New covering codes of radius R, codimension tR and $tR+\frac{R}{2}$, and saturating sets in projective spaces

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Abstract. The length function $\ell_q(r,R)$ is the smallest length of a q-ary linear code of codimension r and covering radius R. In this work we obtain new constructive upper bounds on $\ell_q(r,R)$ for all $R\geq 4$ and $r=tR$ with integer $t\geq 2$, and also for all even $R\geq 2$ and $r=tR+\frac{R}{2}$ with integer $t\geq 1$. The new bounds are provided by new infinite families of covering codes with fixed R and growing codimension. The new bounds improve upon the known ones.

We propose a general regular construction (called "Line-Ovals") of a minimal ρ-saturating $((\rho+1)q+1)$-set in the projective space $\mathrm{PG}(2\rho+1,q)$ for all $\rho\geq 0$. Such a set corresponds to an $[Rq+1,Rq+1-2R]_qR$ locally optimal[1] code of covering radius $R=\rho+1$. In these codes, we investigate combinatorial properties regarding to spherical capsules (including the property to be a surface-covering code[1]) and give corresponding constructions for code codimension lifting. Using the new codes as starting points in these constructions we obtained the desired infinite code families with growing $r=tR$.

In addition, we obtain new 1-saturating sets in the projective plane $\mathrm{PG}(2,q^2)$ and, founding on them, construct infinite code families with fixed even radius $R\geq 2$ and growing codimension $r=tR+\frac{R}{2}$, $t\geq 1$.

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[1]See the definitions at the end of Section 1.1.

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1 Introduction

1.1 Covering codes. The length function

Let $\mathbb{F}_q$ be the Galois field with q elements, $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Let $\mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$. Denote by $[n, n-r]_q$ a q-ary linear code of length n and codimension (redundancy) r, that is a subspace of $\mathbb{F}_q^n$ of dimension $n-r$. The *sphere of radius* R with center c in $\mathbb{F}_q^n$ is the set $\{v : v \in \mathbb{F}_q^n,\ d(v,c) \le R\}$ where $d(v,c)$ is the Hamming distance between the vectors v and c.

Definition 1.1. **(i)** The covering radius of a linear $[n, n-r]_q$ code is the least integer R such that the space $\mathbb{F}_q^n$ is covered by the spheres of radius R centered at the codewords.

(ii) A linear $[n, n-r]_q$ code has covering radius R if every column of $\mathbb{F}_q^r$ is equal to a linear combination of at most R columns of a parity check matrix of the code, and R is the smallest value with this property.

Definitions 1.1(i) and 1.1(ii) are equivalent. Let an $[n, n-r]_q R$ code be an $[n, n-r]_q$ code of covering radius R. An $[n, n-r]_q R$ code of minimum distance d is denoted by $[n, n-r, d]_q R$ code. For an introduction to coverings of vector Hamming spaces over finite fields, see [5, 7].

The covering density μ of an $[n, n-r]_q R$-code is defined as the ratio of the total volume of all q^{n-r} spheres of radius R centered at the codewords to the volume q^n of the space $\mathbb{F}_q^n$. By Definition 1.1(i), we have $\mu \ge 1$. In the other words,

$$\mu = \left(q^{n-r}\sum_{i=0}^{R}(q-1)^i\binom{n}{i}\right)\frac{1}{q^n} = \frac{1}{q^r}\sum_{i=0}^{R}(q-1)^i\binom{n}{i} \ge 1. \tag{1.1}$$

The covering quality of a code is better if its covering density is smaller. For fixed q, r, R, the covering density of an $[n, n-r]_q R$ code decreases with decreasing n.

Codes investigated from the point of view of the covering quality are usually called *covering codes* [7]; see an online bibliography [28], works [5, 9–14, 17–20, 26, 27], and the references therein.

Definition 1.2. [5, 7] *The length function* $\ell_q(r, R)$ is the smallest length of a q-ary linear code of codimension r and covering radius R.

From (1.1), see also Definition 1.1(ii), one obtains an implicit *lower bound on* $\ell_q(r,R)$:

$$\sum_{i=0}^{R}(q-1)^i\binom{\ell_q(r,R)}{i}\geq q^r. \tag{1.2}$$

In particular, for $R=1$ we have $\ell_q(r,1)\geq\frac{q^r-1}{q-1}$. This means that the perfect $\left[\frac{q^r-1}{q-1},\frac{q^r-1}{q-1}-r,3\right]_q 1$ Hamming code achieves the bound and has the covering density equal to one. The same is true for the perfect Golay codes $[23,12,7]_2 3$ and $[11,6,5]_3 2$. In the general case, note that the main term of the sum in (1.2) is $(q-1)^R\binom{\ell_q(r,R)}{R}$. If n is considerable larger than R (this is the natural situation in covering codes investigations) and if q is large enough, we have

$$\sum_{i=0}^{R}(q-1)^i\binom{\ell_q(r,R)}{i}\approx(q-1)^R\binom{\ell_q(r,R)}{R}\approx q^R\frac{(\ell_q(r,R))^R}{R!}\gtrsim q^r,$$
$$\ell_q(r,R)\gtrsim\sqrt[R]{R!}\cdot q^{(r-R)/R},$$

and, in a more general form,

$$\ell_q(r,R)\gtrsim cq^{(r-R)/R}, \tag{1.3}$$

where c is independent of q but it is possible that c depends on r and R.

Let t,s,R^* be integers. Let q' be a prime power. In [11, 13, 14, 17], see also the references therein, for the situations

$$\begin{aligned}&\textbf{(i)}\quad r=tR,\ \text{arbitrary } q,\\&\textbf{(ii)}\ \ R=sR^*,\ r=tR+s,\ q=(q')^{R^*},\\&\textbf{(iii)}\ r\neq tR,\quad q=(q')^R,\end{aligned} \tag{1.4}$$

$[n,n-r]_qR$ covering codes are obtained with lengths of the form

$$n=c_1(r,R)q^{(r-R)/R}+\sum_{i\geq2}c_i(r,R)q^{(r-R)/R-\mu_i},\ c_1(r,R)>1,\ \mu_i>0, \tag{1.5}$$

where all $c_i(r,R)$ are constants independent of q. Also, for $i\geq2$, one usually has $c_i(r,R)\geq 0$, but it is possible that $c_i(r,R)<0$, see for example Propositions 1.6 and 1.7. For growing q, code length n of (1.5) is close (by order) to the bound (1.3) since all $\mu_i>0$.

In this work, we consider the case (i) of (1.4) for $R\geq4$ and the situation (ii) for even R with $R^*=2$. We briefly describe the known results and then improve upon many of them by constructing new codes.

For new codes with $r=tR$ we note and use interesting and useful combinatorial properties connected with the locally optimality, R,ℓ-capsules and R,ℓ-objects.

Definition 1.3. [12] A linear covering code is called *locally optimal* if one cannot remove any column from its parity check matrix without increase in covering radius. A locally optimal code can be called also *non-shortening* in the sense mentioned.

