A] \in GL_1(\mathfrak{m})/GL_m(\mathfrak{m})$ is determined by the homogeneous components of A up to degree $m-1$.

Theorem 6. *i) The group $GL_1(\mathfrak{m})$ acts simply transitively on $\mathrm{Gr}(V)$, by $FA := F \circ A$, for any $F \in \mathrm{Gr}(V)$ and $A \in GL_1(\mathfrak{m})$, and the orbits of the subgroup $GL_m(\mathfrak{m})$ are the fibers of the natural map $\Pi^m : \mathrm{Gr}(V) \to \mathrm{Gr}_m(V)$ defined by (??).*

ii) The map Π^m induces an isomorphism between the orbit space $\mathrm{Gr}(V)/\mathrm{GL}_m(\mathfrak{m})$ and $\mathrm{Gr}_m(V)$.

iii) The quotient group $\mathrm{GL}_1(\mathfrak{m})/GL_m(\mathfrak{m})$ acts simply transitively on $\mathrm{Gr}_m(V)$, by

$$(F_{\bar{H}}[A])^i(x) := \sum_{j=0}^{m-1} f_{j+i,m} F_{\bar{H}}^{j+i}(A^j(x)),\ \forall x \in \mathfrak{m}^i, \tag{6}$$

where $F_{\bar{H}} \in \mathrm{Gr}_m(V)$, $[A] \in \mathrm{GL}_1(\mathfrak{m})/GL_m(\mathfrak{m})$ and $f_{j+i} : \mathrm{gr}^{j+i}_{(m)}(V) \to \mathrm{gr}^i_{(m)}(V)$ are the natural maps.

Proof. Claim i) is easy, claim ii) follows from claim i) and the surjectivity of Π^m. We now prove iii). We define an action of $GL_1(\mathfrak{m})/GL_m(\mathfrak{m})$ on $\mathrm{Gr}_m(V)$ by $\Pi^m(F_H)[A] := \Pi^m(F_H \circ A)$, for any $F_H \in \mathrm{Gr}(V)$ and $[A] \in GL_1(\mathfrak{m})/GL_m(\mathfrak{m})$. It is easy to check that it is a well-defined, simply transitive action. We now prove that it is given by (??). To simplify notation, let $F_{\bar{H}} := \Pi^m(F_H)$. For any $x \in \mathfrak{m}^i$,

$$(F_{\bar{H}}[A])(x) = \Pi^m(F_H A)(x) = \mathrm{pr}^i_{(m)}(F_H \circ A)(x) = \sum_{j=0}^{m-1} (\mathrm{pr}^i_{(m)} \circ F_H^{j+i})(A^j(x)). \tag{7}$$

Consider the left hand side of (??): for any fixed $0 \leq j \leq m-1$,

$$f_{j+i,m} F_{\bar{H}}^{j+i}(A^j(x)) = f_{j+i,m} \circ \mathrm{pr}^{j+i}_{(m)} \circ F_H^{j+i}(A^j(x)) = (\mathrm{pr}^i_{(m)} \circ F_H^{j+i})(A^j(x)), \tag{8}$$

where we used (??) and $f_{j+i,m} \circ \mathrm{pr}^{j+i}_{(m)} = \mathrm{pr}^i_{(m)}|_{V_{j+i}}$. Relation (??) follows from (??) and (??). □

2.3 Tanaka prolongation of a non-positively graded Lie algebra

Let $\mathfrak{m}_0 = \sum_{i=-k}^{-1} \mathfrak{m}^i + \mathfrak{g}^0$ be a non-positively graded Lie algebra, with Lie bracket $[\cdot,\cdot]$. We always assume that the negative part $\mathfrak{m} := \sum_{i=-k}^{-1} \mathfrak{m}^i$ of $\mathfrak{m}_0$ is fundamental, i.e. generated by $\mathfrak{m}^{-1}$. We define inductively a sequence of vector spaces $\mathfrak{g}^r$ ($r \geq 1$), such that, with the notation $\mathfrak{m}_f := \mathfrak{m} + \sum_{r=0}^{f} \mathfrak{g}^r$ ($f \geq 0$), $\mathfrak{g}^r \subset \mathfrak{gl}^r(\mathfrak{m}_{r-1})$. First, let

$$\mathfrak{g}^1 := \{A \in \mathfrak{gl}^1(\mathfrak{m}_0), \quad A[x,y] = [A(x),y] + [x, A(y)],\ \forall x, y \in \mathfrak{m}\}.$$

Next, suppose that $\mathfrak{g}^s \subset \mathfrak{gl}^s(\mathfrak{m}_{s-1})$ are known for any $1 \leq s \leq r$. We define

$$\mathfrak{g}^{r+1} := \{A \in \mathfrak{gl}^{r+1}(\mathfrak{m}_r),\ A[x,y] = [A(x),y] + [x,A(y)] \quad \forall x, y \in \mathfrak{m}\}. \tag{9}$$

In (??) $[\cdot,\cdot] : \mathfrak{m} \times \mathfrak{m}_r \to \mathfrak{m}_r$ extends the Lie bracket $[\cdot,\cdot]$ of $\mathfrak{m}$ and

$$[x,z] = -[z,x] = -z(x), \quad x \in \mathfrak{m},\ z \in \mathfrak{g}^s \subset \mathfrak{gl}^s(\mathfrak{m}_{s-1}),\ s \leq r. \tag{10}$$

Remark that any $A \in \mathfrak{g}^r \subset \mathfrak{gl}^r(\mathfrak{m}_{r-1})$ annihilates the non-negative part $\sum_{i=0}^{r-1} \mathfrak{g}^i$ of $\mathfrak{m}_{r-1}$ and we may consider $\mathfrak{g}^r \subset \mathrm{Hom}^r(\mathfrak{m}, \mathfrak{m}_{r-1})$.

Theorem 7. *[?] The vector space* $(\mathfrak{m}_0)^\infty := \mathfrak{m}_0 + \sum_{r\geq 1} \mathfrak{g}^r$ *has the structure of a graded Lie algebra (called the* **Tanaka prolongation of** $\mathfrak{m}_0$*), with the following Lie bracket:*

i) the Lie bracket of two elements from $\mathfrak{m}_0$ *is their Lie bracket in the Lie algebra* $\mathfrak{m}_0$*;*

ii) the Lie bracket $[x, z]$*, where* $x \in \mathfrak{m}$ *and* $z \in \mathfrak{g}^s$ $(s \geq 1)$ *is given by (??).*

iii) the Lie bracket $[f_1, f_2]$*, where* $f_1 \in \sum_{r\geq 0} \mathfrak{g}^r$ *and* $f_2 \in \sum_{r\geq 1} \mathfrak{g}^r$ *is defined by induction by the condition*

$$[f_1, f_2](x) = [f_1(x), f_2] + [f_1, f_2(x)], \quad f_1 \in \mathfrak{g}^{r_1},\ f_2 \in \mathfrak{g}^{r_2},\ x \in \mathfrak{m}.$$

Definition 8. *Let* $G \subset GL(\mathfrak{m})$ *be a Lie group with Lie algebra* $\mathfrak{g}^0$*. The group* $G^l := \mathrm{Id} + \mathfrak{g}^l \subset \mathrm{End}(\mathfrak{m}_{l-1})$ *with group operation* $(\mathrm{Id} + A)(\mathrm{Id} + B) := \mathrm{Id} + A + B$ *(for any* $A, B \in \mathfrak{g}^l$*) is called the* l**-Tanaka prolongation of** G.

We denote by $G^l GL_{l+1}(\mathfrak{m}_{l-1})$ the subgroup of $GL(\mathfrak{m}_{l-1})$ of all automorphisms of the form $\mathrm{Id} + A^l + A_{l+1}$, where $A^l \in \mathfrak{g}^l \subset \mathfrak{gl}^l(\mathfrak{m}_{l-1})$ and $A_{l+1} \in \mathfrak{gl}_{l+1}(\mathfrak{m}_{l-1})$. The Tanaka prolongation G^l is isomorphic to the quotient of $G^l GL_{l+1}(\mathfrak{m}_{l-1})$ by the normal subgroup $GL_{l+1}(\mathfrak{m}_{l-1})$.

2.4 G-structures

Notation 9. We begin by fixing notation. Our actions on manifolds are always right actions. If a Lie group G acts on a manifold P, we denote by $R_g : P \to P$, $p \to pg$ the action of $g \in G$ on P and by $(\xi^a)^P$ (or simply ξ^a) the fundamental vector field on P generated by $a \in \mathrm{Lie}(G) = \mathfrak{g}$. For any $u \in P$, $a, b \in \mathfrak{g}$ and $g \in G$, $(R_g)_*(\xi^a_u) = (\xi^{\mathrm{Ad}(g^{-1})(a)})_{ug}$ (see e.g. [?], p. 51) and $[\xi^a, \xi^b] = \xi^{[a,b]}$ (see e.g. [?], p. 41). In particular, if $\pi : P \to M$ is a principal G-bundle and $\nu : \mathfrak{g} \to T^v P$ the vertical parallelism, $\nu(a)_u = \nu_u(a) := \xi^a_u$, then $\nu_{ug} = (R_g)_* \circ \nu_u \circ \mathrm{Ad}(g)$.

Let $\pi : P \to M$ be a G-structure with structure group $G \subset GL(V)$. Any $u \in P$ is a frame $u : V \to T_p M$. The action of $g \in G$ on u is given by $ug := u \circ g$. Let $\theta \in \Omega^1(P, V)$ be the soldering form of π, defined by $\theta_u(X) := (u^{-1} \circ \pi_*)(X)$, for any $X \in T_u P$. It is well-known that θ is G-equivariant (see e.g. [?], p. 309-310):

$$R_g^*(\theta) = g^{-1} \circ \theta,\ L_{\xi^A}(\theta) = -A \circ \theta, \quad g \in G,\ A \in \mathfrak{g} \subset \mathfrak{gl}(V). \tag{11}$$

Let ρ be a connection on the G-structure $\pi : P \to M$.

Definition 10. *A* ρ**-twisted** *vector field is a vector field* X_a *on* P *(where* $a \in V$*), such that* $(X_a)_u \in T_u P$ *is the* ρ*-horisontal lift of* $u(a) \in T_{\pi(u)} M$*, for any* $u \in P$.

According to [?] (see p. 356),

$$(R_g)_* X_a = X_{g^{-1}(a)},\ [\xi^B, X_a] = X_{B(a)}, \quad g \in G,\ B \in \mathfrak{g} \subset \mathfrak{gl}(V),\ a \in V. \tag{12}$$

Definition 11. *The* ρ**-torsion function** *is the function*

$$t^\rho : P \to \mathrm{Hom}(\Lambda^2(V), V),\ t^\rho_u(a \wedge b) := (d\theta)_u(X_a, X_b), \quad u \in P,\ a \wedge b \in \Lambda^2(V). \tag{13}$$

Remark that $\theta(X_a) = a$ is constant, for any $a \in V$, and

$$t^\rho_u(a \wedge b) = -\theta_u([X_a, X_b]) = -(u^{-1} \circ \pi_*)([X_a, X_b]_u), \quad u \in P,\ a \wedge b \in \Lambda^2(V). \tag{14}$$

Theorem 12. *i) The torsion function t^ρ is G-equivariant:*

$$t^\rho_{ug}(a \wedge b) = g^{-1} t^\rho_u(g(a) \wedge g(b)), \quad u \in P,\ g \in G,\ a \wedge b \in \Lambda^2(V). \tag{15}$$

ii) For any other connection ρ' on π,

$$t^{\rho'}_u(a \wedge b) = t^\rho_u(a \wedge b) - A(b) + B(a),\ u \in P,\ a \wedge b \in \Lambda^2(V). \tag{16}$$

Above $A, B \in \mathfrak{g} \subset \mathrm{End}(V)$ are given by $\xi^A_u := (X'_a)_u - (X_a)_u$, $\xi^B_u := (X'_b)_u - (X_b)_u$, where X_a, X_b (respectively, X'_a, X'_b) are the ρ-twisted (respectively, the ρ'-twisted) vector fields determined by a, b.

2.5 Tanaka structures

2.5.1 Filtrations of the Lie algebra of vector fields

Let $TM = \mathcal{D}_{-k} \supset \mathcal{D}_{-k+1} \cdots \supset \mathcal{D}_l$ ($l \geq -1$) be a flag of distributions on a manifold M. For any $p \in M$, let $\mathrm{gr}^i(T_pM) = \mathcal{D}^i_p := (\mathcal{D}_i)_p / (\mathcal{D}_{i+1})_p$, $\mathrm{gr}(T_pM) := \sum_i \mathrm{gr}^i(T_pM)$ and $(\mathrm{gr}^i)^{\mathcal{D}} : \mathcal{D}_i \to \mathrm{gr}^i(TM)$ the natural projection. We assume that the non-positive part $\{\mathcal{D}_i,\ i \leq 0\}$ defines a filtration

$$\mathfrak{X}(M) = \Gamma(\mathcal{D}_{-k}) \supset \Gamma(\mathcal{D}_{-k+1}) \supset \cdots \supset \Gamma(\mathcal{D}_0)$$

of the Lie algebra $\mathfrak{X}(M)$ of vector fields on M. Then, for any $p \in M$, $\mathrm{gr}^{<0}(T_pM) := \sum_{i<0} \mathrm{gr}^i(T_pM)$ is a graded Lie algebra, with Lie bracket $\{\cdot,\cdot\}_p$ (or just $\{\cdot,\cdot\}$ when p is understood) induced by the Lie bracket of vector fields. It is called the **symbol algebra of** $\{\mathcal{D}_i\}$ at p. The following lemma will be useful and can be checked directly.

Lemma 13. *Let $f : N \to M$ be a smooth map of constant rank and $\{\mathcal{D}^M_i,\ i \leq 0\}$ a flag of distributions which defines a filtration of the Lie algebra $\mathfrak{X}(M)$. Then $\{\mathcal{D}^N_i = (f_*)^{-1}(\mathcal{D}^M_i),\ i \leq 0\}$ defines a filtration of the Lie algebra $\mathfrak{X}(N)$. For any $X \in \Gamma(\mathcal{D}^N_i)$, $Y \in \Gamma(\mathcal{D}^N_j)$ with $i, j < 0$ and $p \in N$,*

$$(\mathrm{gr}^{i+j})^{\mathcal{D}^M} f_*([X,Y]_p) = \{(\mathrm{gr}^i)^{\mathcal{D}^M} f_*(X_p), (\mathrm{gr}^j)^{\mathcal{D}^M} f_*(Y_p)\}_{f(p)}.$$

2.5.2 Definition of Tanaka structures

Let $\mathfrak{m}_0 = \sum_{i=-k}^{-1} \mathfrak{m}^i + \mathfrak{g}^0$ be a non-positively graded Lie algebra, $(\mathfrak{m}_0)^\infty = \mathfrak{m}_0 + \sum_{i\geq 1} \mathfrak{g}^i$ its Tanaka prolongation and $(\mathfrak{m}_l)^{\geq 0} = \sum_{i=0}^{l} \mathfrak{g}^i$ the non-negative part of $\mathfrak{m}_l = \mathfrak{m}_0 + \sum_{i=1}^{l} \mathfrak{g}^i$.

Definition 14. *A flag of distributions $TM = \mathcal{D}_{-k} \supset \mathcal{D}_{-k+1} \cdots \supset \mathcal{D}_l$ ($l \geq -1$) is a* **filtration of type** $\mathfrak{m}_l$ *if the following conditions are satisfied:*

i) for any $i, j \leq 0$, $[\Gamma(\mathcal{D}_i), \Gamma(\mathcal{D}_j)] \subset \Gamma(\mathcal{D}_{i+j})$;

ii) for any $p \in M$, there is an isomorphism $u^-_p : \mathfrak{m} \to \mathrm{gr}^{<0}(T_pM)$ of graded Lie algebras;

iii) for any $p \in M$, there is a **canonical** *isomorphism $\nu_p : (\mathfrak{m}_l)^{\geq 0} \to \mathrm{gr}^{\geq 0}(T_pM)$ of graded vector spaces.*

The isomorphism $u := u^-_p \oplus \nu_p : \mathfrak{m}_l \to \mathrm{gr}(T_pM)$ is called a **graded frame at** p.

The group $\mathrm{Aut}(\mathfrak{m})$ of automorphisms of the graded Lie algebra $\mathfrak{m}$ acts simply transitively on the set $\mathbb{P}_p$ of graded frames at $p \in P$. We denote by $\pi : \mathbb{P} \to M$ the principal bundle of graded frames. It has structure group $\mathrm{Aut}(\mathfrak{m})$.

Definition 15. *Let $\{\mathcal{D}_i, \ -k \le i \le l\}$ be a filtration of type $\mathfrak{m}_l$ on a manifold M and $G \subset \mathrm{Aut}(\mathfrak{m})$ a Lie subgroup of $\mathrm{Aut}(\mathfrak{m})$. A* **Tanaka G-structure of type $\mathfrak{m}_l$** *on M is a principal G-subbundle $\pi_G : P_G \to M$ of the bundle $\pi : \mathbb{P} \to M$ of graded frames.*

The notion of automorphism of a Tanaka structure is defined in a natural way:

Definition 16. *An automorphism of a Tanaka G-structure $(\mathcal{D}_i, \pi_G : P_G \to M)$ of type $\mathfrak{m}_l$ is a diffeomorphism $f : M \to M$ with the following properties:*

i) it preserves the flag of distributions $\mathcal{D}_i$ (and induces a map $f_* : \mathrm{gr}(TM) \to \mathrm{gr}(TM)$);

ii) for any graded frame $u : \mathfrak{m}_l \to \mathrm{gr}(T_pM)$ from P_G, the composition $f_* \circ u : \mathfrak{m}_l \to \mathrm{gr}(T_{f(p)}M)$ is also a graded frame from P_G.

Let $(\mathcal{D}_i, \pi_G)$ be a Tanaka G-structure of type $\mathfrak{m} = \sum_{i=-k}^{-1} \mathfrak{m}^i$ and $\mathfrak{g}^0 := \mathrm{Lie}(G)$. Since $\mathfrak{g}^0 \subset \mathrm{Der}^0(\mathfrak{m})$, $\mathfrak{m}(\mathfrak{g}^0) := \mathfrak{m} + \mathfrak{g}^0$ is a graded Lie algebra: its Lie bracket $[\cdot, \cdot]$ extends the Lie brackets of $\mathfrak{m}$ and $\mathfrak{g}^0$ and $[a, b] = -[b, a] = -b(a)$, for any $a \in \mathfrak{m}$ and $b \in \mathfrak{g}^0 \subset \mathrm{End}(\mathfrak{m})$. Let $\mathfrak{m}(\mathfrak{g}^0)^\infty := \mathfrak{m}(\mathfrak{g}^0) + \sum_{l \ge 1} \mathfrak{g}^l$ be the Tanaka prolongation of $\mathfrak{m}(\mathfrak{g}^0)$.

Definition 17. *The Tanaka G-structure $(\mathcal{D}_i, \pi_G)$ of type $\mathfrak{m}$ has (finite)* **order** $\bar{l}$ *if $\bar{l}$ is the minimal number such that $\mathfrak{g}^{\bar{l}+1} = 0$.*

3 Statement of the main results

In this paper we aim to prove the following statements:

Theorem 18. *Let $(\mathcal{D}_i, \pi_G : P_G \to M)$ be a Tanaka G-structure of type $\mathfrak{m} = \sum_{i=-k}^{-1} \mathfrak{m}^i$, $\mathfrak{m}(\mathfrak{g}^0)^\infty = \mathfrak{m} + \sum_{i \ge 0} \mathfrak{g}^i$ the Tanaka prolongation of $\mathfrak{m}(\mathfrak{g}^0) = \mathfrak{m} + \mathfrak{g}^0$ (where $\mathfrak{g}^0 = \mathrm{Lie}(G)$) and $G^n = \mathrm{Id} + \mathfrak{g}^n$ the n-prolongation of G. There is a sequence of principal G^n-bundles $\bar{\pi}^{(n)} : \bar{P}^{(n)} \to \bar{P}^{(n-1)}$ $(n \ge 1)$, with the following properties:*

A) *The base $\bar{P}^{(n-1)}$ has a Tanaka $\{e\}$-structure of type $\mathfrak{m}_{n-1}$. This means that there is a flag of distributions $\{T\bar{P}^{(n-1)} = \bar{\mathcal{D}}_{-k}^{(n-1)} \supset \cdots \supset \bar{\mathcal{D}}_{n-1}^{(n-1)}\}$ which satisfies*

$$[\Gamma(\bar{\mathcal{D}}_i^{(n-1)}), \Gamma(\bar{\mathcal{D}}_j^{(n-1)})] \subset \Gamma(\bar{\mathcal{D}}_{i+j}^{(n-1)}), \ i, j \le 0,$$

and for any $\bar{H}^{n-1} \in \bar{P}^{(n-1)}$, there is a canonical graded vector space isomorphism

$$I_{\bar{H}^{n-1}} : \mathfrak{m}_{n-1} \to \mathrm{gr}(T_{\bar{H}^{n-1}} \bar{P}^{(n-1)})$$

of whose restriction to $\mathfrak{m}$ is a Lie algebra isomorphism onto $\mathrm{gr}^{<0}(T_{\bar{H}^{n-1}} \bar{P}^{(n-1)})$.

B) *The principal bundle $\bar{\pi}^{(n)}$ is the quotient of a G-structure $\tilde{\pi}^n : \tilde{P}^n \to \bar{P}^{(n-1)}$, with structure group $G^n GL_{n+1}(\mathfrak{m}_{n-1})$, by the normal subgroup $GL_{n+1}(\mathfrak{m}_{n-1})$. The G-structure $\tilde{\pi}^n$ is a subbundle of the bundle $\mathrm{Gr}(T\bar{P}^{(n-1)}) \to \bar{P}^{(n-1)}$ of adapted gradations of $T\bar{P}^{(n-1)}$ (the latter being a G-structure, whose frames are lifts of the graded frames $I_{\bar{H}^{n-1}}$, $\bar{H}^{n-1} \in \bar{P}^{(n-1)}$). In particular, $\bar{\pi}^{(n)} : \bar{P}^{(n)} = \tilde{P}^n / GL_{n+1}(\mathfrak{m}_{n-1}) \to \bar{P}^{(n-1)}$ is canonically isomorphic* *to a subbundle of the bundle* $\mathrm{Gr}_{n+1}(T\bar{P}^{(n-1)}) \to T\bar{P}^{(n-1)}$ *of* $(n+1)$*-quasi-gradations of* $T\bar{P}^{(n-1)}$.

C) *The torsion function* $t^{\tilde{\rho}}$ *of one (equivalently, any) connection* $\tilde{\rho}$ *on* $\tilde{\pi}^n$ *satisfies* $t^{\tilde{\rho}}_{H^n}(a \wedge b) \in (\mathfrak{m}_{n-1})_{i-1}$*, for any* $H^n \in \tilde{P}^n$ *and* $a \wedge b \in \mathfrak{m}^{-1} \wedge \mathfrak{g}^i$ $(0 \leq i \leq n-1)$*, and*

$$(t^{\tilde{\rho}}_{H^n})^0(a \wedge b) = -[a,b], \quad H^n \in \tilde{P}^n,\ a \wedge b \in \mathfrak{m}^{-1} \wedge (\sum_{i=0}^{n-1} \mathfrak{g}^i). \tag{17}$$

(In (??) $[a,b]$ *denotes the Lie bracket of* a *and* b *in the Lie algebra* $\mathfrak{m}(\mathfrak{g}_0)^\infty$*).*

We reobtain the final result of the Tanaka's prolongation procedure:

Theorem 19. *Let* $(\mathcal{D}_i, \pi_G : P_G \to M)$ *be a Tanaka* G*-structure of type* $\mathfrak{m} = \sum_{i=-k}^{-1}$ *and order* $\bar{l}$*. The* $\bar{l}$*-Tanaka prolongation* $\bar{P}^{(\bar{l})}$ *has a canonical* $\{e\}$*-structure. The automorphism group* $\mathrm{Aut}(\mathcal{D}_i, \pi_G)$ *of* $(\mathcal{D}_i, \pi_G)$ *is isomorphic to the automorphism group of this* $\{e\}$*-structure. It is a finite dimensional Lie group with* $\mathrm{dimAut}(\mathcal{D}_i, \pi_G) \leq \dim(M) + \sum_{i=0}^{\bar{l}} \mathfrak{g}^i$.

The remaining part of the paper is devoted to the proofs of Theorem **??** and **??**.

4 The first prolongation of a Tanaka structure

Let $(\mathcal{D}_i, \pi_G : P_G \to M)$ be a Tanaka G-structure of type $\mathfrak{m} = \mathfrak{m}^{-k} + \cdots + \mathfrak{m}^{-1}$. In this section we define the first principal bundle $\bar{\pi}^{(1)} : \bar{P}^{(1)} \to P$ from Theorem **??**.

4.1 The G-structure $\pi^1 : P^1 \to P$

To simplify notation, we denote by $P := P_G$ the total space of π_G. Let $\nu^0 : \mathfrak{g}^0 \to T^vP$ be the vertical parallelism of π_G, where $\mathfrak{g}^0 = \mathrm{Lie}(G)$. For any $i \leq -1$, let $\mathcal{D}_i^P := (\pi_G)_*^{-1}(\mathcal{D}_i)$ and $\mathcal{D}_0 := T^vP$ the tangent vertical bundle of π_G. The sequence

$$TP = \mathcal{D}^P_{-k} \supset \mathcal{D}^P_{-k+1} \supset \cdots \supset \mathcal{D}^P_{-1} \supset \mathcal{D}^P_0 \tag{18}$$

defines a filtration of the Lie algebra $\mathfrak{X}(P)$ of vector fields on P and the differential $(\pi_G)_*$ induces a symbol algebra isomorphism

$$(\pi_G)_* : \mathrm{gr}^{<0}(T_uP) \to \mathrm{gr}(T_pM), \quad u \in P,\ p = \pi_G(u). \tag{19}$$

The next proposition can be checked directly.

Proposition 20. *Any point* $u \in P$ *defines an isomorphism*

$$\hat{u} = (\pi_G)_*^{-1} \circ u + \nu^0_u : \mathfrak{m}_0 = \mathfrak{m} + \mathfrak{g}^0 \to \mathrm{gr}(T_uP) = \mathrm{gr}^{<0}(T_uP) + T^v_uP.$$

The set of isomorphisms $\{\hat{u},\ u \in P\}$ *is a Tanaka* $\{e\}$*-structure of type* $\mathfrak{m}_0$ *on* P.

From Theorem **??** (applied to gradations), any gradation $H = \{H^i\}$ of T_uP adapted to the filtration (**??**) determines a frame

$$F_H : \mathfrak{m}_0 \to T_uP, \quad F_H := \widehat{H} \circ \hat{u}, \tag{20}$$

which lifts the graded frame $\hat{u} : \mathfrak{m}_0 \to \mathrm{gr}(T_uP)$ (for the definition of $\widehat{H}$, see the comments before Proposition **??**). For any $a \in \mathfrak{g}^0$, $F_H(a) = \widehat{H}((\xi^a)^P_u) = (\xi^a)^P_u$. From Theorem **??** i) we obtain:

Proposition 21. *The principal bundle $\pi^1 : P^1 \to P$ of adapted gradations of TP is a G-structure with structure group $GL_1(\mathfrak{m}_0)$. It consists of all frames of T_uP (for any $u \in P$) which are lifts of the canonical graded frame $\hat{u} : \mathfrak{m}_0 \to \mathrm{gr}(T_uP)$.*

4.2 The action of G on P^1

In this subsection we construct an action of G on P^1 which lifts the action of G on the total space $P = P_G$ of the principal G-bundle π_G. For any $g \in G$, $R_g : P \to P$ preserves the filtration (??) and induces a map $(R_g)_* : \mathrm{gr}(T_uP) \to \mathrm{gr}(T_{ug}P)$, for any $u \in P$. Let

$$\rho : G \to \mathrm{Aut}(\mathfrak{m}_0),\ \rho(g)(a+b) := g(a) + \mathrm{Ad}(g)(b), \quad g \in G,\ a \in \mathfrak{m},\ b \in \mathfrak{g}^0. \tag{21}$$

Proposition 22. *i) For any $u \in P$ and $g \in G$, the frames $\hat{u}$ and $\widehat{ug}$ from Proposition* **??** *are related by*

$$\widehat{ug} = (R_g)_* \circ \hat{u} \circ \rho(g) : \mathfrak{m}_0 \to \mathrm{gr}(T_{ug}P). \tag{22}$$

ii) There is an action of G on P^1, which associates to any frame $F_H : \mathfrak{m}_0 \to T_uP$ from P^1 and $g \in G$ the frame

$$F_Hg := (R_g)_* \circ F_H \circ \rho(g) : \mathfrak{m}_0 \to T_{ug}P. \tag{23}$$

iii) For any $a \in \mathfrak{g}^0$, the fundamental vector field $(\xi^a)^{P^1}$ of the above action of G on P^1, generated by a, is π^1-projectable and $(\pi^1)_*(\xi^a)^{P^1} = (\xi^a)^P$.

Proof. Claim i) follows from the definition of $\hat{u}$, $\widehat{ug}$, and $\nu^0_{ug} = (R_g)_* \circ \nu^0_u \circ \mathrm{Ad}(g)$. For claim ii), one checks that $F_Hg \in P^1$, i.e. is a lift of $\widehat{ug}$ (direct computation, which uses that F_H is a lift of $\hat{u}$ and that ρ is gradation preserving). Claim iii) follows from $R_g \circ \pi^1 = \pi^1 \circ R_g$ (where we use the same notation R_g for the actions of $g \in G$ on P^1 and P). □

Lemma 23. *The soldering form $\theta^1 \in \Omega^1(P^1, \mathfrak{m}_0)$ of π^1 is G-equivariant:*

$$(R_g)^*\theta^1 = \rho(g^{-1}) \circ \theta^1, \quad L_{(\xi^a)^{P^1}}(\theta^1) = -\rho_*(a) \circ \theta^1, \quad g \in G,\ a \in \mathfrak{g}^0. \tag{24}$$

Proof. From the definition of θ^1 and $R_g \circ \pi^1 = \pi^1 \circ R_g$, we obtain, for any $X_H \in T_HP^1$,

$$\begin{aligned}((R_g)^*\theta^1)(X_H) &= \theta^1((R_g)_*X_H) = (F_Hg)^{-1}(\pi^1 \circ R_g)_*(X_H)\\ &= (\rho(g^{-1}) \circ (F_H)^{-1} \circ (R_{g^{-1}})_* \circ (\pi^1 \circ R_g)_*)(X_H)\\ &= (\rho(g^{-1}) \circ (F_H)^{-1} \circ (\pi^1)_*)(X_H) = (\rho(g^{-1}) \circ \theta^1)(X_H).\end{aligned}$$

The second relation (**??**) is the infinitesimal version of the first. □

4.3 The torsion function t^ρ of π^1

Let ρ be a connection on the G-structure $\pi^1 : P^1 \to P$. In this section we study the properties of the torsion function t^ρ, in connection with the gradation of $\mathfrak{m}_0$. Let $\{X_a,\ a \in \mathfrak{m}_0\}$ be the family of ρ-twisted vector fields on P^1 (recall Section **??**). For any $a \in \mathfrak{m}_0$, $(X_a)_H \in T_HP^1$ is the ρ-horisontal lift of $F_H(a) \in T_pP$ (where $\pi^1(H) = p$); when $a \in \mathfrak{g}^0$, $X_a \in \mathfrak{X}(P^1)$ is the ρ-horisontal lift of $(\xi^a)^P \in \mathfrak{X}(P)$.

Proposition 24. *The function $t^\rho : P^1 \to \mathrm{Hom}(\Lambda^2(\mathfrak{m}_0), \mathfrak{m}_0)$ has only components of non-negative homogeneous degree.*

Proof. For any $i \leq 0$, let $\mathcal{D}_i^{P^1} := (\pi^1)_*^{-1}(\mathcal{D}_i^P)$. Since for any $H \in P^1$, $F_H : \mathfrak{m}_0 \to T_uP$ preserves filtrations, $X_a \in \Gamma(\mathcal{D}_i^{P^1})$, for any $a \in (\mathfrak{m}_0)^i$ ($i \leq 0$). Similarly, $X_b \in \Gamma(\mathcal{D}_j^{P^1})$ for any $b \in (\mathfrak{m}_0)^j$ ($j \leq 0$). The sequence $\{\mathcal{D}_i^{P^1},\ i \leq 0\}$ defines a filtration of the Lie algebra $\mathfrak{X}(P^1)$. It follows that $[X_a, X_b] \in \Gamma(\mathcal{D}_{i+j}^{P^1})$ and $t_H^{\rho}(a,b) = -(F_H)^{-1}(\pi^1)_*([X_a,X_b])_H)$ belongs to $(\mathfrak{m}_0)_{i+j}$. □

Theorem 25. *i) For any $a \wedge b \in \Lambda^2(\mathfrak{g}^0)$ and $H \in P^1$, $t_H^{\rho}(a \wedge b) = -[a,b]$.*
ii) For any $a \wedge b \in \Lambda^2(\mathfrak{m}_0)$ and $H \in P^1$, $(t_H^{\rho})^0(a,b) = -[a,b]$.