Let $0 \leq \ell \leq R$. A *spherical* R, ℓ*-capsule* with center c in $\mathbb{F}_q^n$ is the set $\{v : v \in \mathbb{F}_q^n, 0 \leq \ell \leq d(v, c) \leq R\}$ where $d(v, c)$ is the Hamming distance between the vectors v and c, see [9, Rem. 5], [10, Rem. 2.1], [14, Sect, 2].

Definition 1.4. [9], [10, Sect. 2], [14, Sect, 2] Let $0 \leq \ell \leq R$. A linear $[n, n-r]_q R$ code of covering radius R is called an R, ℓ*-object* and is denoted as an $[n, n-r]_q R, \ell$ code if any of following holds.

(i) The space $\mathbb{F}_q^n$ is covered by the R, ℓ-capsules centered at the codewords.

(ii) Every column of the space $\mathbb{F}_q^r$ (including the zero column) is equal to a linear combination with *nonzero coefficients* of at least ℓ and at most R distinct columns of a parity-check matrix of the code.

(iii) Every coset of the code (including the code itself) contains a weight w word of the space $\mathbb{F}_q^n$ such that $\ell \leq w \leq R$.

Definitions 1.4(i), 1.4(ii), and 1.4(iii) are equivalent. In [9, 10, 14] widened definitions of R, ℓ-objects are considered. But for this work, Definition 1.4 is sufficient.

Note that the R, R-capsule is the surface of the sphere of radius R.

Definition 1.5. An $[n, n-r]_q R, R$ code is called *surface-covering code* of radius R.

The value of ℓ is important for code codimension lifting constructions, see Section 4.

1.2 The known results

Codes with radius $R = 2, 3$ and codimension $r = tR$ are widely investigated for arbitrary q, see [11], [14, Sects. 4, 5], [17, Ths. 9,12]. At the same time, codes with $R \geq 4$ and $r = tR$ are investigated insufficiently; moreover, the known results on these codes are obtained by use of codes with $R = 2, 3$ in the so-called direct sum construction [14, Sect. 4.2]. The following results on codes with $R \geq 4$ and $r = tR$ are described in the literature.

Proposition 1.6. [13, Sect. 2], [14, Ths. 6.1,6.2] *The following constructive upper bounds on the length function (provided by infinite families of codes with growing codimension) hold:*

$$\ell_q(r, R) \leq Rq^{(r-R)/R} + \left\lceil \frac{R}{3} \right\rceil q^{(r-2R)/R} + \delta_q(r, R), \; R \geq 4, \; r = tR, \; t \geq 2, \tag{1.6}$$

where values of $\delta_q(r, R)$ *with* $w = 2R \pmod 3$ *are as follows:*

$\delta_q(r, R) = 0,$ $\quad q \geq 4,$ $\quad r = 2R$ $\quad$ [14, Th. 6.1];

$\delta_q(r,R)=0,$	$q=16,\ q\geq 23,$	$r=3R$	[14, eq. (6.1)], [17];
$\delta_q(r,R)=2w(q^{(r-3R)/R}+1),$	$q=4,5,9,$	$r=4R$	[14, eq. (6.1)], [11];
$\delta_q(r,R)=w(q^{(r-3R)/R}+1),$	$q\geq 7,\ q\neq 9,$	$r=4R,6R$	[14, eq. (6.1)], [17];
$\delta_q(r,R)=wq^{(r-3R)/R},$	$q=5,9,$	$r\geq 5R,\ r\neq 6R$	[14, Th. 6.2];
$\delta_q(r,R)=0,$	$q\geq 7,\ q\neq 9,$	$r\geq 5R,\ r\neq 6R$	[14, Th. 6.2].

The following results on codes with even covering radius $R\geq 2$ and codimension $r=tR+\frac{R}{2}$ are described in the literature.

Proposition 1.7. [11, Ex. 6, eq. (33)], [13], [14, Sects. 4.4, 7] *Let q' be a prime power. Let the covering radius $R\geq 2$ be even. Let the code codimension be $r=tR+\frac{R}{2}$ with integer t. The following constructive upper bounds on the length function (provided by infinite families of codes with growing codimension) hold:*

$$\ell_q(r,R)\leq \frac{R}{2}\left(3-\frac{1}{\sqrt{q}}\right)q^{(r-R)/R}+\frac{R}{2}\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor,\ q=(q')^2\geq 16,\ t\geq 1; \tag{1.7}$$

$$\ell_q(r,R)\leq R\left(1+\frac{1}{\sqrt[4]{q}}+\frac{1}{\sqrt{q}}\right)q^{(r-R)/R}+\frac{R}{2}\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor,\ q=(q')^4,\ t\geq 1; \tag{1.8}$$

$$\ell_q(r,R)\leq R\left(1+\frac{1}{\sqrt[6]{q}}+\frac{1}{\sqrt[3]{q}}+\frac{1}{\sqrt{q}}\right)q^{(r-R)/R}+R\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor,\ q=(q')^6, \tag{1.9}$$

$q'\leq 73$ *prime*, $t\geq 1,\ t\neq 4,6.$

Problem 1. *Improve the known bounds on the length function $\ell_q(r,R)$ collected in*
(i) *Proposition* 1.6 *where* $R\geq 4,\ r=tR,\ t\geq 2,$
(ii) *Proposition* 1.7 *where $R\geq 2$ is even,* $r=tR+\frac{R}{2},\ t\geq 1.$

1.3 Saturating sets in projective spaces

Effective methods to obtain upper bounds on $\ell_q(r,R)$ are connected with *saturating sets in projective spaces.*

Let $\mathrm{PG}(N,q)$ be the N-dimensional projective space over the field $\mathbb{F}_q$; see [21–23] for an introduction to the projective spaces over finite fields, see also [19, 22, 26, 27] for connections between coding theory and Galois geometries.

Definition 1.8. (i) A point set $\mathcal{S}\subseteq\mathrm{PG}(N,q)$ is *ρ-saturating* if for any point A of $\mathrm{PG}(N,q)\setminus\mathcal{S}$ there exist $\rho+1$ points in $\mathcal{S}$ generating a subspace of $\mathrm{PG}(N,q)$ containing A, and ρ is the smallest value with such property.

(ii) A point set $\mathcal{S}\subseteq\mathrm{PG}(N,q)$ is *ρ-saturating* if every point $A\in\mathrm{PG}(N,q)$ (in the homogeneous coordinates) can be written as a linear combination of at most $\rho+1$ points of $\mathcal{S}$, and ρ is the smallest value with such property (cf. Definition 1.1(ii)).

Definitions 1.8(i) and 1.8(ii) are equivalent.

Saturating sets are considered, for instance, in [5, 6, 10, 12–17, 19, 20, 24, 26, 27, 30]. In the literature, saturating sets are also called "saturated sets", "spanning sets", "dense sets".

Let $s_q(N, \rho)$ be *the smallest size of a ρ-saturating set* in $\mathrm{PG}(N, q)$.