Proof. Let $a, b \in \mathfrak{g}^0$. Then X_a, X_b are the ρ-horisontal lifts of the fundamental vector fields $(\xi^a)^P$ and $(\xi^b)^P$ on P. Thus, $[X_a, X_b]$ is π^1-projectable and $(\pi^1)_*[X_a,X_b] = [(\xi^a)^P, (\xi^b)^P] = (\xi^{[a,b]})^P$. We obtain

$$t_H^{\rho}(a,b) = -(F_H)^{-1}(\pi^1)_*([X_a,X_b]_H) = -(F_H)^{-1}(\xi^{[a,b]})^P = -[a,b].$$

Claim i) follows.

For claim ii), we distinguish two cases: I) $a, b \in \mathfrak{m}$; II) $a \in \mathfrak{g}^0$, $b \in \mathfrak{m}$.

Let $a \in \mathfrak{m}^i$ and $b \in \mathfrak{m}^j$ ($i, j < 0$). Then $X_a \in \Gamma(\mathcal{D}_i^{P^1})$, $X_b \in \Gamma(\mathcal{D}_j^{P^1})$ and $[X_a, X_b] \in \Gamma(\mathcal{D}_{i+j}^{P^1})$. Being a lift of $\hat{u} : \mathfrak{m}_0 \to \mathrm{gr}(T_uP)$, the frame $F_H : \mathfrak{m}_0 \to T_uP$ is filtration preserving and satisfies

$$(\mathrm{gr}^s)^{\mathcal{D}^P} \circ F_H|_{(\mathfrak{m}_0)_s} = \hat{u} \circ \pi_{(\mathfrak{m}_0)^s}|_{(\mathfrak{m}_0)_s},\ \pi_{(\mathfrak{m}_0)^s} \circ (F_H)^{-1}|_{\mathcal{D}_s^P} = \hat{u}^{-1} \circ (\mathrm{gr}^s)^{\mathcal{D}^P}. \tag{25}$$

Using $(\pi^1)_*([X_a,X_b]_H) \in (\mathcal{D}_{i+j}^P)_u$ and the second relation (**??**), we obtain

$$(t_H^{\rho})^0(a \wedge b) = -\pi_{(\mathfrak{m}_0)^{i+j}}(F_H)^{-1}(\pi^1)_*([X_a,X_b]_H) = -\hat{u}^{-1} \circ (\mathrm{gr}^{i+j})^{\mathcal{D}^P}(\pi^1)_*([X_a,X_b]_H). \tag{26}$$

On the other hand, from Lemma **??**, $(\pi^1)_*(X_H^a) = F_H(a)$, $(\pi^1)_*(X_H^b) = F_H(b)$ and the first relation (**??**), we obtain

$$(\mathrm{gr}^{i+j})^{\mathcal{D}^P}(\pi^1)_*([X_a,X_b]_H) = \{((\mathrm{gr}^i)^{\mathcal{D}^P} \circ F_H)(a), ((\mathrm{gr}^j)^{\mathcal{D}^P} \circ F_H)(b)\} = \{\hat{u}(a), \hat{u}(b)\}. \tag{27}$$

Using that $\hat{u} : \mathfrak{m} \to \mathrm{gr}^{<0}(T_uP)$ is a Lie algebra isomorphism, we deduce, from (**??**) and (**??**), that $(t_H^{\rho})^0(a \wedge b) = -[a,b]$, as needed.

It remains to consider $a \in \mathfrak{g}^0$ and $b \in \mathfrak{m}$. For this we use the action of G on P^1, defined in Subsection **??**. From Proposition **??**, $(\xi^a)^{P^1}$ is π^1-projectable and $(\pi^1)_*(\xi^a)^{P^1} = (\xi^a)^P$. Since $a \in \mathfrak{g}^0$, X_a is the ρ-horisontal lift of $(\xi^a)^P$. Therefore, the vector field $Y := X_a - (\xi^a)^{P^1}$ is π^1-vertical. We write

$$t_H^{\rho}(a \wedge b) = -\theta^1([X_a,X_b]_H) = -\theta^1([(\xi^a)^{P^1}, X_b]_H) - \theta^1([Y,X_b]_H). \tag{28}$$

We need to compute the right hand side of (**??**). From Lemma **??** and $\theta^1(X_b) = b$,

$$\theta^1([(\xi^a)^{P^1}, X_b]_H) = -(L_{(\xi^a)^{P^1}}\theta^1)_H(X_b) = \rho_*(a)(b) = a(b). \tag{29}$$

In order to compute $\theta^1([Y,X_b]_H)$, we write $Y = \sum_s f_s(\xi^{A_s})^{P^1}$, where f_s are functions on P^1 and $\{A_s\}$ is a basis of $\mathfrak{gl}_1(\mathfrak{m}_0)$. Then

$$\begin{aligned} \theta^1([Y,X_b]_H) &= -\sum_s \theta_H^1(X_b(f_s)(\xi^{A_s})^{P^1} + f_s[X_b,(\xi^{A_s})^{P^1}]) \\ &= -\sum_s f_s(H)\theta_H^1([X_b,(\xi^{A_s})^{P^1}]) = -\sum_s f_s(H) L_{(\xi^{A_s})^{P^1}}(\theta^1)(X_b) \\ &= \sum_s f_s(H)A_s(b), \end{aligned} \tag{30}$$

where in the second equality we used that $(\xi^{A_s})^{P^1}$ is π^1-vertical (hence annihilated by θ^1), in the third equality we used that $\theta^1(X_b) = b$ is constant and in the last equality we used the second relation (**??**). From (**??**), (**??**) and (**??**), we obtain

$$t^{\rho}_H(a \wedge b) = -a(b) - A(b), \quad a \in \mathfrak{g}^0 \subset \mathfrak{gl}(\mathfrak{m}),\ b \in \mathfrak{m}, \tag{31}$$

where $A = \sum_s f_s(H) A_s \in \mathfrak{gl}_1(\mathfrak{m}_0)$ is uniquely determined by $(X_a)_H - (\xi^a)^{P^1}_H = (\xi^A)^{P^1}_H$. Assume now that $b \in \mathfrak{m}^i$. From (**??**), $t^{\rho}_H(a \wedge b) \in (\mathfrak{m}_0)_i$ and (by projecting (**??**) onto $\mathfrak{m}^i$) $(t^{\rho}_H)^0(a, b) = -a(b) = -[a, b]$ as required. $\square$

4.4 Variation of the torsion t^{ρ} of π^1

Let ρ be a connection on $\pi^1 : P^1 \to P$.

Proposition 26. *i) The degree zero homogeneous component of* $t^{\rho} : P^1 \to \mathrm{Hom}(\mathfrak{g}^0 \wedge \mathfrak{m}, \mathfrak{m}_0)$ *is independent of* ρ.

ii) The degree zero and one homogeneous components of $t^{\rho} : P^1 \to \mathrm{Hom}(\Lambda^2(\mathfrak{m}), \mathfrak{m})$ *are independent of* ρ.

Proof. Let ρ' be another connection on π^1. For any $a \in \mathfrak{m}_0$, the ρ and ρ'-twisted vector fields X_a and X'_a, at a point $H \in P^1$, are related by $(X'_a)_H = (X_a)_H + (\xi^A)^{P^1}_H$, where $A \in \mathfrak{gl}_1(\mathfrak{m}_0)$ (the Lie algebra of the structure group $GL_1(\mathfrak{m}_0)$ of π^1). Similarly, for any $b \in \mathfrak{m}_0$, $(X'_b)_H = (X_b)_H + (\xi^B)^{P^1}_H$, where $B \in \mathfrak{gl}_1(\mathfrak{m}_0)$. From Theorem **??**,

$$t^{\rho'}_H(a \wedge b) = t^{\rho}_H(a \wedge b) - A(b) + B(a). \tag{32}$$

Let $a \in \mathfrak{g}^0$ and $b \in \mathfrak{m}^i$ $(i < 0)$. Then $B(a) = 0$, $\deg(A(b)) \geq i + 1$. We obtain that the $\mathfrak{m}^i$-component of $A(b) - B(a)$ vanishes. Claim i) follows. Let $a \in \mathfrak{m}^i$ and $b \in \mathfrak{m}^j$ with $i, j < 0$. Then $\deg(A(b)) \geq j + 1 > i + j + 1$ and $\deg(B(a)) \geq i + 1 > i + j + 1$. We obtain that the $\mathfrak{m}^{i+j}$ and $\mathfrak{m}^{i+j+1}$-components of $A(b) - B(a)$ vanish. Claim ii) follows. $\square$

We denote by $\mathrm{Tor}(\mathfrak{m}_0) := \mathrm{Hom}(\Lambda^2(\mathfrak{m}), \mathfrak{m}_0)$ the **space of torsions**. It is a graded vector space, with gradation $\mathrm{Tor}^m(\mathfrak{m}_0) = \sum_{i,j} \mathrm{Hom}(\mathfrak{m}^i \wedge \mathfrak{m}^j, (\mathfrak{m}_0)^{i+j+m})$. For any $H \in P^1$, we denote by $(t^{\rho}_H)^m$ the projection of t^{ρ}_H onto $\mathrm{Tor}^m(\mathfrak{m}_0)$.

Definition 27. *Let* ρ *be a connection on the* G*-structure* $\pi^1 : P^1 \to P$ *associated to the Tanaka structure* $\pi_G : P_G \to M$. *The function*

$$t^1 : P^1 \to \mathrm{Tor}^1(\mathfrak{m}_0), \quad P^1 \ni H \to t^1_H := (t^{\rho}_H)^1 \in \mathrm{Tor}^1(\mathfrak{m}_0)$$

is called the **torsion function of the Tanaka structure** $(\mathcal{D}_i, \pi_G)$.

Proposition 28. *The torsion function is independent of the choice of* ρ. *It is given by:*

$$t^1_H(a, b) = -\pi_{\mathfrak{m}^{i+j+1}}(F_H)^{-1}(\pi^1)_*([X_a, X_b]_H), \quad H \in P^1,\ a \in \mathfrak{m}^i,\ b \in \mathfrak{m}^j,\ (i, j < 0), \tag{33}$$

where X_a, $X_b \in \mathcal{X}(P^1)$ *are* ρ*-twisted vector fields.*

Proof. The first claim follows from Proposition **??** ii). Relation (**??**) follows from (**??**). $\square$

Proposition 29. *For any $H \in P^1$ and $A = \mathrm{Id} + A_1 \in GL_1(\mathfrak{m}_0)$,*

$$t^1_{HA} = t^1_H + \partial A.$$

Above, $\partial A \in \mathrm{Hom}(\Lambda^2\mathfrak{m}, \mathfrak{m}_0)$ is given by

$$(\partial A)(a \wedge b) := A^1_1([a,b]) - [A^1_1(a), b] - [a, A^1_1(b)],\ a \wedge b \in \Lambda^2(\mathfrak{m}). \tag{34}$$

Proof. The inverse A^{-1} of A is of the form $A^{-1} = \mathrm{Id} + \tilde{A}_1$, where $\tilde{A}_1 \in \mathfrak{gl}_1(\mathfrak{m}_0)$ and $\tilde{A}^1_1 = -A^1_1$. We choose a connection ρ on π^1. From Theorem ??, for any $a \wedge b \in \Lambda^2(\mathfrak{m})$,

$$\begin{aligned} t^\rho_{HA}(a \wedge b) &= A^{-1} t^\rho_H(A(a) \wedge A(b)) \\ &= t^\rho_H(a \wedge b) + t^\rho_H(a \wedge A_1(b)) + t^\rho_H(A_1(a) \wedge b) + t^\rho_H(A_1(a) \wedge A_1(b)) \\ &+ \tilde{A}_1\left(t^\rho_H(a \wedge b) + t^\rho_H(a \wedge A_1(b)) + t^\rho_H(A_1(a) \wedge b) + t^\rho_H(A_1(a) \wedge A_1(b))\right). \end{aligned}$$

Let $a \in \mathfrak{m}^i$ and $b \in \mathfrak{m}^j$ $(i, j < 0)$. Projecting the above equality to $\mathfrak{m}^{i+j+1}$ and using that t^ρ has only components of non-negative homogeneous degree (see Proposition ??), we obtain

$$t^1_{HA}(a \wedge b) = t^1_H(a \wedge b) + (t^\rho_H)^0(a \wedge A^1_1(b)) + (t^\rho_H)^0(A^1_1(a) \wedge b) + \tilde{A}^1_1 t^0_H(a \wedge b).$$

Using $\tilde{A}^1_1 = -A^1_1$ and $(t^\rho_H)^0(a \wedge b) = -[a,b]$, for any $a, b \in \mathfrak{m}_0$ (see Theorem ??), we obtain our claim. □

4.5 The first prolongation

Let $(\mathcal{D}_i, \pi : P_G \to M)$ be a Tanaka G-structure of type $\mathfrak{m} = \sum_{i=-k}^{-1} \mathfrak{m}^i$ and $t^1 : P^1 \to \mathrm{Tor}^1(\mathfrak{m}_0)$ its torsion function (see Definition ??). Let

$$\partial : \mathfrak{gl}_1(\mathfrak{m}_0) \to \mathrm{Tor}^1(\mathfrak{m}_0),\ (\partial A)(a \wedge b) = A^1([a,b]) - [A^1(a), b] - [a, A^1(b)],\ a \wedge b \in \Lambda^2(\mathfrak{m}). \tag{35}$$

Fix a complement W of $\partial(\mathfrak{gl}_1(\mathfrak{m}_0))$ in $\mathrm{Tor}^1(\mathfrak{m}_0)$.

Proposition 30. *The bundle $\tilde{\pi}^1 : \tilde{P}^1 := (t^1)^{-1}(W) \to P$ is a G-structure with structure group $G^1 GL_2(\mathfrak{m}_0)$. The torsion function $t^{\tilde{\rho}}$ of any connection $\tilde{\rho}$ on $\tilde{\pi}^1$ satisfies $t^{\tilde{\rho}}_H(a \wedge b) \in \mathfrak{m}^{-1} + \mathfrak{g}^0$, for any $H \in \tilde{P}^1$ and $a \wedge b \in \mathfrak{m}^{-1} \wedge \mathfrak{g}^0$, and*

$$(t^{\tilde{\rho}}_H)^0(a \wedge b) = -[a,b],\ a \wedge b \in \mathfrak{m}^{-1} \wedge \mathfrak{g}^0.$$

Proof. The first claim follows from Proposition ?? and $\mathrm{Ker}(\partial) = \mathfrak{gl}_2(\mathfrak{m}_0) + \mathfrak{g}^1$. The second claim follows from Proposition ?? and Theorem ?? (extend $\tilde{\rho}$ to a connection on π^1). □

Let $\bar{P}^{(1)} := \tilde{P}^1 / GL_2(\mathfrak{m}_0)$. The map $\bar{\pi}^{(1)} : \bar{P}^{(1)} \to P$ induced by $\tilde{\pi}^1$ is a principal bundle with structure group G^1.

Definition 31. *The principal G^1-bundle $\bar{\pi}^{(1)} : \bar{P}^{(1)} \to P$ is called the* **first prolongation of the Tanaka structure** *$(\mathcal{D}_i, \pi_G)$.*

The next proposition concludes the first induction step from the proof of Theorem ??.

Proposition 32. *The principal bundle $\bar{\pi}^{(1)} : \bar{P}^{(1)} \to P$ satisfies properties A), B) and C) from Theorem ??. In particular, it is canonically isomorphic to a subbundle of the bundle $\mathrm{Gr}_2(TP) \to P$ of 2-quasi-gradations of TP.*

Proof. From Proposition ??, property A) is satisfied. Properties B) and C) follow from the definition of $\bar{\pi}^{(1)}$ and Proposition ??. The statement about quasi-gradations follows from Theorem ?? ii). □

5 The G-structure $\pi^{n+1} : P^{n+1} \to \bar{P}^{(n)}$

We now assume that the principal bundles $\bar{\pi}^{(i)} : \bar{P}^{(i)} \to \bar{P}^{(i-1)}$ from Theorem **??** are given, for any $i \leq n$. Our goal is to construct the principal bundle $\bar{\pi}^{(n+1)} : \bar{P}^{(n+1)} \to \bar{P}^{(n)}$ from this theorem. In particular, $\bar{P}^{(n)}$ needs to have a Tanaka $\{e\}$-structure of type $\mathfrak{m}_n$. This is induced from $\bar{P}^{(n-1)}$, as follows.

Lemma 33. *The manifold $\bar{P}^{(n)}$ has a Tanaka $\{e\}$-structure of type $\mathfrak{m}_n$. The flag of distributions $\{\bar{\mathcal{D}}_i^{(n)},\ -k \leq i \leq n\}$ of this Tanaka structure is $\bar{\mathcal{D}}_i^{(n)} := (\bar{\pi}^{(n)})_*^{-1}(\bar{\mathcal{D}}_i^{(n-1)})$ $(-k \leq i \leq n-1)$ and $\bar{\mathcal{D}}_n^{(n)} := T^v \bar{P}^{(n)} = \mathrm{Ker}(\bar{\pi}^{(n)})_*$. For any $\bar{H}^n \in \bar{P}^{(n)}$, the canonical graded frame*

$$I_{\bar{H}^n} : \mathfrak{m}_n = \mathfrak{m}_{n-1} + \mathfrak{g}^n \to \mathrm{gr}(T_{\bar{H}^n}\bar{P}^{(n)}) = \sum_{-k \leq i \leq n-1} \mathrm{gr}^i(T_{\bar{H}^n}\bar{P}^{(n)}) + T^v_{\bar{H}^n}\bar{P}^{(n)}$$

is given by

$$I_{\bar{H}^n}|_{\mathfrak{m}_{n-1}} := (\bar{\pi}^{(n)})_*^{-1} \circ I_{\bar{H}^{n-1}}, \quad I_{\bar{H}^n}|_{\mathfrak{g}^n} := \nu^n_{\bar{H}^n}, \tag{36}$$

where

$$(\bar{\pi}^{(n)})_* : \sum_{-k \leq i \leq n-1} \mathrm{gr}^i(T_{\bar{H}^n}\bar{P}^{(n)}) \to \mathrm{gr}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)})$$

is the isomorphism induced by the differential of $\bar{\pi}^{(n)}$, $\bar{H}^{n-1} = \bar{\pi}^{(n)}(\bar{H}^n)$, and $\nu^n_{\bar{H}^n} : \mathfrak{g}^n \to T^v_{\bar{H}^n}\bar{P}^{(n)}$ is the vertical parallelism of $\bar{\pi}^{(n)}$.

Proof. The only non-trivial fact to check is that $I_{\bar{H}^n} : \mathfrak{m} \to \mathrm{gr}^{<0}(T_{\bar{H}^n}\bar{P}^{(n)})$ preserves Lie brackets. For this, we use that both $(\bar{\pi}^{(n)})_* : \mathrm{gr}^{<0}(T_{\bar{H}^n}\bar{P}^{(n)}) \to \mathrm{gr}^{<0}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)})$ and $I_{\bar{H}^{n-1}} : \mathfrak{m} \to \mathrm{gr}^{<0}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)})$ have this property. $\square$

In the next sections we shall consider various adapted gradations and quasi-gradations of $T\bar{P}^{(n)}$ or $T\bar{P}^{(n-1)}$. They are always considered with respect to the filtrations of the Tanaka structures of these manifolds.

5.1 Definition and basic properties of π^{n+1}

An important role in the prolongation procedure plays a G-structure $\pi^{n+1} : P^{n+1} \to \bar{P}^{(n)}$ which we are going to define in this subsection. Let $\bar{H}^n \in \bar{P}^{(n)}$ and $H^{n+1} = \{(H^{n+1})^i,\ -k \leq i \leq n\}$ an adapted gradation of $T_{\bar{H}^n}\bar{P}^{(n)}$. It projects to an adapted gradation $(\bar{\pi}^{(n)})_*(H^{n+1}) := \{(\bar{\pi}^{(n)})_*(H^{n+1})^i,\ -k \leq i \leq n-1\}$ of $T_{\bar{H}^{n-1}}\bar{P}^{(n-1)}$ (remark that $(H^{n+1})^n = T^v\bar{P}^{(n)}$ projects trivially to $T_{\bar{H}^{n-1}}\bar{P}^{(n-1)}$). The adapted gradations H^{n+1} and $(\bar{\pi}^{(n)})_*(H^{n+1})$ define frames which lift the canonical graded frames $I_{\bar{H}^n}$ and $I_{\bar{H}^{n-1}}$ respectively (see Theorem **??**, applied to gradations and lifts):

$$\begin{aligned} F_{H^{n+1}} &= \widehat{H^{n+1}} \circ I_{\bar{H}^n} : \mathfrak{m}_n \to T_{\bar{H}^n}\bar{P}^{(n)} \\ F_{(\bar{\pi}^{(n)})_*(H^{n+1})} &= \widehat{(\bar{\pi}^{(n)})_*(H^{n+1})} \circ I_{\bar{H}^{n-1}} : \mathfrak{m}_{n-1} \to T_{\bar{H}^{n-1}}\bar{P}^{(n-1)}. \end{aligned} \tag{37}$$

As usual, $F^i_{H^{n+1}} := F_{H^{n+1}}|_{(\mathfrak{m}_n)^i}$ $(i \leq n)$ and similarly $F^i_{(\bar{\pi}^{(n)})_*(H^{n+1})} := F_{(\bar{\pi}^{(n)})_*(H^{n+1})}|_{(\mathfrak{m}_{n-1})^i}$ $(i \leq n-1)$. Recall that $\bar{P}^{(n)} \subset \mathrm{Gr}_{n+1}(T\bar{P}^{(n-1)})$.

Definition 34. *The manifold P^{n+1} is the set of all adapted gradations H^{n+1} of $T_{\bar{H}^n}\bar{P}^{(n)}$ (for any $\bar{H}^n \in \bar{P}^{(n)}$), whose projection $(\bar{\pi}^{(n)})_*(H^{n+1})$ to $T_{\bar{H}^{n-1}}\bar{P}^{(n-1)}$ is compatible with the quasi-gradation $\bar{H}^n \in \mathrm{Gr}_{n+1}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)})$ (where $\bar{H}^{n-1} := \bar{\pi}^{(n)}(\bar{H}^n)$). The map $\pi^{n+1} : P^{n+1} \to \bar{P}^{(n)}$ is the natural projection.*

More precisely, we set

$$P^{n+1} = \cup_{\bar{H}^n \in \bar{P}^{(n)}} \{H^{n+1} \in \mathrm{Gr}(T_{\bar{H}^n}\bar{P}^{(n)}),\ \Pi^{n+1}(\bar{\pi}^{(n)})_*(H^{n+1}) = \bar{H}^n\}$$

where $\Pi^{n+1} : \mathrm{Gr}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)}) \to \mathrm{Gr}_{n+1}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)})$ is the map (**??**). Using the first relation (**??**), we identify any $H^{n+1} \in P^{n+1}$ with the associated frame $F_{H^{n+1}}$. The next lemma describes P^{n+1} as a submanifold of the frame manifold of $\bar{P}^{(n)}$. In Lemma **??** ii) below the map $F_{\bar{H}^n}$ is the $(n+1)$-lift of $I_{\bar{H}^{n-1}}$ determined by $\bar{H}^n \in \mathrm{Gr}_{n+1}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)})$ (according to Theorem **??**):

$$F_{\bar{H}^n} = (F^i_{\bar{H}^n}), \quad F^i_{\bar{H}^n} = (\widehat{\bar{H}^n})^i \circ I_{\bar{H}^{n-1}} : (\mathfrak{m}_{n-1})^i \to \mathrm{gr}^i_{(n+1)}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)}),\ -k \leq i \leq n-1. \tag{38}$$

Lemma 35. *i) Let $H^{n+1} = \{(H^{n+1})^i,\ -k \leq i \leq n\}$ be an adapted gradation of $T_{\bar{H}^n}\bar{P}^{(n)}$ and $(\bar{\pi}^{(n)})_*(H^{n+1})$ its projection to $T_{\bar{H}^{n-1}}\bar{P}^{(n-1)}$. The associated frames $F_{(\bar{\pi}^{(n)})_*(H^{n+1})}$ and $F_{H^{n+1}}$ defined by (**??**) are related by*

$$F_{(\bar{\pi}^{(n)})_*(H^{n+1})} = (\bar{\pi}^{(n)})_* \circ F_{H^{n+1}}|_{\mathfrak{m}_{n-1}}. \tag{39}$$

ii) The fiber of π^{n+1} over $\bar{H}^n \in \bar{P}^{(n)}$ consists of all $H^{n+1} \in \mathrm{Gr}(T_{\bar{H}^n}\bar{P}^{(n)})$ whose associated frame $F_{H^{n+1}}$ satisfies: for any $-k \leq i \leq n-1$ and $x \in (\mathfrak{m}_{n-1})^i$,

$$\mathrm{pr}^i_{(n+1)}(\bar{\pi}^{(n)})_* F^i_{H^{n+1}}(x) = F^i_{\bar{H}^n}(x), \tag{40}$$

where $\mathrm{pr}^i_{(n+1)} : (\bar{\mathcal{D}}^{(n-1)}_i)_{\bar{H}^{n-1}} \to \mathrm{gr}^i_{(n+1)}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)})$ is the natural projection. In particular, $(\bar{\pi}^{(n)})_* F_{H^{n+1}} = F_{\bar{H}^n}$ on $(\mathfrak{m}_{n-1})_{-1}$.

Proof. From the definitions of $\widehat{H^{n+1}}$ and $\widehat{(\bar{\pi}^{(n)})_*(H^{n+1})}$,

$$\widehat{(\bar{\pi}^{(n)})_*(H^{n+1})} \circ (\bar{\pi}^{(n)})_*|_{\mathrm{gr}^{\leq n-1}(T_{\bar{H}^n}\bar{P}^{(n)})} = (\bar{\pi}^{(n)})_* \circ \widehat{H^{n+1}}|_{\mathrm{gr}^{\leq n-1}(T_{\bar{H}^n}\bar{P}^{(n)})}. \tag{41}$$

Relation (**??**) follows from (**??**), (**??**) and $I_{\bar{H}^n}|_{\mathfrak{m}_{n-1}} = (\bar{\pi}^{(n)})^{-1}_* \circ I_{\bar{H}^{n-1}}$.

For claim ii), let $H^{n+1} \in \mathrm{Gr}(T_{\bar{H}^n}\bar{P}^{(n)})$. Then $H^{n+1} \in P^{n+1}$ if and only if $(\bar{\pi}^{(n)})_*(H^{n+1}) \in \mathrm{Gr}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)})$ is compatible with the quasi-gradation $\bar{H}^n \in \mathrm{Gr}_{n+1}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)})$. From Proposition **??** ii), this condition is equivalent to

$$\mathrm{pr}^i_{(n+1)} \circ \widehat{(\bar{\pi}^{(n)})_*(H^{n+1})}^{\,i} = (\widehat{\bar{H}^n})^i,\ i \leq n-1. \tag{42}$$

Composing (**??**) with $I_{\bar{H}^{n-1}}$ and using the relations (**??**) and (**??**), we obtain that (**??**) is equivalent to $\mathrm{pr}^i_{(n+1)} \circ F^i_{(\bar{\pi}^{(n)})_*(H^{n+1})} = F^i_{\bar{H}^n}$, or, from (**??**), to (**??**). $\square$

Below any $A \in \mathrm{Hom}(\sum_{i=0}^{n-1} \mathfrak{g}^i, \mathfrak{g}^n)$ acts on $\mathfrak{m}_n$, by annihilating $\mathfrak{m}$ and $\mathfrak{g}^n$.

Proposition 36. *The projection $\pi^{n+1} : P^{n+1} \to \bar{P}^{(n)}$ is a G-structure with structure group $\bar{G} := \mathrm{Id} + \mathfrak{gl}_{n+1}(\mathfrak{m}_n) + \mathrm{Hom}(\sum_{i=0}^{n-1} \mathfrak{g}^i, \mathfrak{g}^n)$.*

Proof. Let H^{n+1}, $\tilde{H}^{n+1} \in (\pi^{n+1})^{-1}(\bar{H}^n)$ be two adapted gradations of $T_{\bar{H}^n}\bar{P}^{(n)}$, whose projections to $T_{\bar{H}^{n-1}}\bar{P}^{(n-1)}$ are compatible with the quasi-gradation $\bar{H}^n \in \mathrm{Gr}_{n+1}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)})$. From Lemma **??** ii), for any $x \in (\mathfrak{m}_{n-1})^i$, $i \leq n-1$,

$$F^i_{H^{n+1}}(x) - F^i_{\tilde{H}^{n+1}}(x) \in (\bar{\pi}^{(n)})^{-1}_*(\bar{\mathcal{D}}^{(n-1)}_{i+n+1})_{\bar{H}^{n-1}} = (\bar{\mathcal{D}}^{(n)}_{i+n+1})_{\bar{H}^n} + T^v_{\bar{H}^n}\bar{P}^{(n)}. \tag{43}$$

Note that $T^v_{\bar{H}^n}\bar{P}^{(n)} \subset (\bar{\mathcal{D}}^{(n)}_{i+n+1})_{\bar{H}^n}$ when $i \leq -1$ and $(\bar{\mathcal{D}}^{(n)}_{i+n+1})_{\bar{H}^n} = 0$ when $i \geq 0$. Also,

$$F^n_{H^{n+1}} = F^n_{\tilde{H}^{n+1}} : (\mathfrak{m}_n)^n = \mathfrak{g}^n \to (\bar{\mathcal{D}}^{(n)}_n)_{\bar{H}^n} = T^v_{\bar{H}^n}\bar{P}^{(n)} \tag{44}$$

is the vertical parallelism of $\bar{\pi}^{(n)}$. From relations (**??**) and (**??**) we obtain $\mathrm{Id} + A := F^{-1}_{H^{n+1}} \circ F_{\tilde{H}^{n+1}} \in \bar{G}$. □

5.2 An action of $G^nGL_{n+1}(\mathfrak{m}_{n-1})$ on P^{n+1}

In this subsection we define an action of $G^nGL_{n+1}(\mathfrak{m}_{n-1})$ on P^{n+1}, naturally related to the action of G^n on the total space $\bar{P}^{(n)}$ of the principal G^n-bundle $\bar{\pi}^{(n)}$. Consider the group homomorphism

$$\mathrm{Pr} : G^nGL_{n+1}(\mathfrak{m}_{n-1}) \to G^n, \quad g = \mathrm{Id} + A^n + A_{n+1} \to \mathrm{Pr}(g) := \bar{g} = \mathrm{Id} + A^n.$$

Let $\rho^n : G^nGL_{n+1}(\mathfrak{m}_n) \to \mathrm{Aut}(\mathfrak{m}_n)$ be the trivial extension to $\mathfrak{m}_n = \mathfrak{m}_{n-1} + \mathfrak{g}^n$ of the natural (left) action of $G^nGL_{n+1}(\mathfrak{m}_{n-1}) \subset GL(\mathfrak{m}_{n-1})$ on $\mathfrak{m}_{n-1}$. We define an action of $G^nGL_{n+1}(\mathfrak{m}_{n-1})$ on the frame manifold of $\bar{P}^{(n)}$: for any $g \in G^nGL_{n+1}(\mathfrak{m}_{n-1})$ and frame $F : \mathfrak{m}_n \to T_{\bar{H}^n}\bar{P}^{(n)}$,

$$Fg := (R_{\bar{g}})_* \circ F \circ \rho^n(g) : \mathfrak{m}_n \to T_{\bar{H}^n\bar{g}}\bar{P}^{(n)}. \tag{45}$$

Proposition 37. *The action (**??**) preserves P^{n+1} and*

$$(\pi^{n+1})_*((\xi^a)^{P^{n+1}}) = (\xi^{\bar{a}})^{\bar{P}^{(n)}}, \quad \forall a \in \mathfrak{g}^n + \mathfrak{gl}_{n+1}(\mathfrak{m}_{n-1}). \tag{46}$$

*(In (**??**) $\bar{a} \in \mathfrak{g}^n$ denotes the $\mathfrak{g}^n$-component of a).*

Proof. Let $H^{n+1} \in P^{n+1}$ and $F_{H^{n+1}} : \mathfrak{m}_n \to T_{\bar{H}^n}\bar{P}^{(n)}$ the associated frame. We need to prove that for any $g \in G^nGL_{n+1}(\mathfrak{m}_{n-1})$, the frame $F_{H^{n+1}}g$ related to $F_{H^{n+1}}$ as in (**??**), belongs to P^{n+1}, i.e. satisfies the following conditions:

I) it is a lift of $I_{\bar{H}^n\bar{g}} : \mathfrak{m}_n \to \mathrm{gr}(T_{\bar{H}^n\bar{g}}\bar{P}^{(n)})$, i.e. is filtration preserving and

$$((\mathrm{gr}^i)^{\bar{\mathcal{D}}^{(n)}} \circ (F_{H^{n+1}}g))(x) = (I_{\bar{H}^n\bar{g}} \circ \pi_{(\mathfrak{m}_n)^i})(x), \; x \in (\mathfrak{m}_n)_i, \; i \leq n-1. \tag{47}$$

(This means that $F_{H^{n+1}}g$ is the frame associated to an adapted gradation of $T_{\bar{H}^n\bar{g}}\bar{P}^{(n)}$).