If q-ary positions of a column of an $r \times n$ parity check matrix of an $[n, n-r]_qR$ code are treated as homogeneous coordinates of a point in $\mathrm{PG}(r-1, q)$ then this parity check matrix defines an $(R-1)$-saturating set of size n in $\mathrm{PG}(r-1, q)$ [6, 10, 13, 14, 16, 19, 20, 24, 26, 27]. So, there is a *one-to-one correspondence between $[n, n-r]_qR$ codes and $(R-1)$-saturating n-sets in* $\mathrm{PG}(r-1, q)$. Therefore,

$$\ell_q(r, R) = s_q(r-1, R-1).$$

Recall that the results of Proposition 1.6 are based on direct sum of codes of radius $R = 2, 3$. The following geometrical constructions make an important contribution to the structures of the best codes with $R = 2, 3$:

• "oval plus line" [6, p. 104], [10, Th. 5.1]; the construction obtains an 1-saturating $(2q+1)$-set in $\mathrm{PG}(3, q)$ that corresponds to an $[2q+1, (2q+1)-4]_q2$ code with $r = 2R$;

• "two ovals plus line" [16, Sect. 4]; the construction obtains a 2-saturating $(3q+1)$-set in $\mathrm{PG}(5, q)$ that corresponds to a $[3q+1, (3q+1)-6]_q3$ code with $r = 2R$.

Problem 2. [14, Sect. 6.1] *For all $\rho \geq 3$ obtain a general construction of a ρ-saturating $((\rho+1)q+1)$-set in $\mathrm{PG}(2\rho+1, q)$ that corresponds to an $[Rq+1, Rq+1-2R]_qR$ code with $R = \rho+1$. In other words, prove (constructively) that $s_q(2\rho+1, \rho) \leq (\rho+1)q+1$ and thereby prove that $\ell_q(2R, R) \leq Rq+1$.*

Note that for $n < (\rho+1)q+1 = Rq+1$, no examples of ρ-saturating n-sets in $\mathrm{PG}(2\rho+1, q)$ (resp. $[n, n-2R]_qR$ codes with $R = \rho+1$) seem to be known. Moreover, in [14, Prop. 4.2], it is proved that $\ell_4(4, 2) = s_4(3, 1) = 2 \cdot 4 + 1$. This strengthens the interest to Problem 2 and gives rise to the following.

Problem 3. [14, Sects. 4, 5] *Determining whether $\ell_q(2R, R) = Rq+1$, equivalently whether $s_q(2\rho+1, \rho) = (\rho+1)q+1$.*

Definition 1.9. A ρ-saturating set in $\mathrm{PG}(N, q)$ is *minimal* if it does not contain a smaller ρ-saturating set in $\mathrm{PG}(N, q)$.

If the positions of a column of a parity check matrix of an $[n, n-r]_qR$ locally optimal code are considered as homogeneous coordinates of a point in $\mathrm{PG}(r-1, q)$ then this parity check matrix defines a minimal $(R-1)$-saturating n-set in $\mathrm{PG}(r-1, q)$ [12]. So, there is a a one-to-one correspondence between $[n, n-r]_qR$ *locally optimal* codes and *minimal* $(R-1)$-saturating n-sets in $\mathrm{PG}(r-1, q)$.

If for the solution of Problem 2 we obtain minimal $((\rho+1)q+1)$-sets in $\mathrm{PG}(2\rho+1,q)$ (resp. locally optimal $[Rq+1, Rq+1-2R]_qR$ codes), this advances the solution of Problem 3.

Note that the codes providing the bounds of Proposition 1.7 are based on 1-saturating sets in the projective plane of square order. Improvements of these bounds could be connected with new 1-saturating sets of relatively small sizes.

Problem 4. *In* $\mathrm{PG}(2,q^2)$*, construct new* 1*-saturating sets with sizes smaller than the known ones.*

1.4 The goals and the structure of the paper

The goals of this paper:

• solve Problem 2 and with the help of the new $[Rq+1, Rq+1-2R]_qR$ codes solve Problem 1(i) regarding codes of covering radius $R \geq 4$ and codimension tR;

• solve Problem 4 and with the help of the new 1-saturating sets solve Problem 1(ii) regarding codes with even covering radius $R \geq 2$ and codimension $tR + \frac{R}{2}$.

The paper is organized as follows. In Section 2 we collect the main results of the paper. In Section 3, we propose a construction "line plus ρ ovals" for ρ-saturating sets in $\mathrm{PG}(2\rho+1,q)$ and codes of codimension $2R$. This solves Problem 2. In Section 4, we describe two constructions from the family of the so-called "q^m-concatenating constructions" for code codimension lifting. The constructions are convenient for $[n, n-r]_qR, \ell$ codes with $\ell \in \{R-1, R\}$. In Section 5, we prove that the codes obtained in Section 3 have $\ell = R$ for odd q and $\ell = R-1$ for even q. (So, for odd q we have surface-covering codes.) Then we use these codes as starting ones for the constructions of Section 4. As the result, we obtained new infinite code families with fixed radius $R \geq 4$ and growing codimension tR. This solves Problem 1(i) for the most part. In Section 6, using recent results on double blocking set, we obtain new 1-saturating sets in $\mathrm{PG}(2,q^2)$ that solve in part Problem 4. Then basing on these sets, we obtain new infinite code families for all fixed even radii $R \geq 2$ and growing codimension $tR + \frac{R}{2}$. This solves in part Problem 1(ii).

2 The main results

The main results of this paper are as follows:

• Problem 2 is solved, see Section 3. For all $\rho \geq 0$ we propose a general regular construction ("Line-Ovals") of a minimal ρ-saturating $((\rho+1)q+1)$-set in $\mathrm{PG}(2\rho+1,q)$. This set corresponds to an $[Rq+1, Rq+1-2R]_qR$ locally optimal code with $R = \rho+1$. Thereby we have proved that $s_q(2\rho+1,\rho) \leq (\rho+1)q+1$ and, equivalently, $\ell_q(2R,R) \leq Rq+1$. The minimality of the obtained ρ-saturating set allows to hope that Problem 3 can be solved.

• Problem 1(i) is solved for the most part, see Sections 4 and 5. We described two constructions for code codimension lifting. Using the $[Rq+1, Rq+1-2R]_qR$ codes as a start for these constructions, we obtained infinite code families with fixed radius $R \geq 4$ and growing codimension tR. These families improve the known results collected in Proposition 1.6 apart from $t=3$. New bounds on the length function obtained in this paper are given in Theorem 2.1 based on Theorems 3.8, 3.10, 5.3, 5.4.

Theorem 2.1. *Let t be a growing integer. For the length function $\ell_q(r,R)$ and for the smallest size $s_q(r-1,R-1)$ of a $(R-1)$-saturating set in the projective space $\mathrm{PG}(r-1,q)$ the following constructive upper bounds (provided by infinite families of codes) hold:*

$$\ell_q(r,R) = s_q(r-1,R-1) \leq Rq^{(r-R)/R} + q^{(r-2R)/R} + \Delta_q(r,R), \ r = tR,$$

where for $m_1 = \lceil \log_q(R+1) \rceil + 1$ we have

(i) $\Delta_q(r,R) = 0$ *if* $t=2,\ q=4$ *and* $q \geq 7,\ R \geq 4$;

(ii) $\Delta_q(r,R) = 0$ *if* $t=2,\ q=5,\ R=4,5$;

(iii) $\Delta_q(r,R) = 0$ *if* $t \geq \lceil \log_q R \rceil + 3,\ q \geq 7$ *odd*, $R \geq 4$;

(iv) $\Delta_q(r,R) = \sum\limits_{j=2}^{t} q^{(r-jR)/R}$ *if* $m_1 + 2 < t < 3m_1 + 2,\ q \geq 8$ *even*, $R \geq 4$;

(v) $\Delta_q(r,R) = \sum\limits_{j=2}^{m_1+2} q^{(r-jR)/R}$ *if* $t = m_1 + 2$ *and* $t \geq 3m_1 + 2,\ q \geq 8$ *even*, $R \geq 4$.