II) the adapted gradation from I) belongs to P^{n+1}, i.e. (from Lemma **??**),

$$\mathrm{pr}^i_{(n+1)}(\bar{\pi}^{(n)})_*F^i_{H^{n+1}g}(x) = F^i_{\bar{H}^n\bar{g}}(x), \; \forall x \in (\mathfrak{m}_{n-1})^i, \; i \leq n-1.$$

Since $G^nGL_{n+1}(\mathfrak{m}_{n-1}) \subset GL_1(\mathfrak{m}_n)$ and $(R_{\bar{g}})_* : T_{\bar{H}^{n-1}}\bar{P}^{(n)} \to T_{\bar{H}^{n-1}\bar{g}}\bar{P}^{(n)}$ preserve filtrations, $F_{H^{n+1}}g$ preserves filtrations as well. Using the definition of $F_{H^{n+1}}g$, that $R_{\bar{g}}$

preserves filtrations, $(R_{\bar{g}^{-1}})_* \circ I_{\bar{H}^n \bar{g}} = I_{\bar{H}^n}$ (which follows from (**??**) and the fact that $\mathfrak{g}^n$ is abelian), we obtain that (**??**) is equivalent to

$$((\mathrm{gr}^i)^{\bar{\mathcal{D}}^{(n)}} \circ F_{H^{n+1}})(\rho^n(g)(x)) = (I_{\bar{H}^n} \circ \pi_{(\mathfrak{m}_n)^i})(x),\ x \in (\mathfrak{m}_n)_i,\ i \leq n. \tag{48}$$

Using that $\rho^n(g)(x) \in (\mathfrak{m}_n)_i$ and $F_{H^{n+1}}$ lifts $I_{\bar{H}^n}$, we obtain that (**??**) is equivalent to $\pi_{(\mathfrak{m}_n)^i}(\rho^n(g)(x) - x) = 0$, which holds from the definition of ρ^n. Condition I) is proved.

Condition II) can be checked in a similar way, using

$$F^i_{\bar{H}^n \bar{g}}(x) = F^i_{\bar{H}^n}(x) + (f_{i+n,n+1} \circ F^{i+n}_{\bar{H}^n})(A^n x),\ x \in (\mathfrak{m}_{n-1})^i,\ i \leq n-1$$

where $A^n := \bar{g} - \mathrm{Id} \in \mathfrak{g}^n$ and $f_{i+n,n+1} : \mathrm{gr}^{i+n}_{(n+1)}(T\bar{P}^{(n-1)}) \to \mathrm{gr}^i_{(n+1)}(T\bar{P}^{(n-1)})$ is the natural map (see Theorem **??** ii)). We proved that (**??**) defines an action on P^{n+1}. Relation (**??**) follows from $\pi^{n+1} \circ R_g = R_{\bar{g}} \circ \pi^{n+1}$, for any $g \in G^n GL_{n+1}(\mathfrak{m}_{n-1})$. □

Let $\theta^{n+1} : TP^{n+1} \to \mathfrak{m}_n$ be the soldering form of the G-structure π^{n+1}:

$$\theta^{n+1}(X) = (F_{H^{n+1}})^{-1}((\pi^{n+1})_* X), \quad \forall X \in T_{H^{n+1}} P^{n+1}.$$

From relation (**??**), it is $\bar{G}$-equivariant. The next lemma shows that θ^{n+1} is equivariant also with respect to the actions ρ^n and (**??**) of $G^n GL_{n+1}(\mathfrak{m}_{n-1})$ on $\mathfrak{m}_n$ and P^{n+1} respectively.

Lemma 38. *For any $g \in G^n GL_{n+1}(\mathfrak{m}_{n-1})$ and $a \in \mathfrak{g}^n + \mathfrak{gl}_{n+1}(\mathfrak{m}_{n-1})$,*

$$(R_g)^*(\theta^{n+1}) = \rho^n(g^{-1}) \circ \theta^{n+1}, \quad L_{(\xi^a)^{P^{n+1}}}(\theta^{n+1}) = -(\rho^n)_*(a) \circ \theta^{n+1}. \tag{49}$$

Proof. Like in the proof of Lemma **??**, for any $g \in G^n GL_{n+1}(\mathfrak{m}_{n-1})$,

$$(R_g)^*(\theta^{n+1})(X_{H^{n+1}}) = \theta^{n+1}((R_g)_*(X_{H^{n+1}})) = (F_{H^{n+1}} g)^{-1}((\pi^{n+1} \circ R_g)_*(X_{H^{n+1}})).$$

From $F_{H^{n+1}} g = (R_{\bar{g}})_* \circ F_{H^{n+1}} \circ \rho^n(g)$ and $\pi^{n+1} \circ R_g = R_{\bar{g}} \circ \pi^{n+1}$, we obtain the first relation (**??**). The second relation (**??**) is the infinitesimal version of the first.

□

6 The torsion function of π^{n+1}

In this section we prove the following theorem.

Theorem 39. *Let ρ be a connection on the G-structure π^{n+1} and t^ρ its torsion function.*

i) Then $t^\rho : P^{n+1} \to \mathrm{Hom}((\mathfrak{m}^{-1} + \mathfrak{g}^n) \wedge \mathfrak{m}_n, \mathfrak{m}_n)$ has only homogeneous components of non-negative reduced degree, i.e. for any $H^{n+1} \in P^{n+1}$ and $-k \leq i \leq n$,

$$t^\rho_{H^{n+1}}(\mathfrak{m}^{-1} \wedge (\mathfrak{m}_n)^i) \subset (\mathfrak{m}_n)_{i-1},\ t^\rho_{H^{n+1}}(\mathfrak{g}^n \wedge (\mathfrak{m}_n)^i) \subset (\mathfrak{m}_n)_{\min\{n+i,n\}}.$$

ii) For any $H^{n+1} \in P^{n+1}$,

$$(t^\rho_{H^{n+1}})^0(a \wedge b) = -[a, b],\ \forall a \wedge b \in \mathfrak{m}^{-1} \wedge \mathfrak{m}_n + \mathfrak{g}^n \wedge \mathfrak{m}. \tag{50}$$

We divide the proof of the above theorem into three parts (Subsections **??**, **??** and **??**), according to the $\mathrm{Hom}(\mathfrak{m}^{-1} \wedge \mathfrak{m}, \mathfrak{m}_n)$, $\mathrm{Hom}(\mathfrak{m}^{-1} \wedge (\sum_{i=0}^{n-1} \mathfrak{g}^i), \mathfrak{m}_n)$ and $\mathrm{Hom}(\mathfrak{g}^n \wedge \mathfrak{m}_n, \mathfrak{m}_n)$-valued components of t^ρ. Along the proof we shall use the following notation: for any $a \in \mathfrak{m}_n$, the ρ-twisted vector field on P^{n+1} determined by a will be denoted by X^{n+1}_a; for any a, b belonging to $\mathfrak{m}$ or $\mathfrak{g}^i$, $[a, b]$ will always denote (as in the statement of Theorem **??** above) their Lie bracket in the Tanaka prolongation $\mathfrak{m}(\mathfrak{g}^0)^\infty$.

6.1 The $\mathrm{Hom}(\mathfrak{m}^{-1} \wedge \mathfrak{m}, \mathfrak{m}_n)$-valued component

Proposition 40. *The torsion function $t^\rho : P^{n+1} \to \mathrm{Hom}(\mathfrak{m}^{-1} \wedge \mathfrak{m}, \mathfrak{m}_n)$ has only homogeneous components of non-negative degree. For any $H^{n+1} \in P^{n+1}$,*

$$(t^\rho_{H^{n+1}})^0(a \wedge b) = -[a, b], \ \forall a \wedge b \in \mathfrak{m}^{-1} \wedge \mathfrak{m}. \tag{51}$$

Proof. The argument is similar to the proof of Proposition **??** and Theorem **??** ii). The sequence $\mathcal{D}_i^{n+1} := (\pi^{n+1})_*^{-1} \bar{\mathcal{D}}_i^{(n)}$ $(i \le 0)$ is a filtration of $\mathfrak{X}(P^{n+1})$. For any $a \in \mathfrak{m}^{-1}$ and $b \in \mathfrak{m}^i$, $X_a^{n+1} \in \Gamma(\mathcal{D}^{n+1}_{-1})$, $X_b^{n+1} \in \Gamma(\mathcal{D}_i^{n+1})$ and $[X_a^{n+1}, X_b^{n+1}] \in \Gamma(\mathcal{D}^{n+1}_{i-1})$. Since $F_{H^{n+1}} : \mathfrak{m}_n \to T_{\bar{H}^n}\bar{P}^{(n)}$ is filtration preserving, we obtain that $t^\rho_{H^{n+1}}(a \wedge b) = -(F_{H^{n+1}})^{-1}(\pi^{n+1})_*([X_a^{n+1}, X_b^{n+1}]) \in (\mathfrak{m}_n)_{i-1}$, which proves the first statement. We now prove (**??**). Since $F_{H^{n+1}}$ is a lift of $I_{\bar{H}^n} : \mathfrak{m}_n \to \mathrm{gr}(T_{\bar{H}^n}\bar{P}^{(n)})$, for any $-k \le s \le n$,

$$((\mathrm{gr}^s)^{\bar{\mathcal{D}}^{(n)}} \circ F_{H^{n+1}})|_{(\mathfrak{m}_n)_s} = I_{\bar{H}^n} \circ \pi_{(\mathfrak{m}_n)^s}|_{(\mathfrak{m}_n)_s}, \ (\pi_{(\mathfrak{m}_n)^s} \circ (F_{H^{n+1}})^{-1})|_{\bar{\mathcal{D}}_s^{(n)}} = (I_{\bar{H}^n})^{-1} \circ (\mathrm{gr}^s)^{\bar{\mathcal{D}}^{(n)}}.$$

From these relations and $(\pi^{n+1})_*([X_a^{n+1}, X_b^{n+1}]_{H^{n+1}}) \in (\bar{\mathcal{D}}^{(n)}_{i-1})_{\bar{H}^n}$, we obtain:

$$\begin{aligned}
(t^\rho_{H^{n+1}})^0(a \wedge b) &= -(\pi_{(\mathfrak{m}_n)^{i-1}} \circ (F_{H^{n+1}})^{-1} \circ (\pi^{n+1})_*)([X_a^{n+1}, X_b^{n+1}]_{H^{n+1}}) \\
&= -((I_{\bar{H}^n})^{-1} \circ (\mathrm{gr}^{i-1})^{\bar{\mathcal{D}}^{(n)}} \circ (\pi^{n+1})_*)([X_a^{n+1}, X_b^{n+1}]_{H^{n+1}}) \\
&= -(I_{\bar{H}^n})^{-1}\{((\mathrm{gr}^{-1})^{\bar{\mathcal{D}}^{(n)}} \circ F_{H^{n+1}})(a), ((\mathrm{gr}^i)^{\bar{\mathcal{D}}^{(n)}} \circ F_{H^{n+1}})(b)\} \\
&= -(I_{\bar{H}^n})^{-1}\{I_{\bar{H}^n}(a), I_{\bar{H}^n}(b)\} = -[a, b]
\end{aligned}$$

(we used Lemma **??** and that $I_{\bar{H}^n}|_{\mathfrak{m}} : \mathfrak{m} \to \mathrm{gr}^{<0}(T_{\bar{H}^n}\bar{P}^{(n)})$ is a Lie algebra isomorphism). □

6.2 The $\mathrm{Hom}(\mathfrak{m}^{-1} \wedge (\sum_{i=0}^{n-1} \mathfrak{g}^i), \mathfrak{m}_n)$-valued component of t^ρ

Since $\bar{\pi}^{(n)} : \bar{P}^{(n)} \to \bar{P}^{(n-1)}$ satisfies the conditions from Theorem **??**, it is the quotient of a G-structure $\tilde{\pi}^n : \tilde{P}^n \to \bar{P}^{(n-1)}$ with structure group $G^n GL_{n+1}(\mathfrak{m}_{n-1})$, by the normal subgroup $GL_{n+1}(\mathfrak{m}_{n-1})$. In particular, $\bar{P}^{(n)} = \tilde{P}^n / GL_{n+1}(\mathfrak{m}_{n-1})$ and the fundamental vector field $(\xi^c)^{\tilde{P}^n} \in \mathfrak{X}(\tilde{P}^n)$ generated by $c \in \mathfrak{g}^n + \mathfrak{gl}_{n+1}(\mathfrak{m}_{n-1})$ projects to the fundamental vector field $(\xi^{\bar{c}})^{\bar{P}^{(n)}} \in \mathfrak{X}(\bar{P}^{(n)})$ generated by $\bar{c} \in \mathfrak{g}^n$ (the $\mathfrak{g}^n$-component of c). Let $\tilde{\rho}$ be a connection on the G-structure $\tilde{\pi}^n$ and $X_a^n \in \mathfrak{X}(\tilde{P}^n)$ the $\tilde{\rho}$-twisted vector fields $(a \in \mathfrak{m}_{n-1})$. From (**??**), for any $A \in G^n GL_{n+1}(\mathfrak{m}_{n-1})$ and $c \in \mathfrak{g}^n + \mathfrak{gl}_{n+1}(\mathfrak{m}_{n-1})$,

$$(R_A)_*(X_a^n) = X^n_{A^{-1}(a)}, \quad [(\xi^c)^{\tilde{P}^n}, X_a^n] = X^n_{c(a)}. \tag{52}$$

The first relation (**??**) implies that X_a^n is $GL_{n+1}(\mathfrak{m}_{n-1})$-invariant, for any $a \in (\mathfrak{m}_{n-1})_{-1}$ (because $A|_{(\mathfrak{m}_{n-1})_{-1}} = \mathrm{Id}$, for any $A \in GL_{n+1}(\mathfrak{m}_{n-1})$) and descends to a vector field $\widehat{X}_a^n$ on $\bar{P}^{(n)}$. The following lemma collects the main properties of the vector fields $\widehat{X}_a^n$.