The new bounds of Theorem 2.1 are better than the known ones of Proposition 1.6. In particular, in Proposition 1.6, the coefficient for $q^{(r-2R)/R}$ is $\lceil \frac{R}{3} \rceil$, whereas in Theorem 2.1 it is equal to 1 or 2, see (i)–(iii) and (iv)–(v), respectively. Note that in the cases (iv)–(v), the coefficient is equal to 2 since the term with $j=2$ of the sum in $\Delta_q(r,R)$ is $q^{(r-2R)/R}$.

• Problem 4 is solved in part, see Section 6.

Throughout the paper we use the following notation:

$$\phi(q) \text{ is the order of the largest proper subfield of } \mathbb{F}_q; \tag{2.1}$$

$$f_q(r,R) = \begin{cases} 0 & \text{if} \quad r \neq \frac{9R}{2}, \frac{13R}{2} \\ q^{(r-3R)/R-0.5} + q^{(r-4R)/R-0.5} & \text{if} \quad r = \frac{9R}{2}, \frac{13R}{2} \end{cases}. \tag{2.2}$$

In Theorem 6.3(v),(vi), using recent results on double blocking set, it is shown that in $\mathrm{PG}(2,q)$ there are 1-saturating sets of the following sizes:

$$2\sqrt{q} + 2\frac{\sqrt{q}-1}{\phi(\sqrt{q}\,)-1}, \qquad q = p^{2h},\ h \geq 2,\ p \geq 3 \text{ prime};$$

$$2\sqrt{q}+2\frac{\sqrt{q}}{p}+2, \qquad\qquad q=p^{2h},\ h\geq 2,\ p\geq 7 \text{ prime}.$$

The new 1-saturating sets have smaller sizes than the known ones, see Remark 6.4.

• Problem 1(ii) is solved in part, see Section 6. Using the new 1-saturating sets in $PG(2,q)$, we obtained infinite families of codes with covering radius $R=2$, see Theorem 6.9, and, basing on them, we constructed infinite code families with fixed even radius $R\geq 2$ and growing codimension $tR+\frac{R}{2}$, see Theorem 6.11 that gives rise to Theorem 2.2.

Theorem 2.2. *Assume that p is prime, $q=p^{2\eta}$, $\eta\geq 2$, covering radius $R\geq 2$ is even, and code codimension is $r=tR+\frac{R}{2}$ with growing integer $t\geq 1$. Let $\phi(\sqrt{q}\,)$ and $f_q(r,R)$ be as in* (2.1), (2.2). *The following constructive upper bounds on the length function (provided by infinite families of codes with growing codimension) hold:*

(i) $\ell_q(r,R)\leq R\left(1+\dfrac{\sqrt{q}-1}{\sqrt{q}(\phi(\sqrt{q}\,)-1)}\right)q^{(r-R)/R}+R\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor+\dfrac{R}{2}f_q(r,R),\ p\geq 3;$

(ii) $\ell_q(r,R)\leq R\left(1+\dfrac{1}{p}+\dfrac{1}{\sqrt{q}}\right)q^{(r-R)/R}+R\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor+\dfrac{R}{2}f_q(r,R),\ p\geq 7.$

If $\sqrt{q}=p^{\eta}$ with $\eta\geq 3$ odd, the new bounds of Theorem 2.2 are better than the known ones of Proposition 1.7. For example, if $q=p^6$, $\eta=3$, then the bound of Theorem 2.2(ii) is by $Rq^{(r-R)/R-1/3}$ smaller than the known one of (1.9). Also, the new bound holds for all $p\geq 7$ whereas in (1.9) $p\leq 73$. Moreover, if $\eta\geq 5$ odd, the known bounds (1.7) have the main term $\frac{3}{2}Rq^{(r-R)/R}$ whereas for the new bounds it is $Rq^{(r-R)/R}$.

3 Construction "Line-Ovals" for ρ-saturating sets in $\mathrm{PG}(2\rho+1,q)$ and codes of codimension $2R$

Notation. Throughout the paper we denote by x_i, $i=0,1,\dots,N$, the homogeneous coordinates of points of $PG(N,q)$. In the other words, a point $(x_0x_1\dots x_N)\in\mathrm{PG}(N,q)$. The leftmost nonzero coordinate is equal to 1. In general, **by default**, $x_i\in\mathbb{F}_q$. If $x_i\in\mathbb{F}_q^*$, we denote it as $\widehat{x}_i$. If $(x_i\dots x_{i+m})\neq(0\dots 0)$, we denote it as $\overline{x_i\dots x_{i+m}}$. Also, we can write explicit values 0,1 for some coordinates or denote coordinates by letters values of which is explained later.

3.1 The construction

Let $\mathbb{F}_q=\{a_1=0,a_2,\dots,a_q\}$ be the Galois field of order q. Let $\mathbb{F}_q^*=\mathbb{F}_q\setminus\{0\}=\{a_2,\dots,a_q\}$. Denote $\Sigma_\rho=\mathrm{PG}(2\rho+1,q)$. Let Σ_u be the $(2u+1)$-subspace of Σ_ρ such that

$$\Sigma_u=\{(\underbrace{x_0x_1\dots x_{2u+1}}_{2u+2}\underbrace{0\dots0}_{2\rho-2u}):x_i\in\mathbb{F}_q\},\ u=0,1,\dots,\rho.$$

In Σ_u, let π_u be the plane such that

$$\pi_u = \{(\underbrace{0\ldots0}_{2u-1}x_{2u-1}x_{2u}x_{2u+1}\underbrace{0\ldots0}_{2\rho-2u}) : x_i \in \mathbb{F}_q\} \subset \Sigma_u,\ u = 1, 2, \ldots, \rho.$$

In π_u, let A_u^0 and A_u^∞ be the points of the form

$$A_u^0 = (\underbrace{0\ldots0}_{2u-1}100\underbrace{0\ldots0}_{2\rho-2u}) \in \pi_u,\ A_u^\infty = (\underbrace{0\ldots0}_{2u-1}001\underbrace{0\ldots0}_{2\rho-2u}) \in \pi_u,\ u = 1, 2, \ldots, \rho.$$

In π_u, let $\mathcal{C}_u$ and $\mathcal{C}_u^*$ be the conic and the truncated one, respectively, of the form

$$\mathcal{C}_u = \mathcal{C}_u^* \cup \{A_u^0, A_u^\infty\},\ \mathcal{C}_u^* = \{(\underbrace{0\ldots0}_{2u-1}1aa^2\underbrace{0\ldots0}_{2\rho-2u}) : a \in \mathbb{F}_q^*\},\ u = 1, 2, \ldots, \rho.$$

Let T_u be the nucleus of $\mathcal{C}_u$, if q is even, or the intersection of the tangents to $\mathcal{C}_u$ in A_u^0 and A_u^∞, if q is odd, so that

$$T_u = (\underbrace{0\ldots0}_{2u-1}010\underbrace{0\ldots0}_{2\rho-2u}) \in \pi_u,\ u = 1, 2, \ldots, \rho.$$

Finally, in Σ_0, let A_0^0 and A_0^∞ be the points of the form $A_0^0 = (10\underbrace{0\ldots0}_{2\rho})$, $A_0^\infty = (01\underbrace{0\ldots0}_{2\rho})$.