Lemma 41. *i) For any $a \in \mathfrak{m}^{-1}$ and $b \in \mathfrak{g}^i$ (with $0 \le i \le n-1$),*

$$[\widehat{X}_a^n, \widehat{X}_b^n] = \widehat{X}^n_{[a,b]} \ \mathrm{mod}(\bar{\mathcal{D}}_i^{(n)}). \tag{53}$$

ii) For any $c \in \mathfrak{g}^n \subset \mathfrak{gl}(\mathfrak{m}_{n-1})$, $a \in \mathfrak{m}^{-1}$ and $b \in \sum_{i=0}^{n-1} \mathfrak{g}^i$,

$$[(\xi^c)^{\bar{P}^{(n)}}, \widehat{X}_a^n] = \widehat{X}^n_{c(a)}, \ [(\xi^c)^{\bar{P}^{(n)}}, \widehat{X}_b^n] = 0. \tag{54}$$

iii) Let $H^{n+1} \in P^{n+1}$, $\bar{H}^n = \pi^{n+1}(H^{n+1}) \in \bar{P}^{(n)}$ and $a \in (\mathfrak{m}_{n-1})_{-1}$. Then

$$F_{H^{n+1}}(a) = (\widehat{X}_a^n)_{\bar{H}^n} \ \mathrm{mod}(T^v_{\bar{H}^n}\bar{P}^{(n)}). \tag{55}$$

Proof. i) Let $H^n \in \tilde{P}^n$, $F_{H^n} : \mathfrak{m}_{n-1} \to T_{\bar{H}^{n-1}}\bar{P}^{(n-1)}$ the associated frame and $a \in \mathfrak{m}^{-1}$, $b \in \mathfrak{g}^i$ with $0 \leq i \leq n-1$. From the property C) in Theorem **??**, we know that $t^{\tilde{\rho}}_{H^n}(a \wedge b) = -(F_{H^n})^{-1}(\tilde{\pi}^n)_*([X^n_a, X^n_b]_{H^n})$ belongs to $(\mathfrak{m}_{n-1})_{i-1}$ and its projection onto $(\mathfrak{m}_{n-1})^{i-1}$ is equal to $-[a,b]$. Using $(\tilde{\pi}^n)_*((X^n_{[a,b]})_{H^n}) = F_{H^n}([a,b])$ we obtain

$$(F_{H^n})^{-1}(\tilde{\pi}^n)_*\left([X^n_a, X^n_b]_{H^n} - (X^n_{[a,b]})_{H^n}\right) = (F_{H^n})^{-1}(\tilde{\pi}^n)_*([X^n_a, X^n_b]_{H^n}) - [a,b] \in (\mathfrak{m}_{n-1})_i.$$

Thus, $(\tilde{\pi}^n)_*([X^n_a, X^n_b]_{H^n} - (X^n_{[a,b]})_{H^n})$ belongs to $F_{H^n}((\mathfrak{m}_{n-1})_i) = (\bar{\mathcal{D}}^{(n-1)}_i)_{\bar{H}^{n-1}}$. But since $a, b, [a,b] \in (\mathfrak{m}_{n-1})_{-1}$, the vector fields X^n_a, X^n_b and $X^n_{[a,b]}$ project to $\bar{P}^{(n)}$ and

$$(\tilde{\pi}^n)_*([X^n_a, X^n_b]_{H^n} - (X^n_{[a,b]})_{H^n}) = (\bar{\pi}^{(n)})_*([\widehat{X}^n_a, \widehat{X}^n_b]_{\bar{H}^n} - (\widehat{X}^n_{[a,b]})_{\bar{H}^n}).$$

We deduce that

$$(\bar{\pi}^{(n)})_*([\widehat{X}^n_a, \widehat{X}^n_b]_{\bar{H}^{n-1}} - (\widehat{X}^n_{[a,b]})_{\bar{H}^{n-1}}) \in (\bar{\mathcal{D}}^{(n-1)}_i)_{\bar{H}^{n-1}},$$

which implies (**??**), because $(\bar{\pi}^{(n)})^{-1}_*(\bar{\mathcal{D}}^{(n-1)}_i) = \bar{\mathcal{D}}^{(n)}_i$.

ii) In order to prove (**??**), let $c \in \mathfrak{g}^n \subset \mathfrak{gl}^n(\mathfrak{m}_{n-1})$ and $a \in (\mathfrak{m}_{n-1})_{-1}$. The vector fields X^n_a, $X^n_{c(a)}$ and $(\xi^c)^{\tilde{P}^n}$ on $\tilde{P}^n$ project to the vector fields $\widehat{X}^n_a$, $\widehat{X}^n_{c(a)}$ and $(\xi^c)^{\bar{P}^{(n)}}$ on $\bar{P}^{(n)}$ (and $c(a) = 0$, $X^n_{c(a)} = 0$, for any $c \in \mathfrak{g}^n$ and $a \in (\mathfrak{m}_{n-1})_0$). Claim ii) follows by projecting the second relation (**??**) on $\bar{P}^{(n)}$.

iii) Let $H^{n+1} \in P^{n+1}$, $\bar{H}^n = \pi^{n+1}(H^{n+1}) \in \bar{P}^{(n)}$ and choose $H^n \in \tilde{P}^n$ which projects to $\bar{H}^n$. For any $a \in (\mathfrak{m}_{n-1})_{-1}$,

$$(\bar{\pi}^{(n)})_*((\widehat{X}^n_a)_{\bar{H}^n}) = (\tilde{\pi}^n)_*((X^n_a)_{H^n}) = F_{H^n}(a) = F_{\bar{H}^n}(a), \tag{56}$$

where in the last equality we used that $H^n \in (\tilde{\pi}^n)^{-1}(\bar{H}^{n-1}) \subset \mathrm{Gr}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)})$ is compatible with $\bar{H}^n \in \mathrm{Gr}_{n+1}(T_{\bar{H}^{n-1}}\bar{P}^{(n-1)})$ (in particular, $F_{H^n} = F_{\bar{H}^n}$ on $(\mathfrak{m}_{n-1})_{-1}$). On the other hand, since $H^{n+1} \in P^{n+1}$, $(\bar{\pi}^{(n)})_* F_{H^{n+1}}(a) = F_{\bar{H}^n}(a)$ (from Lemma **??** and $a \in (\mathfrak{m}_{n-1})_{-1}$). We obtain $(\bar{\pi}^{(n)})_*((\widehat{X}^n_a)_{\bar{H}^n}) = (\bar{\pi}^{(n)})_* F_{H^{n+1}}(a)$, which implies (**??**). □

Proposition 42. *The function $t^\rho : P^{n+1} \to \mathrm{Hom}(\mathfrak{m}^{-1} \wedge (\sum_{i=0}^{n-1}\mathfrak{g}^i), \mathfrak{m}_n)$ has only homogeneous components of non-negative degree. For any $H^{n+1} \in P^{n+1}$,*

$$(t^\rho_{H^{n+1}})^0(a \wedge b) = -[a,b], \quad a \wedge b \in \mathfrak{m}^{-1} \wedge (\sum_{i=0}^{n-1}\mathfrak{g}^i). \tag{57}$$

Proof. Let $a, b \in (\mathfrak{m}_{n-1})_{-1}$. From relation (**??**), $(\pi^{n+1})_*((X^{n+1}_a)_{H^{n+1}}) = F_{H^{n+1}}(a) = (\widehat{X}^n_a)_{\bar{H}^n}$ and similarly $(\pi^{n+1})_*((X^{n+1}_b)_{H^{n+1}}) = (\widehat{X}^n_b)_{\bar{H}^n}$ modulo $T^v_{\bar{H}^n}\bar{P}^{(n)}$. Therefore, there are $A, B \in \mathfrak{gl}_{n+1}(\mathfrak{m}_n) + \mathrm{Hom}(\sum_{i=0}^{n-1}\mathfrak{g}^i, \mathfrak{g}^n)$ (the Lie algebra of the structure group $\bar{G}$ of π^{n+1}) and $c, d \in \mathfrak{g}^n$ (the Lie algebra of the structure group of $\bar{\pi}^{(n)}$), such that

$$\begin{aligned}
(X^{n+1}_a)_{H^{n+1}} &= (\widetilde{\widehat{X}^n_a})_{H^{n+1}} + (\widetilde{(\xi^c)^{\bar{P}^{(n)}}})_{H^{n+1}} + (\xi^A)^{P^{n+1}}_{H^{n+1}},\\
(X^{n+1}_b)_{H^{n+1}} &= (\widetilde{\widehat{X}^n_b})_{H^{n+1}} + (\widetilde{(\xi^d)^{\bar{P}^{(n)}}})_{H^{n+1}} + (\xi^B)^{P^{n+1}}_{H^{n+1}}
\end{aligned} \tag{58}$$

(for a vector field $Z \in \mathfrak{X}(\bar{P}^{(n)})$, we denote by $\tilde{Z}$ its ρ-horisontal lift to P^{n+1}). Then

$$\begin{aligned}
t^\rho_{H^{n+1}}(a \wedge b) &= (d\theta^{n+1})_{H^{n+1}}(\widetilde{\widehat{X}^n_a} + \widetilde{(\xi^c)^{\bar{P}^{(n)}}} + (\xi^A)^{P^{n+1}}, \widetilde{\widehat{X}^n_b} + \widetilde{(\xi^d)^{\bar{P}^{(n)}}} + (\xi^B)^{P^{n+1}})\\
&= (\widetilde{\widehat{X}^n_a} + \widetilde{(\xi^c)^{\bar{P}^{(n)}}} + (\xi^A)^{P^{n+1}})_{H^{n+1}}(f) - (\widetilde{\widehat{X}^n_b} + \widetilde{(\xi^d)^{\bar{P}^{(n)}}} + (\xi^B)^{P^{n+1}})_{H^{n+1}}(g)\\
&- \theta^{n+1}([\widetilde{\widehat{X}^n_a} + \widetilde{(\xi^c)^{\bar{P}^{(n)}}} + (\xi^A)^{P^{n+1}}, \widetilde{\widehat{X}^n_b} + \widetilde{(\xi^d)^{\bar{P}^{(n)}}} + (\xi^B)^{P^{n+1}}]_{H^{n+1}}),
\end{aligned}$$

where

$$f(H^{n+1}) := \theta^{n+1}_{H^{n+1}}(\widetilde{\widehat{X}^n_b} + \widetilde{(\xi^d)^{\bar{P}(n)}} + (\xi^B)^{P^{n+1}}) = (F_{H^{n+1}})^{-1}(\widehat{X}^n_b + (\xi^d)^{\bar{P}(n)}) \equiv b$$
$$g(H^{n+1}) := \theta^{n+1}_{H^{n+1}}(\widetilde{\widehat{X}^n_a} + \widetilde{(\xi^c)^{\bar{P}(n)}} + (\xi^A)^{P^{n+1}}) = (F_{H^{n+1}})^{-1}(\widehat{X}^n_a + (\xi^c)^{\bar{P}(n)}) \equiv a$$

and the sign '$\equiv$' means modulo $\mathfrak{g}^n$. (We used (**??**), $(F_{H^{n+1}})^{-1}((\xi^c)^{\bar{P}(n)}) = c \in \mathfrak{g}^n$ and $(F_{H^{n+1}})^{-1}((\xi^d)^{\bar{P}(n)}) = d \in \mathfrak{g}^n$). We obtain

$$\begin{aligned} t^\rho_{H^{n+1}}(a\wedge b) &\equiv -\theta^{n+1}([\widetilde{\widehat{X}^n_a} + \widetilde{(\xi^c)^{\bar{P}(n)}} + (\xi^A)^{P^{n+1}}, \widetilde{\widehat{X}^n_b} + \widetilde{(\xi^d)^{\bar{P}(n)}} + (\xi^B)^{P^{n+1}}]_{H^{n+1}}) \\ &\equiv -(F_{H^{n+1}})^{-1}([\widehat{X}^n_a + (\xi^c)^{\bar{P}(n)}, \widehat{X}^n_b + (\xi^d)^{\bar{P}(n)}]_{\bar{H}^n}) \\ &\equiv -(F_{H^{n+1}})^{-1}([\widehat{X}^n_a, \widehat{X}^n_b]_{\bar{H}^n} + [(\xi^c)^{\bar{P}(n)}, \widehat{X}^n_b]_{\bar{H}^n} + [\widehat{X}^n_a, (\xi^d)^{\bar{P}(n)}]_{\bar{H}^n}) \\ &\equiv -(F_{H^{n+1}})^{-1}([\widehat{X}^n_a, \widehat{X}^n_b]_{\bar{H}^n} + (\widehat{X}^n_{c(b)})_{\bar{H}^n} - (\widehat{X}^n_{d(a)})_{\bar{H}^n}), \end{aligned} \tag{59}$$

where $\bar{H}^n = \pi^{n+1}(H^{n+1})$ and we used (**??**) (we remark that $c(b) = 0$ when $b \in \sum_{i=0}^{n-1} \mathfrak{g}^i$ and similarly for $d(a)$). Suppose now that $a \in \mathfrak{m}^{-1}$ and that $b \in \mathfrak{g}^i$ (with $0 \le i \le n-1$). Using (**??**), (**??**), (**??**) and $c(b) = 0$ we obtain

$$\begin{aligned} t^\rho_{H^{n+1}}(a\wedge b) &= -(F_{H^{n+1}})^{-1}(\widehat{X}^n_{[a,b]} - \widehat{X}^n_{d(a)}) \mod (\mathfrak{m}_n)_i \\ &= -[a,b] + d(a) \mod (\mathfrak{m}_n)_i. \end{aligned}$$

Since $d \in \mathfrak{g}^n \subset \mathfrak{gl}^n(\mathfrak{m}_{n-1})$, $d(a) \in \mathfrak{g}^{n-1}$. Also, $[a,b] = -b(a) \in \mathfrak{g}^{i-1}$. We deduce that $t^\rho_{H^{n+1}} \in \mathrm{Hom}(\mathfrak{m}^{-1} \wedge (\sum_{i=0}^{n-1} \mathfrak{g}^i), \mathfrak{m}_n)$ has only components of non-negative homogeneous degree and relation (**??**) holds, for any $a \wedge b \in \mathfrak{m}^{-1} \wedge (\sum_{i=0}^{n-1} \mathfrak{g}^i)$. □

6.3 The $\mathrm{Hom}(\mathfrak{g}^n \wedge \mathfrak{m}_n, \mathfrak{m}_n)$-valued component

This is the last component of the torsion function t^ρ which needs to be studied, in order to conclude the proof of Theorem **??**.

Proposition 43. *The function $t^\rho : P^{n+1} \to \mathrm{Hom}(\mathfrak{g}^n \wedge \mathfrak{m}_n, \mathfrak{m}_n)$ has non-negative reduced homogeneous components and satisfies*

$$(t^\rho_{H^{n+1}})^0(a \wedge b) = -[a,b], \quad \forall a \wedge b \in \mathfrak{g}^n \wedge \mathfrak{m}.$$

Proof. Let $a \in \mathfrak{g}^n$ and $b \in \mathfrak{m}_n$. Recall that $G^n GL_{n+1}(\mathfrak{m}_{n-1})$ acts on P^{n+1} and the fundamental vector field $(\xi^a)^{P^{n+1}}$ of this action, generated by $a \in \mathfrak{g}^n \subset \mathfrak{g}^n + \mathfrak{gl}_{n+1}(\mathfrak{m}_{n-1})$, is π^{n+1}-projectable and $(\pi^{n+1})_*(\xi^a)^{P^{n+1}} = (\xi^a)^{\bar{P}(n)}$ (see Proposition **??**). On the other hand, $X^{n+1}_a \in \mathfrak{X}(P^{n+1})$ is the ρ-horisontal lift of $(\xi^a)^{\bar{P}(n)}$. We obtain that $Y := X^{n+1}_a - (\xi^a)^{P^{n+1}}$ is π^{n+1}-vertical. We write

$$\begin{aligned} t^\rho_{H^{n+1}}(a\wedge b) &= -\theta^{n+1}([X^{n+1}_a, X^{n+1}_b]_{H^{n+1}}) \\ &= -\theta^{n+1}([(\xi^a)^{P^{n+1}}, X^{n+1}_b]_{H^{n+1}}) - \theta^{n+1}([Y, X^{n+1}_b]_{H^{n+1}}). \end{aligned} \tag{60}$$

We need to compute the last row from the right hand side of (**??**). For the first term, we use Lemma **??** and that $\theta^{n+1}(X^{n+1}_b) = b$ is constant:

$$\theta^{n+1}([(\xi^a)^{P^{n+1}}, X^{n+1}_b]) = -L_{(\xi^a)^{P^{n+1}}}(\theta^{n+1})(X^{n+1}_b) = (\rho^n)_*(a)(b). \tag{61}$$

To compute the second term, we remark that, since Y is π^{n+1}-vertical, there is $A \in \mathfrak{gl}_{n+1}(\mathfrak{m}_n) + \mathrm{Hom}(\sum_{i=0}^{n-1} \mathfrak{g}^i, \mathfrak{g}^n)$, such that $Y_{H^{n+1}} = (\xi^A)^{P^{n+1}}_{H^{n+1}}$. The soldering form θ^{n+1} of π^{n+1} is $\bar{G}$-equivariant (see relation (**??**)). Like in the computation (**??**) from the proof of Theorem **??**, we obtain $\theta^{n+1}([Y, X_b^{n+1}]_{H^{n+1}}) = A(b)$. This fact, together with (**??**) and (**??**), imply that

$$t^{\rho}_{H^{n+1}}(a \wedge b) = -(\rho^n)_*(a)(b) - A(b), \quad a \in \mathfrak{g}^n,\ b \in \mathfrak{m}_n.$$

If $b \in \mathfrak{m}^i$ (with $i \leq -1$) then $(\rho^n)_*(a)(b) = a(b) \in (\mathfrak{m}_{n-1})^{i+n}$ and $A(b) \in (\mathfrak{m}_n)_{i+n+1}$. We deduce that $t^{\rho}_{H^{n+1}}(a \wedge b) \in (\mathfrak{m}_n)_{i+n}$ and $(t^{\rho}_{H^{n+1}})^0(a \wedge b) = -a(b) = -[a, b]$. If $b \in \mathfrak{g}^j$ $(0 \leq j \leq n)$ then $(\rho^n)_*(a)(b) = 0$ and $t^{\rho}_{H^{n+1}}(a \wedge b) = -A(b) \in \mathfrak{g}^n$. □

The proof of Theorem **??** is now completed.