Also, let $\mathcal{L}_0$ and $\mathcal{L}_0^*$ be the line and the truncated one, respectively, such that

$$\mathcal{L}_0 = \mathcal{L}_0^* \cup \{A_0^0, A_0^\infty\} \subset \Sigma_0,\ \mathcal{L}_0^* = \{(1a\underbrace{0\ldots0}_{2\rho}) : a \in \mathbb{F}_q^*\} \subset \Sigma_0.$$

Construction S. ("Line-Ovals") Let $\rho \geq 0$. Let $\mathcal{S}_\rho$ be a point $((\rho+1)q+1)$-subset of Σ_ρ. Let P_j be the j-th point of $\mathcal{S}_\rho$, $j = 1, 2, \ldots, (\rho+1)q+1$. We construct S_ρ as follows:

$$\mathcal{S}_\rho = \{A_0^0\} \cup \mathcal{L}_0^* \cup \bigcup_{u=1}^{\rho}\left(\mathcal{C}_u^* \cup \{T_u\}\right) \cup \{A_\rho^\infty\} = \left\{P_1, P_2, \ldots, P_{(\rho+1)q+1}\right\} \tag{3.1}$$

$$= \left\{\begin{array}{c||c||c|c||c|c||c||c|c||c|c||c}
1 & 1\ldots1 & 0\ldots0 & 0 & 0\ldots0 & 0 & \ldots & 0\ \ldots\ 0 & 0 & 0\ldots0 & 0 & 0\\
0 & a_2\ldots a_q & 1\ \ldots\ 1 & 0 & 0\ldots0 & 0 & \ldots & 0\ \ldots\ 0 & 0 & 0\ldots0 & 0 & 0\\
0 & 0\ldots0 & a_2\ldots a_q & 1 & 0\ldots0 & 0 & \ldots & 0\ \ldots\ 0 & 0 & 0\ldots0 & 0 & 0\\
0 & 0\ldots0 & a_2^2\ldots a_q^2 & 0 & 1\ \ldots\ 1 & 0 & \ldots & 0\ \ldots\ 0 & 0 & 0\ldots0 & 0 & 0\\
0 & 0\ldots0 & 0\ldots0 & 0 & a_2\ldots a_q & 1 & \ldots & 0\ \ldots\ 0 & 0 & 0\ldots0 & 0 & 0\\
0 & 0\ldots0 & 0\ldots0 & 0 & a_2^2\ldots a_q^2 & 0 & \ldots & 0\ \ldots\ 0 & 0 & 0\ldots0 & 0 & 0\\
 & \ldots & \ldots & & \ldots & & \ldots & \ldots & & \ldots & & \\
0 & 0\ldots0 & 0\ldots0 & 0 & 0\ldots0 & 0 & \ldots & 1\ \ldots\ 1 & 0 & 0\ldots0 & 0 & 0\\
0 & 0\ldots0 & 0\ldots0 & 0 & 0\ldots0 & 0 & \ldots & a_2\ \ldots\ a_q & 1 & 0\ldots0 & 0 & 0\\
0 & 0\ldots0 & 0\ldots0 & 0 & 0\ldots0 & 0 & \ldots & a_2^2\ \ldots\ a_q^2 & 0 & 1\ \ldots\ 1 & 0 & 0\\
0 & 0\ldots0 & 0\ldots0 & 0 & 0\ldots0 & 0 & \ldots & 0\ \ldots\ 0 & 0 & a_2\ldots a_q & 1 & 0\\
0 & 0\ldots0 & 0\ldots0 & 0 & 0\ldots0 & 0 & \ldots & 0\ \ldots\ 0 & 0 & a_2^2\ldots a_q^2 & 0 & 1\\
- & -\ -\ - & -\ -\ - & - & -\ -\ - & - & - & -\ -\ - & - & -\ -\ - & - & -\\
A_0^0 & \mathcal{L}_0^* & \mathcal{C}_1^* & T_1 & \mathcal{C}_2^* & T_2 & \ldots & \mathcal{C}_{\rho-1}^* & T_{\rho-1} & \mathcal{C}_\rho^* & T_\rho & A_\rho^\infty
\end{array}\right\}.$$

The points P_j of $\mathcal{S}_\rho$ have the form

$$P_1=(10\underbrace{0\ldots0}_{2\rho})=A_0^0;\quad P_j=(1a_j\underbrace{0\ldots0}_{2\rho}),\ a_j\in\mathbb{F}_q^*,\quad j=2,3,\ldots,q; \tag{3.2}$$
$$P_{uq+j-1}=(\underbrace{0\ldots0}_{2u-1}1a_ja_j^2\underbrace{0\ldots0}_{2\rho-2u}),\ a_j\in\mathbb{F}_q^*,\quad u=1,2,\ldots,\rho,\ j=2,3,\ldots,q;$$
$$P_{(u+1)q}=(\underbrace{0\ldots0}_{2u-1}010\underbrace{0\ldots0}_{2\rho-2u})=T_u,\ u=1,2,\ldots,\rho;\quad P_{(\rho+1)q+1}=A_\rho^\infty.$$

Example 3.1. By (3.1), $S_0=\{A_0^0\}\cup\mathcal{L}_0^*\cup\{A_0^\infty\}$, $S_1=\{A_0^0\}\cup\mathcal{L}_0^*\cup\mathcal{C}_1^*\cup\{T_1,A_1^\infty\}$, $S_2=\{A_0^0\}\cup\mathcal{L}_0^*\cup\mathcal{C}_1^*\cup\{T_1\}\cup\mathcal{C}_2^*\cup\{T_2,A_2^\infty\}$. By (3.1), (3.2), we have

$$\mathcal{S}_0=\left\{\begin{array}{c||c||c} 1 & 1\ \ldots\ 1 & 0\\ 0 & a_2\ldots a_q & 1\\ - & -\ -\ - & -\\ A_0^0 & \mathcal{L}_0^* & A_0^\infty \end{array}\right\},\ \mathcal{S}_1=\left\{\begin{array}{c||c||c|c||c} 1 & 1\ \ldots\ 1 & 0\ \ldots\ 0 & 0 & 0\\ 0 & a_2\ldots a_q & 1\ \ldots\ 1 & 0 & 0\\ 0 & 0\ \ldots\ 0 & a_2\ldots a_q & 1 & 0\\ 0 & 0\ \ldots\ 0 & a_2^2\ldots a_q^2 & 0 & 1\\ - & -\ -\ - & -\ -\ - & - & -\\ A_0^0 & \mathcal{L}_0^* & \mathcal{C}_1^* & T_1 & A_1^\infty \end{array}\right\},$$