7 Variation of the torsion t^{ρ} of π^{n+1}

In this section we define and study the $(n+1)$-torsion of the Tanaka structure $(\mathcal{D}_i, \pi_G)$. We preserve the setting from Section **??**. In particular, ρ is a connection on the G-structure $\pi^{n+1} : P^{n+1} \to \bar{P}^{(n)}$.

Proposition 44. *i) Let $0 \leq i \leq n-1$. The map*

$$t^{\rho} : P^{n+1} \to \mathrm{Hom}(\mathfrak{m}^{-1} \wedge \mathfrak{g}^i, (\mathfrak{m}_n)^{-i+1} + \cdots + (\mathfrak{m}_n)^{n-1})$$

is independent of the connection ρ.

ii) Let $i \leq n+1$. The homogeneous component $(t^{\rho})^i$ of degree i of $t^{\rho} : P^{n+1} \to$ $\mathrm{Hom}(\mathfrak{m}^{-1} \wedge \mathfrak{m}, \mathfrak{m}_n)$ is independent of the connection ρ.

Proof. Consider another connection ρ' on $\pi^{n+1} : P^{n+1} \to \bar{P}^{(n)}$. From Theorem **??** ii), for any $H^{n+1} \in P^{n+1}$ and $a, b \in \mathfrak{m}_n$, there are $A, B \in \mathfrak{gl}_{n+1}(\mathfrak{m}_n) + \mathrm{Hom}(\sum_{i=0}^{n-1} \mathfrak{g}^i, \mathfrak{g}^n)$, such that

$$t^{\rho'}_{H^{n+1}}(a \wedge b) = t^{\rho}_{H^{n+1}}(a \wedge b) - A(b) + B(a). \tag{62}$$

If $a \in \mathfrak{m}^{-1}$ and $b \in \mathfrak{g}^i$ $(0 \leq i \leq n-1)$ then $A(b), B(a) \in \mathfrak{g}^n$ and, from (**??**), we obtain claim i). Let $a \in \mathfrak{m}^{-1}$ and $b \in \mathfrak{m}^j$. Then $\deg A(b) \geq n+1+j > (-1+j)+i$ and $\deg B(a) = n > (-1+j)+i$, for any $i \leq n+1$ (because $j < 0$). Relation (**??**) again implies claim ii). □

Definition 45. *i) The vector space* $\mathrm{Tor}^{n+1}(\mathfrak{m}_n) := \mathrm{Hom}^{n+1}(\mathfrak{m}^{-1} \wedge \mathfrak{m}, \mathfrak{m}_n) + \sum_{i=0}^{n-1} \mathrm{Hom}(\mathfrak{m}^{-1} \wedge \mathfrak{g}^i, \mathfrak{g}^{n-1})$ *is called the* **space of** $(n+1)$**-torsions**.

ii) Let ρ be a connection on $\pi^{n+1} : P^{n+1} \to \bar{P}^{(n)}$. The function

$$\bar{t}^{(n+1)} : P^{n+1} \to \mathrm{Tor}^{n+1}(\mathfrak{m}_n)$$

defined by

$$\bar{t}^{(n+1)}_{H^{n+1}}(a \wedge b) = \begin{cases} (t^{\rho}_{H^{n+1}})^{n+1}(a \wedge b), & a \wedge b \in \mathfrak{m}^{-1} \wedge \mathfrak{m} \\ (t^{\rho}_{H^{n+1}})^{n-i}(a \wedge b), & a \wedge b \in \mathfrak{m}^{-1} \wedge \mathfrak{g}^i, \end{cases} \tag{63}$$

for any $H^{n+1} \in P^{n+1}$ and $0 \leq i \leq n-1$, is called the $(n+1)$**-torsion of the Tanaka structure** $(\mathcal{D}_i, \pi_G)$. *In (**??**) the expression $(t^{\rho}_{H^{n+1}})^{n-i}(a \wedge b)$, for $a \wedge b \in \mathfrak{m}^{-1} \wedge \mathfrak{g}^i$, denotes the projection of $t^{\rho}_{H^{n+1}}(a \wedge b)$ on $\mathfrak{g}^{n-1}$.*

From Proposition **??**, $\bar{t}^{(n+1)}$ is independent of the choice of ρ.

Theorem 46. *For any* $H^{n+1} \in P^{n+1}$ *and* $\mathrm{Id} + A \in \bar{G}$,

$$\bar{t}^{(n+1)}_{H^{n+1}(\mathrm{Id}+A)} = \bar{t}^{(n+1)}_{H^{n+1}} + \partial^{(n+1)} A.$$

Above

$$\partial^{(n+1)} : \mathfrak{gl}_{n+1}(\mathfrak{m}_n) + \sum_{i=0}^{n-1} \mathrm{Hom}(\mathfrak{g}^i, \mathfrak{g}^n) \to \mathrm{Tor}^{n+1}(\mathfrak{m}_n)$$

maps $\mathfrak{gl}_{n+1}(\mathfrak{m}_n)$ *into* $\mathrm{Hom}^{n+1}(\mathfrak{m}^{-1} \wedge \mathfrak{m}, \mathfrak{m}_n)$ *and* $\mathrm{Hom}(\mathfrak{g}^i, \mathfrak{g}^n)$ *into* $\mathrm{Hom}(\mathfrak{m}^{-1} \wedge \mathfrak{g}^i, \mathfrak{g}^{n-1})$ *(*$0 \leq i \leq n-1$*) and is defined by*

$$(\partial^{(n+1)} A_{n+1})(a \wedge b) := A^{n+1}_{n+1}([a, b]) - [A^{n+1}_{n+1}(a), b] - [a, A^{n+1}_{n+1}(b)],\ a \wedge b \in \mathfrak{m}^{-1} \wedge \mathfrak{m}$$
$$(\partial^{(n+1)} A^{n-i})(a \wedge b) := -[a, A^{n-i}(b)],\ a \wedge b \in \mathfrak{m}^{-1} \wedge \mathfrak{g}^i,$$

for any $A_{n+1} \in \mathfrak{gl}_{n+1}(\mathfrak{m}_n)$ *and* $A^{n-i} \in \mathrm{Hom}(\mathfrak{g}^i, \mathfrak{g}^n)$.

Proof. From Theorem **??**,

$$t^{\rho}_{H^{n+1}(\mathrm{Id}+A)}(a \wedge b) = (\mathrm{Id} + B) t^{\rho}_{H^{n+1}}((\mathrm{Id} + A)(a) \wedge (\mathrm{Id} + A)(b)),\ a, b \in \mathfrak{m}_n, \tag{64}$$

where $\mathrm{Id} + B := (\mathrm{Id} + A)^{-1}$. If $A = A_{n+1} + \sum_{i=1}^{n} A^i$ and $B = B_{n+1} + \sum_{i=1}^{n} B^i$, with $A_{n+1}, B_{n+1} \in \mathfrak{gl}_{n+1}(\mathfrak{m}_n)$ and $A^i, B^i \in \mathrm{Hom}(\mathfrak{g}^{n-i}, \mathfrak{g}^n)$, then $B^i = -A^i$ $(1 \leq i \leq n)$ and $B^{n+1}_{n+1} = -A^{n+1}_{n+1}$ (easy check). We write (**??**) in the equivalent form

$$\begin{aligned} &t^{\rho}_{H^{n+1}(\mathrm{Id}+A)}(a \wedge b) = t^{\rho}_{H^{n+1}}(a \wedge b) + t^{\rho}_{H^{n+1}}(a \wedge A(b)) + t^{\rho}_{H^{n+1}}(A(a) \wedge b) \\ &+ t^{\rho}_{H^{n+1}}(A(a) \wedge A(b)) \\ &+ B\left(t^{\rho}_{H^{n+1}}(a \wedge b) + t^{\rho}_{H^{n+1}}(a \wedge A(b)) + t^{\rho}_{H^{n+1}}(A(a) \wedge b) + t^{\rho}_{H^{n+1}}(A(a) \wedge A(b))\right). \end{aligned}$$

Suppose now that $a \in \mathfrak{m}^{-1}$ and $b \in \mathfrak{m}^i$ $(i < 0)$. The above equality becomes

$$\begin{aligned} &t^{\rho}_{H^{n+1}(\mathrm{Id}+A)}(a \wedge b) = t^{\rho}_{H^{n+1}}(a \wedge b) + t^{\rho}_{H^{n+1}}(a \wedge A_{n+1}(b)) + t^{\rho}_{H^{n+1}}(A_{n+1}(a) \wedge b) \\ &+ t^{\rho}_{H^{n+1}}(A_{n+1}(a) \wedge A_{n+1}(b)) \\ &+ B\left(t^{\rho}_{H^{n+1}}(a \wedge b) + t^{\rho}_{H^{n+1}}(a \wedge A_{n+1}(b)) + t^{\rho}_{H^{n+1}}(A_{n+1}(a) \wedge b)\right) \\ &+ B\left(t^{\rho}_{H^{n+1}}(A_{n+1}(a) \wedge A_{n+1}(b))\right). \end{aligned} \tag{65}$$

Since $a \in \mathfrak{m}^{-1}$ and $A_{n+1}(a) \in \mathfrak{g}^n$, all arguments of $t^{\rho}_{H^{n+1}}$, in the right hand side of (**??**), belong to $(\mathfrak{m}^{-1} + \mathfrak{g}^n) \wedge \mathfrak{m}_n$. From Theorem **??**,

$$t^{\rho}_{H^{n+1}}(a \wedge b) \in (\mathfrak{m}_n)_{i-1},\ t^{\rho}_{H^{n+1}}(a \wedge A_{n+1}(b)) \in (\mathfrak{m}_n)_{i+n},\ t^{\rho}_{H^{n+1}}(A_{n+1}(a) \wedge b) \in (\mathfrak{m}_n)_{i+n}.$$

Also, since $A_{n+1}(a) \in \mathfrak{g}^n$ and $A_{n+1}(b) \in (\mathfrak{m}_n)_{i+n+1}$,

$$t^{\rho}_{H^{n+1}}(A_{n+1}(a) \wedge A_{n+1}(b)) \in (\mathfrak{m}_n)_{\min\{n, 2n+i+1\}}.$$

We project (**??**) on $(\mathfrak{m}_n)^{i+n}$. The term $t^{\rho}_{H^{n+1}}(A_{n+1}(a) \wedge A_{n+1}(b))$ brings no contribution (because $i + n < \min\{n, 2n + i + 1\}$). We obtain

$$\begin{aligned} (t^{\rho}_{H^{n+1}(\mathrm{Id}+A)})^{n+1}(a \wedge b) &= (t^{\rho}_{H^{n+1}})^{n+1}(a \wedge b) + (t^{\rho}_{H^{n+1}})^0(a \wedge A_{n+1}(b)) \\ &+ (t^{\rho}_{H^{n+1}})^0(A^{n+1}_{n+1}(a) \wedge b) + \pi_{(\mathfrak{m}_n)^{i+n}} B t^{\rho}_{H^{n+1}}(a \wedge b). \end{aligned} \tag{66}$$

From $t^\rho_{H^{n+1}}(a\wedge b) \in (\mathfrak{m}_n)_{-1+i}$, $B(\sum_{j=0}^{n-1}\mathfrak{g}^j) \subset \mathfrak{g}^n$, and $B|\mathfrak{m} = B_{n+1}|\mathfrak{m}$, we obtain

$$\pi_{(\mathfrak{m}_n)^{i+n}} B t^\rho_{H^{n+1}}(a\wedge b) = B^{n+1}_{n+1}(t^\rho_{H^{n+1}})^0(a\wedge b). \tag{67}$$

Using $B^{n+1}_{n+1} = -A^{n+1}_{n+1}$, relations (**??**), (**??**) and (**??**), we obtain, for any $a\wedge b \in \mathfrak{m}^{-1}\wedge\mathfrak{m}$,

$$(t^\rho_{H^{n+1}(\mathrm{Id}+A)})^{n+1}(a\wedge b) = (t^\rho_{H^{n+1}})^{n+1}(a\wedge b) + (\partial A^{n+1})(a\wedge b). \tag{68}$$

In a similar way, we prove that, for any $a\wedge b\in\mathfrak{m}^{-1}\wedge\mathfrak{g}^i$,

$$(t^\rho_{H^{n+1}(\mathrm{Id}+A)})^{n-i}(a\wedge b) = (t^\rho_{H^{n+1}})^{n-i}(a\wedge b) - [a, A^{n-i}(b)]. \tag{69}$$

Relations (**??**) and (**??**) imply our claim. $\square$

8 Definition of $\bar\pi^{(n+1)} : \bar P^{(n+1)} \to \bar P^{(n)}$

Consider the map $\partial^{(n+1)}$ from Theorem **??** and let W^{n+1} be a complement of $\mathrm{Im}(\partial^{(n+1)})$ in $\mathrm{Tor}^{n+1}(\mathfrak{m}_n)$.

Proposition 47. *i) The natural projection $\tilde\pi^{n+1} : \tilde P^{n+1} = (\bar t^{(n+1)})^{-1}(W^{n+1}) \subset P^{n+1} \to \bar P^{(n)}$ is a G-structure, with structure group $G = G^{n+1}GL_{n+2}(\mathfrak{m}_n)$.*

ii) Let $\tilde\rho$ be a connection on $\tilde\pi^{n+1}$. For any $H^{n+1}\in\tilde P^{n+1}$ and $a\wedge b\in\mathfrak{m}^{-1}\wedge\mathfrak{g}^i$ ($0\le i\le n$), $t^{\tilde\rho}_{H^{n+1}}(a\wedge b)\in(\mathfrak{m}_n)_{i-1}$ and

$$(t^{\tilde\rho}_{H^{n+1}})^0(a\wedge b) = -[a,b],\quad H^{n+1}\in\tilde P^{n+1},\ a\wedge b\in\mathfrak{m}^{-1}\wedge(\sum_{i=0}^n\mathfrak{g}^i).$$

Proof. Any $A^{n-i}\in\mathrm{Hom}(\mathfrak{g}^i,\mathfrak{g}^n)$ $(0\le i\le n-1)$ with $\partial^{(n+1)}(A^{n-i}) = 0$, i.e.

$$[a, A^{n-i}(b)] = -A^{n-i}(b)(a) = 0,\quad \forall a\in\mathfrak{m}^{-1},\ b\in\mathfrak{g}^i,$$

vanishes identically (because $A^{n-i}(b)\in\mathfrak{g}^n\subset\mathrm{Hom}(\mathfrak{m},\mathfrak{m}_{n-1})$ satisfies $A^{n-i}(b)[x,y] = [A^{n-i}(b)(x),y]+[x,A^{n-i}(b)(y)]$, for any $x,y\in\mathfrak{m}$, and $\mathfrak{m}^{-1}$ generates $\mathfrak{m}$; so, if $A^{n-i}(b)|_{\mathfrak{m}^{-1}} = 0$, for any $b\in\mathfrak{g}^n$, then $A^{n-i}(b)=0$ and $A^{n-i}=0$). We proved that $\partial^{(n+1)}|_{\sum_{i=0}^{n-1}\mathrm{Hom}(\mathfrak{g}^i,\mathfrak{g}^n)}$ is injective. Similarly, any $A_{n+1}\in\mathfrak{gl}_{n+1}(\mathfrak{m}_n)$ which satisfies $\partial^{(n+1)}(A_{n+1}) = 0$, i.e.

$$A^{n+1}_{n+1}([a,b]) = [A^{n+1}_{n+1}(a), b] + [a, A^{n+1}_{n+1}(b)],\quad \forall a\in\mathfrak{m}^{-1},\ b\in\mathfrak{m},$$

satisfies this relation for any $a,b\in\mathfrak{m}$. It follows that $\mathrm{Ker}\left(\partial^{(n+1)}|_{\mathfrak{gl}_{n+1}(\mathfrak{m}_n)}\right) = \mathfrak{g}^{n+1} + \mathfrak{gl}_{n+2}(\mathfrak{m}_n)$. Claim i) follows. Claim ii) follows from Theorem **??** ii) (extend $\tilde\rho$ to a connection on π^{n+1}).

$\square$

We can finally define the map $\bar\pi^{(n+1)} : \bar P^{(n+1)} \to \bar P^{(n)}$ we are looking for. Namely, let $\bar P^{(n+1)} := \tilde P^{n+1}/GL_{n+2}(\mathfrak{m}_n)$ and $\bar\pi^{(n+1)} : \bar P^{(n+1)}\to\bar P^{(n)}$ the map induced by $\tilde\pi^{n+1}$.

Proposition 48. *The map $\bar\pi^{(n+1)} : \bar P^{(n+1)}\to\bar P^{(n)}$ satisfies properties A), B), and C) from Theorem **??** (with n replaced by $n+1$).*

Proof. From Lemma **??**, property A) is satisfied. Property B) is satisfied by construction and property C) follows from Proposition **??**. From Theorem **??**, $\bar\pi^{(n+1)}$ is canonically isomorphic to a subbundle of the bundle $\mathrm{Gr}_{n+2}(T\bar P^{(n)})$ of $(n+2)$-quasi-gradations of $T\bar P^{(n)}$. $\square$

9 Proof of Theorem ??

In this section we prove Theorem **??**. In Subsection **??** we construct the canonical frame required by Theorem **??**. In Subsection **??** we prove the statements about the automorphism groups.

9.1 The canonical frame of $\bar{P}^{(\bar{l})}$.

Proposition 49. *Let $(\mathcal{D}_i, \pi_G)$ be a Tanaka G-structure of type $\mathfrak{m} = \sum_{i=-k}^{-1} \mathfrak{m}^i$ and finite order $\bar{l}$. Then the Tanaka prolongation $\bar{P}^{(\bar{l})}$ has a canonical frame F^{can}.*

Proof. Since $\mathfrak{g}^{\bar{l}+1} = 0$, also $\mathfrak{g}^s = 0$ for any $s \geq \bar{l}+1$ and $\bar{\pi}^{(s)} : \bar{P}^{(s)} \to \bar{P}^{(s-1)}$ is a diffeomorphism. Moreover, for such an s, $\bar{\mathcal{D}}^{(s)}_{\bar{l}+1} = 0$ (at any $\bar{H}^s \in \bar{P}^{(s)}$, $(\bar{\mathcal{D}}^{(s)}_{\bar{l}+1})_{\bar{H}^s}$ is isomorphic to $(\mathfrak{m}_s)_{\bar{l}+1} = \mathfrak{g}^{\bar{l}+1} + \cdots + \mathfrak{g}^s$, which is trivial). We obtain that $\mathrm{Gr}_m(T\bar{P}^{(s)}) = \mathrm{Gr}(T\bar{P}^{(s)})$ for any $s \geq \bar{l}+1$ and $m \geq k+\bar{l}+1$ (see our comments after Definition **??**). For any f, t with $f \geq t+1$ we denote by $\bar{\pi}^{(f,t+1)} : \bar{P}^{(f)} \to \bar{P}^{(t)}$ the composition $\bar{\pi}^{(t+1)} \circ \cdots \circ \bar{\pi}^{(f)}$.

We need to construct a canonical isomorphism $F^{\mathrm{can}}_{\bar{H}^{\bar{l}}} : \mathfrak{m}_{\bar{l}} \to T_{\bar{H}^{\bar{l}}} \bar{P}^{(\bar{l})}$, for any $\bar{H}^{\bar{l}} \in \bar{P}^{(\bar{l})}$. Let $\bar{H}^{k+\bar{l}+1} := (\bar{\pi}^{(k+\bar{l}+1,\bar{l}+1)})^{-1}(\bar{H}^{\bar{l}}) \in \bar{P}^{(k+\bar{l}+1)}$. By our construction of Tanaka prolongations, $\bar{P}^{(k+\bar{l}+1)} \subset \mathrm{Gr}_{k+\bar{l}+2}(T\bar{P}^{(k+\bar{l})})$. From the above, $\mathrm{Gr}_{k+\bar{l}+2}(T\bar{P}^{(k+\bar{l})}) = \mathrm{Gr}(T\bar{P}^{(k+\bar{l})})$ and we obtain that $\bar{P}^{(k+\bar{l}+1)} \subset \mathrm{Gr}(T\bar{P}^{(k+\bar{l})})$. In particular, $\bar{H}^{k+\bar{l}+1}$ defines a gradation of $T_{\bar{H}^{k+\bar{l}}} \bar{P}^{(k+\bar{l})}$ or a frame $F_{\bar{H}^{k+\bar{l}+1}} = \widehat{\bar{H}^{k+\bar{l}+1}} \circ I_{\bar{H}^{k+\bar{l}}} : \mathfrak{m}_{k+\bar{l}} = \mathfrak{m}_{\bar{l}} \to T_{\bar{H}^{k+\bar{l}}} \bar{P}^{(k+\bar{l})}$, where $\bar{H}^{k+\bar{l}} := \bar{\pi}^{(k+\bar{l}+1)}(\bar{H}^{k+\bar{l}+1})$ and $I_{\bar{H}^{k+\bar{l}}} : \mathfrak{m}_{\bar{l}} \to \mathrm{gr}(T_{\bar{H}^{k+\bar{l}}} \bar{P}^{(k+\bar{l})})$ is the graded frame from the Tanaka $\{e\}$-structure of $\bar{P}^{(k+\bar{l})}$. We define $F^{\mathrm{can}}_{\bar{H}^{\bar{l}}} := (\bar{\pi}^{(k+\bar{l},\bar{l}+1)})_* \circ F_{\bar{H}^{k+\bar{l}+1}} : \mathfrak{m}_{\bar{l}} \to T_{\bar{H}^{\bar{l}}} \bar{P}^{(\bar{l})}$. □

9.2 The automorphism group $\mathrm{Aut}(\mathcal{D}_i, \pi_G)$

The proof of the remaining part of Theorem **??** is based on the behaviour of the automorphisms of a Tanaka structure, under the prolongation procedure:

Proposition 50. *Let $(\mathcal{D}_i, \pi_G : P = P_G \to M)$ be a Tanaka G-structure of type $\mathfrak{m}$. The group of automorphisms $\mathrm{Aut}(\mathcal{D}_i, \pi_G)$ of $(\mathcal{D}_i, \pi_G)$ is isomorphic to the group of automorphisms of the Tanaka $\{e\}$-structure on $P = P_G$ (see Proposition **??**) and to the group of automorphisms $\mathrm{Aut}(\tilde{\pi}^n)$ of the G-structures $\tilde{\pi}^n : \tilde{P}^n \to \bar{P}^{(n-1)}$, $n \geq 1$.*

Proof. The argument is similar to the one used in Theorem 3.2 of [?] (in the setting of prolongation of G-structures) and is based on the naturality of our construction. One first shows that any $f \in \mathrm{Aut}(\mathcal{D}_i, \pi_G)$ induces an automorphism $f_G : P \to P$ of the Tanaka $\{e\}$-structure of P, by $f_G(u) := f_* \circ u$, for any graded frame $u : \mathfrak{m} \to \mathrm{gr}(T_p M)$ which belongs to P, and that $f \mapsto f_G$ is an isomorphism betweeen these Tanaka structure automorphism groups. Next, one notices (from definitions) that the automorphisms of the Tanaka $\{e\}$-structure of P coincide with the automorphisms of the G-structure $\tilde{\pi}^1 : \tilde{P}^1 \to P$.

It remains to prove that $\mathrm{Aut}(\tilde{\pi}^n)$ is isomorphic to $\mathrm{Aut}(\tilde{\pi}^{n+1})$, for any $n \geq 1$. Any $\bar{f}^{(n-1)} \in \mathrm{Aut}(\tilde{\pi}^n)$ induces a map $f_{\tilde{P}^n} : \tilde{P}^n \to \tilde{P}^n$, defined by $f_{\tilde{P}^n}(F_{H^n}) := (\bar{f}^{(n-1)})_* \circ F_{H^n}$, for any $F_{H^n} \in \tilde{P}^n$. The map $f_{\tilde{P}^n}$ commutes with the action of $G^n GL_{n+1}(\mathfrak{m}_{n-1})$ (hence, also with the action of $GL_{n+1}(\mathfrak{m}_{n-1})$) on $\tilde{P}^n$ and induces a map $\bar{f}^{(n)} : \bar{P}^{(n)} \to \bar{P}^{(n)}$ which belongs to $\mathrm{Aut}(\tilde{\pi}^{n+1})$ (easy check). For the converse, let $\bar{f}^{(n)} \in \mathrm{Aut}(\tilde{\pi}^{n+1})$, i.e. $\bar{f}^{(n)} : \bar{P}^{(n)} \to \bar{P}^{(n)}$ is a diffeomorphism, such that, for any frame $F_{H^{n+1}} : \mathfrak{m}_n \to T_{\bar{H}^n} \bar{P}^{(n)}$ which belongs to $\tilde{P}^{n+1}$, $(\bar{f}^{(n)})_* \circ F_{H^{n+1}} : \mathfrak{m}_n \to T_{\bar{f}^{(n)}(\bar{H}^n)} \bar{P}^{(n)}$ also belongs to $\tilde{P}^{n+1}$. Since the frames from $\tilde{P}^{n+1}$ are filtration preserving, both $F_{H^{n+1}}$ and $(\bar{f}^{(n)})_* \circ F_{H^{n+1}}$, therefore also $\bar{f}^{(n)}$, are filtration preserving. Since the frames from $\tilde{P}^{n+1}$, restricted to $\mathfrak{g}^n$, coincide with the vertical parallelism of $\bar{\pi}^{(n)} : \bar{P}^{(n)} \to \bar{P}^{(n-1)}$, we obtain that $(\bar{f}^{(n)})_*(\xi^v)^{\bar{P}^{(n)}} = (\xi^v)^{\bar{P}^{(n)}}$, for any $v \in \mathfrak{g}^n$. Therefore, there is $\bar{f}^{(n-1)} : \bar{P}^{(n-1)} \to \bar{P}^{(n-1)}$ such that $\bar{\pi}^{(n)} \circ \bar{f}^{(n)} = \bar{f}^{(n-1)} \circ \bar{\pi}^{(n)}$. We check that $\bar{f}^{(n-1)}$ induces $\bar{f}^{(n)}$. For this, we use: for any $x \in (\mathfrak{m}_{n-1})^i$,

$$\begin{aligned} &\mathrm{pr}^i_{(n+1)}(\bar{\pi}^{(n)})_* F^i_{H^{n+1}}(x) = F^i_{\bar{H}^n}(x), \\ &\mathrm{pr}^i_{(n+1)}(\bar{\pi}^{(n)})_*(\bar{f}^{(n)})_* F^i_{H^{n+1}}(x) = F^i_{\bar{f}^{(n)}(\bar{H}^n)}(x), \end{aligned} \tag{70}$$

where $\bar{H}^{n-1} = \bar{\pi}^{(n)}(\bar{H}^n)$. (Relations (**??**) follow from $F_{H^{n+1}}, (\bar{f}^{(n)})_* \circ F_{H^{n+1}} \in \tilde{P}^{n+1}$ and Lemma **??**). Since $\bar{\pi}^{(n)} \circ \bar{f}^{(n)} = \bar{f}^{(n-1)} \circ \bar{\pi}^{(n)}$ and $\bar{f}^{(n)}$, $\bar{f}^{(n-1)}$ are filtration preserving, we obtain from relations (**??**) that $F_{\bar{f}^{(n)}(\bar{H}^n)} = (\bar{f}^{(n-1)})_* \circ F_{\bar{H}^n}$, i.e. $\bar{f}^{(n)}$ is induced by $\bar{f}^{(n-1)}$, as required. It is easy to see that $\bar{f}^{(n-1)} \in \mathrm{Aut}(\tilde{\pi}^n)$. □

Proposition 51. *Let $(\mathcal{D}_i, \pi_G)$ be a Tanaka G-structure of type $\mathfrak{m} = \sum_{i=-k}^{-1} \mathfrak{m}^i$ and finite order $\bar{l}$ and F^{can} the canonical frame of $\bar{P}^{(\bar{l})}$. Then $\mathrm{Aut}(\mathcal{D}_i, \pi_G)$ is isomorphic to $\mathrm{Aut}(\bar{P}^{(\bar{l})}, F^{\mathrm{can}})$. It is a Lie group with $\mathrm{dimAut}(\mathcal{D}_i, \pi_G) \le \dim(M) + \sum_{i=0}^{\bar{l}} \mathfrak{g}^i$.*

Proof. The argument from Proposition **??** provides a canonical frame (an absolute parallelism) $(F^{\mathrm{can}})'$ on any prolongation $\bar{P}^{(\bar{l}')}$ (with $\bar{l}' \ge \bar{l}$), isomorphic to F^{can} by means of the map $\bar{\pi}^{(\bar{l}', \bar{l}+1)}$. Let $\bar{l}' \ge \bar{l}$ sufficiently large such that $\pi^{\bar{l}'} : P^{\bar{l}'} \to \bar{P}^{(\bar{l}'-1)}$ is an $\{e\}$-structure (recall Definition **??** for $\pi^{\bar{l}'}$). Then, for any $s \ge \bar{l}'$, $P^s = \tilde{P}^s = \bar{P}^{(s)}$ and $\pi^s = \tilde{\pi}^s = \bar{\pi}^{(s)}$ is an $\{e\}$-structure. Any $\bar{H}^s \in \bar{P}^{(s)}$ $(s \ge \bar{l}')$ is a frame $F_{\bar{H}^s} : \mathfrak{m}_{\bar{l}} \to T_{\bar{\pi}^{(s)}(\bar{H}^s)} \bar{P}^{(s-1)}$. By construction of the prolongations, the preimage $(\pi^{s+1})^{-1}(\bar{H}^s) \in P^{s+1}$ is the unique frame $F_{(\pi^{s+1})^{-1}(\bar{H}^s)} : \mathfrak{m}_{\bar{l}} \to T_{\bar{H}^s} \bar{P}^{(s)}$ given by

$$(\bar{\pi}^{(s)})_* \circ F_{(\pi^{s+1})^{-1}(\bar{H}^s)} = F_{\bar{H}^s}, \quad s \ge \bar{l}'. \tag{71}$$

Consider now the canonical frame $(F^{\mathrm{can}})'$ of $\bar{P}^{(\bar{l}')}$. From the proof of Proposition **??**, it is defined by

$$(F^{\mathrm{can}})'_{\bar{H}^{\bar{l}'}} = (\bar{\pi}^{(k+\bar{l}', \bar{l}'+1)})_* \circ F_{(\bar{\pi}^{(k+\bar{l}'+1, \bar{l}'+1)})^{-1}(\bar{H}^{\bar{l}'})}. \tag{72}$$

We will show that $(F^{\mathrm{can}})'$ is the $\{e\}$-structure $\pi^{\bar{l}'+1}$ of $\bar{P}^{(\bar{l}')}$. From relation (**??**), we need to check that $(\bar{\pi}^{(\bar{l}')})_* \circ (F^{\mathrm{can}})'_{\bar{H}^{\bar{l}'}} = F_{\bar{H}^{\bar{l}'}}$, for any $\bar{H}^{\bar{l}'} \in \bar{P}^{(\bar{l}')}$, or, using relation (**??**), $(\bar{\pi}^{(k+\bar{l}', \bar{l}')})_* \circ F_{\bar{H}^{k+\bar{l}'+1}} = F_{\bar{\pi}^{(k+\bar{l}'+1, \bar{l}'+1)}(\bar{H}^{k+\bar{l}'+1})}$, for any $\bar{H}^{k+\bar{l}'+1} \in \bar{P}^{k+\bar{l}'+1}$. The latter relation follows easily from (**??**). We obtain that $(F^{\mathrm{can}})'$ coincides with the absolute parallelism $\pi^{\bar{l}'+1}$ on $\bar{P}^{(\bar{l}')}$. From Proposition **??**, $\mathrm{Aut}(\bar{P}^{(\bar{l}')}, (F^{\mathrm{can}})')$ (or $\mathrm{Aut}(\bar{P}^{(\bar{l})}, F^{\mathrm{can}})$) is isomorphic to $\mathrm{Aut}(\mathcal{D}_i, \pi_G)$. From Kobayashi theorem (see Theorem 3.2 of [**?**], p. 15), these groups are Lie groups of dimension at most $\dim(M) + \sum_{i=0}^{\bar{l}} \mathfrak{g}^i$. □

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