$$\mathcal{S}_2=\left\{\begin{array}{c||c||c|c||c|c||c} 1 & 1\ \ldots\ 1 & 0\ \ldots\ 0 & 0 & 0\ \ldots\ 0 & 0 & 0\\ 0 & a_2\ldots a_q & 1\ \ldots\ 1 & 0 & 0\ \ldots\ 0 & 0 & 0\\ 0 & 0\ \ldots\ 0 & a_2\ldots a_q & 1 & 0\ \ldots\ 0 & 0 & 0\\ 0 & 0\ \ldots\ 0 & a_2^2\ldots a_q^2 & 0 & 1\ \ldots\ 1 & 0 & 0\\ 0 & 0\ \ldots\ 0 & 0\ \ldots\ 0 & 0 & a_2\ldots a_q & 1 & 0\\ 0 & 0\ \ldots\ 0 & 0\ \ldots\ 0 & 0 & a_2^2\ldots a_q^2 & 0 & 1\\ - & -\ -\ - & -\ -\ - & - & -\ -\ - & - & -\\ A_0^0 & \mathcal{L}_0^* & \mathcal{C}_1^* & T_1 & \mathcal{C}_2^* & T_2 & A_2^\infty \end{array}\right\}.$$

3.2 Saturation of Construction S for $0\le\rho\le2$

We say that a point $A\in\mathrm{PG}(N,q)$ is *ρ-covered* by a set $\mathcal{S}$ if A is a linear combination of less than or equal to $\rho+1$ points of a $\mathcal{S}$. A subset $\mathcal{G}\subset\mathrm{PG}(N,q)$ is *ρ-covered* by $\mathcal{S}$ if all points of $\mathcal{G}$ are ρ-covered by $\mathcal{S}$.

Definition 3.2. Let $\mathcal{S}$ be a ρ-saturating set in $\mathrm{PG}(N,q)$. A point $A\in\mathcal{S}$ is *ρ-essential* if $\mathcal{S}\setminus\{A\}$ is no longer a ρ-saturating set. A point $A\in\mathcal{S}$ is *ρ-essential* for a set $\widetilde{\mathcal{M}_\rho}(A)\subset\mathrm{PG}(N,q)$ if all points of $\widetilde{\mathcal{M}_\rho}(A)$ are not ρ-covered by $\mathcal{S}\setminus\{A\}$. We denote by $\mathcal{M}_\rho(A)$ a set such that $\widetilde{\mathcal{M}_\rho}(A)\subseteq\mathcal{M}_\rho(A)\subset\mathrm{PG}(N,q)$.

Note that by Definition 1.8, a 0-saturating set in $PG(N,q)$ is the whole space.

The following proposition is obvious.

Proposition 3.3. *Let $q \geq 3$. Let $\Sigma_0 = \mathrm{PG}(1,q)$. Let the set $\mathcal{S}_0 \subset \Sigma_0$ be as in* (3.1), (3.2) *see also Example* 3.1. *Then it holds that*

(i) *The $(q+1)$-set $\mathcal{S}_0$ is a minimal 0-saturating set in Σ_0.*

(ii) *The point A_0^∞ of $\mathcal{S}_0$ is 0-essential for the set $\widetilde{\mathcal{M}}_0(A_0^\infty)$ such that*

$$\widetilde{\mathcal{M}}_0(A_0^\infty) = \mathcal{M}_0(A_0^\infty) = \{A_0^\infty\} = \{(01)\}. \tag{3.3}$$

(iii) *The q-set $\mathcal{S}_0 \setminus \{A_0^\infty\}$ is 1-saturating in Σ_0.*

Lemma 3.4. **(i)** *Let $q = 4$ or $q \geq 7$. Then all points of $\pi_u \setminus \{A_u^0, A_u^\infty\}$ are 1-covered by $\mathcal{C}_u^* \cup \{T_u\}$, $u = 1, \dots, \rho$.*

(ii) *Let $q \geq 4$. Then all points of $\pi_\rho \setminus \{A_\rho^0\}$ are 1-covered by $\mathcal{C}_\rho^* \cup \{T_\rho, A_\rho^\infty\}$.*

Proof. **(i)** If q is even, every point of a plane outside of a hyperoval $\mathcal{C}_u \cup \{T_u\}$ lies on $(q+2)/2$ its bisecants. If q is odd, every point of a plane outside of a conic $\mathcal{C}_u$ lies on at least $(q-1)/2$ its bisecants. At most two of aforementioned bisecants will be removed if one removes A_u^0, A_u^∞ from $\mathcal{C}_u$. Thus, for $q = 4$ and $q \geq 7$, every point of $\pi_u \setminus \{A_u^0, A_u^\infty\}$ lies on at least one bisecant of $\mathcal{C}_u^* \cup \{T_u\}$.

(ii) The proof is similar to the case (i) taking into account that here we remove only one point A_ρ^0 from $\mathcal{C}_\rho$. □

Lemma 3.5. *Let $q \geq 4$, $\rho \geq 2$. Then it holds that*

(i) *The point $A_u^\infty = A_{u+1}^0$, $u = 1, \dots, \rho - 1$, is 2-covered by $\mathcal{C}_u^*$ as well as by $\mathcal{C}_{u+1}^*$.*

(ii) *The plane π_u, $u = 1, \dots, \rho$, is 2-covered by $\mathcal{C}_u^*$.*

Proof. Any three points of a conic generate the plane in which it lies. As $q \geq 4$, we have $\#\mathcal{C}_u^* \geq 3$. □

Proposition 3.6. *Let $q = 4$ or $q \geq 7$. Let $\Sigma_1 = \mathrm{PG}(3,q)$. Let the set $\mathcal{S}_1 \subset \Sigma_1$ be as in* (3.1), (3.2), *see also Example* 3.1. *Let $\mathcal{M}_0(A_0^\infty)$ be as in* (3.3). *Then it holds that*

(i) *The $(2q+1)$-set $\mathcal{S}_1$ is a minimal 1-saturating set in Σ_1.*

(ii) *The point A_1^∞ of $\mathcal{S}_1$ is 1-essential for the set $\widetilde{\mathcal{M}}_1(A_1^\infty)$ such that*

$$\widetilde{\mathcal{M}}_1(A_1^\infty) = \mathcal{M}_1(A_1^\infty) = \{(x_0 \dots x_3) : (x_0 x_1) \notin \mathcal{M}_0(A_0^\infty), (x_2 x_3) = (0\widehat{x}_3)\}. \tag{3.4}$$

(iii) *The $2q$-set $\mathcal{S}_1 \setminus \{A_1^\infty\}$ is 2-saturating in Σ_1.*

Proof. **(i)** By Proposition 3.3(iii) and Lemma 3.4, Σ_0 (points $(x_0 x_1 00)$) and π_1 (points $(0 x_1 x_2 x_3)$) are 1-covered by $\{A_0^0\} \cup \mathcal{L}_0^* \cup \mathcal{C}_1^* \cup \{T_1, A_1^\infty\}$. So, we should consider points of the form

$$B = (\widehat{x}_0 x_1 \overline{x_2 x_3}) = (1 x_1 \overline{x_2 x_3}) \in \Sigma_1 \setminus (\Sigma_0 \cup \pi_1). \tag{3.5}$$

We show that B in (3.5) is a linear combination of at most 2 points of $\mathcal{S}_1$.

1) Let $(x_0x_1) \in \mathcal{M}_0(A_0^\infty)$.
By the hypothesis, $(x_0x_1) = (01)$. By (3.5), we have no such points B.

2) Let $(x_0x_1) \notin \mathcal{M}_0(A_0^\infty)$.
By the hypothesis, (x_0x_100) is 0-covered by $S_0 \setminus \{A_0^\infty\}$, i.e. $(x_0x_100) = (1x_100) \in \{A_0^0\} \cup \mathcal{L}_0^*$. For B of (3.5), we have

$$B = (x_0x_10\widehat{x}_3) = (x_0x_100) + \widehat{x}_3(0001) = (x_0x_100) + \widehat{x}_3A_1^\infty; \qquad (3.6)$$
$$B = (x_0x_1\widehat{x}_20) = (x_0x_100) + \widehat{x}_2(0010) = (x_0x_100) + \widehat{x}_2T_1;$$
$$B = (x_0x_1\widehat{x}_2\widehat{x}_3) = (x_0z00) + \frac{\widehat{x}_2^2}{\widehat{x}_3}(01yy^2),\ z = x_1 - \frac{\widehat{x}_2^2}{\widehat{x}_3},\ y = \frac{\widehat{x}_3}{\widehat{x}_2}.$$

Note that $(x_0z00) = (1z00)$ is 0-covered by $S_0 \setminus \{A_0^\infty\}$ for any z.

From (3.6), we see that all points of S_1 are 1-essential.

(ii) The assertion follows from (3.6).

(iii) We have, cf. (3.6), $(1x_10\widehat{x}_3) = (1z00) + (010\widehat{x}_3)$, where $z = x_1 - 1$ and $(010\widehat{x}_3) \in \pi_1 \setminus \{A_1^0, A_1^\infty\}$ is 1-covered by $\mathcal{C}_1^* \cup \{T_1\}$, see Lemma 3.4. □

Proposition 3.7. *Let $q = 4$ or $q \geq 7$. Let $\Sigma_2 = \mathrm{PG}(5, q)$. Let the set $\mathcal{S}_2 \subset \Sigma_2$ be as in* (3.1)*,* (3.2)*, see also Example* 3.1*. Let $\mathcal{M}_1(A_1^\infty)$ be as in* (3.4)*. Then it holds that*

(i) *The $(3q+1)$-set $\mathcal{S}_2$ is a minimal 2-saturating set in Σ_2.*

(ii) *The point A_2^∞ of $\mathcal{S}_2$ is 2-essential for the set $\widetilde{\mathcal{M}_2}(A_2^\infty)$ such that*

$$\widetilde{\mathcal{M}_2}(A_2^\infty) \subset \mathcal{M}_2(A_2^\infty) = \{(x_0 \dots x_5) : (x_0 \dots x_3) \notin \mathcal{M}_1(A_1^\infty), (x_4x_5) = (0\widehat{x}_5)\}. \qquad (3.7)$$

(iii) *The $3q$-set $\mathcal{S}_2 \setminus \{A_2^\infty\}$ is 3-saturating in Σ_2.*

Proof. **(i)** By Propositions 3.3, 3.6 and Lemmas 3.4, 3.5, we have the following: Σ_0 (points (x_0x_10000)) is 1-covered by $\{A_0^0\} \cup \mathcal{L}_0^*$; π_1 (points $(0x_1x_2x_300)$) and π_2 (points $(000x_3x_4x_5)$) are 2-covered by $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$, respectively; $\pi_2 \setminus \{A_2^0\}$ is 1-covered by $\mathcal{C}_2^* \cup \{T_2, A_2^\infty\}$; Σ_1 (points $(x_0x_1x_2x_300)$) is 2-covered by $\mathcal{S}_1 \setminus \{A_1^\infty\}$. Recall that $\Sigma_0 \cup \pi_1 \subset \Sigma_1$. So, we should consider points of the form

$$B = (\overline{x_0x_1x_2}x_3\overline{x_4x_5}) \in \Sigma_2 \setminus (\Sigma_1 \cup \pi_2). \qquad (3.8)$$

We show that B in (3.8) is a linear combination of at most 3 points of $\mathcal{S}_2$.

1) Let $(x_0 \dots x_3) \in \mathcal{M}_1(A_1^\infty)$.
By the hypothesis and by (3.4), (3.8), we have

$$(x_0x_1) \notin \mathcal{M}_0(A_0^\infty),\ B = (x_0x_10\widehat{x}_3\overline{x_4x_5}) = (x_0x_10000) + (000\widehat{x}_3\overline{x_4x_5}),$$

where (x_0x_10000) is 0-covered by $S_0 \setminus \{A_0^\infty\}$ and $(000\widehat{x}_3\overline{x_4x_5}) \in \pi_2 \setminus \{A_2^0, A_2^\infty\}$ is 1-covered by $\mathcal{C}_2^* \cup \{T_2\}$, see Lemma 3.4.

2) Let $(x_0 \dots x_3) \notin \mathcal{M}_1(A_1^\infty)$.
By the hypothesis, $(x_0 \dots x_300)$ is 1-covered by $S_1 \setminus \{A_1^\infty\}$. We can write

$$B = (x_0 \dots x_30\widehat{x}_5) = (x_0 \dots x_300) + \widehat{x}_5(000001) = (x_0 \dots x_300) + \widehat{x}_5A_2^\infty; \qquad (3.9)$$

$$B = (x_0 \dots x_3 \widehat{x}_4 0) = (x_0 \dots x_3 00) + \widehat{x}_4(000010) = (x_0 \dots x_3 00) + \widehat{x}_4 T_2; \tag{3.10}$$

$$B = (x_0 \dots x_3 \widehat{x}_4 \widehat{x}_5) = (x_0 x_1 x_2 z 00) + \frac{\widehat{x}_4^2}{\widehat{x}_5}(0001yy^2),\ z = x_3 - \frac{\widehat{x}_4^2}{\widehat{x}_5},\ y = \frac{\widehat{x}_5}{\widehat{x}_4}. \tag{3.11}$$

In (3.9), (3.10), B is a linear combination of at most $(1+1)+1 = 3$ points. If $(x_0x_1x_2z) \notin \mathcal{M}_1(A_1^\infty)$, then the representation (3.11) is the needed linear combination. If $(x_0x_1x_2z) \in \mathcal{M}_1(A_1^\infty)$ whereas $(x_0 \dots x_3) \notin \mathcal{M}_1(A_1^\infty)$, then the only possible situation is $(x_0x_1) \notin \mathcal{M}_0(A_0^\infty)$ with $(x_2x_3) = (00)$, see (3.4). In this case,

$$B = (x_0 x_1 00 \widehat{x}_4 \widehat{x}_5) = (1 x_1 00 \widehat{x}_4 \widehat{x}_5) = (1x_1 0000) + (0000 \widehat{x}_4 \widehat{x}_5), \tag{3.12}$$

where $(1x_10000)$ is 0-covered by $\{A_0^0\} \cup \mathcal{L}_0^*$ and $(0000\widehat{x}_4\widehat{x}_5) \in \pi_2 \setminus \{A_2^0, A_2^\infty\}$ is 1-covered by $\mathcal{C}_2^* \cup \{T_2\}$, see Lemma 3.4. Thus, B in (3.12) is a linear combination of at most $(0+1)+(1+1) = 3$ points.

From (3.9)–(3.12) we see that all points of $\mathcal{S}_2 \setminus \mathcal{S}_1$ are 2-essential. Also, we take into account that $\mathcal{S}_1$ is a *minimal* 1-saturating set.

(ii) The assertion follows from (3.9). For some (but not for all) points in (3.9) we could avoid use of A_2^∞; this explains the sign "$\subset$" in (3.7). For example, let $B = (001\widehat{x}_3 0 \widehat{x}_5) \notin \mathcal{M}_1(A_1^\infty)$. Then $B = (001000) + \widehat{x}_3 \left(00010\frac{\widehat{x}_5}{\widehat{x}_3}\right)$, where $(001000) = T_1$ and $\left(00010\frac{\widehat{x}_5}{\widehat{x}_3}\right) \in \pi_2 \setminus \{A_2^0, A_2^\infty\}$ is 1-covered by $\mathcal{C}_2^* \cup \{T_2\}$, see Lemma 3.4. However, if $B = (00100\widehat{x}_5) \notin \mathcal{M}_1(A_1^\infty)$, we are not able to avoid use of A_2^∞.

(iii) We have, cf. (3.9), $B = (x_0 \dots x_3 0 \widehat{x}_5) = (x_0x_1x_2z00) + (00010\widehat{x}_5)$, where $z = x_3 - 1$ and $(00010\widehat{x}_5) \in \pi_2 \setminus \{A_2^0, A_2^\infty\}$ is 1-covered by $\mathcal{C}_2^* \cup \{T_2\}$, see Lemma 3.4. This representation of B is the needed linear combination of at most $(1+1)+(1+1) = 4$ columns if $(x_0x_1x_2z) \notin \mathcal{M}_1(A_1^\infty)$ whence $(x_0x_1x_2z00)$ is 1-covered by $\mathcal{S}_1 \setminus \{A_1^\infty\}$.

But if $(x_0x_1x_2z) \in \mathcal{M}_1(A_1^\infty)$, then by (3.4), $(x_0x_1) \notin \mathcal{M}_0(A_0^\infty)$ and we have, similarly to (3.12), $B = (1x_1000\widehat{x}_5) = (1x_10000) + \widehat{x}_5(000001)$, where $(1x_10000)$ is 0-covered by $\{A_0^0\} \cup \mathcal{L}_0^*$ and $(000001) = A_2^\infty \in \pi_2$ is 2-covered by $\mathcal{C}_2^*$, see Lemma 3.5. □

3.3 Saturation of Construction S for any ρ

Theorem 3.8. *Let $q = 4$ or $q \geq 7$. Let $\Upsilon \geq 1$. Let $\Sigma_\rho = \mathrm{PG}(2\rho+1, q)$. Let $\mathcal{S}_\rho$ be a point $((\rho+1)q+1)$-subset of Σ_ρ as in Construction S of (3.1), (3.2). Then it holds that*

(i) *The $((\rho+1)q+1)$-set $\mathcal{S}_\rho$ is a minimal ρ-saturating set in Σ_ρ, $\rho = 0, 1, \dots, \Upsilon$.*

(ii) *The point A_ρ^∞ of $\mathcal{S}_\rho$ is ρ-essential for the set $\widetilde{\mathcal{M}}_\rho(A_\rho^\infty)$ such that*

$$\begin{aligned}
&\widetilde{\mathcal{M}}_0(A_0^\infty) = \mathcal{M}_0(A_0^\infty) = \{(01)\},\\
&\widetilde{\mathcal{M}}_1(A_1^\infty) = \mathcal{M}_1(A_1^\infty) = \{(x_0 \dots x_3) : (x_0x_1) \notin \mathcal{M}_0(A_0^\infty), (x_2x_3) = (0\widehat{x}_3)\},\\
&\widetilde{\mathcal{M}}_\rho(A_\rho^\infty) \subset \mathcal{M}_\rho(A_\rho^\infty) = \{(x_0 \dots x_{2\rho+1}) : (x_0 \dots x_{2\rho-1}) \notin \mathcal{M}_{\rho-1}(A_{\rho-1}^\infty),\\
&(x_{2\rho}x_{2\rho+1}) = (0\widehat{x}_{2\rho+1})\},\ \rho = 2, 3, \dots, \Upsilon.
\end{aligned} \tag{3.13}$$

(iii) *The* $(\rho+1)q$*-set* $\mathcal{S}_\rho \setminus \{A_\rho^\infty\}$ *is* $(\rho+1)$*-saturating in* Σ_ρ, $\rho = 0, 1, \ldots, \Upsilon$.

Proof. We prove by induction on Υ.

For $\Upsilon = 3$ the theorem is proved in Propositions 3.3, 3.6, 3.7.

Assumption: let the assertions (i)–(iii) *hold for some* $\Upsilon \geq 3$.

We show that under Assumption, the assertions hold for $\Gamma = \Upsilon + 1$.

(i) By Propositions 3.3, 3.6, 3.7, Lemmas 3.4, 3.5, and Assumption, we have the following: Σ_0 (points $(x_0x_10\ldots0)$) is 1-covered by $\{A_0^0\} \cup \mathcal{L}_0^*$; $\pi_1 \setminus \{A_1^\infty\}$, $\pi_u \setminus \{A_u^0, A_u^\infty\}$, $u = 2, 3, \ldots, \Gamma$, are 1-covered by $\{A_0^0\} \cup \mathcal{L}_0^* \cup \bigcup\limits_{u=1}^{\Gamma} (\mathcal{C}_u^* \cup \{T_u\})$; $\pi_\Gamma \setminus \{A_\Gamma^0\}$ is 1-covered by $\mathcal{C}_\Gamma^* \cup \{T_\Gamma, A_\Gamma^\infty\}$; π_1 (points $(0x_1x_2x_30\ldots0)$), π_2 (points $(000x_3x_4x_50\ldots0)$), …, π_Γ (points $(0\ldots0x_{2\Gamma-1}x_{2\Gamma}x_{2\Gamma+1})$) are 2-covered by $\mathcal{C}_1^*$, $\mathcal{C}_2^*$, …, $\mathcal{C}_\Gamma^*$, respectively; Σ_Υ is Γ-covered by $\mathcal{S}_\Upsilon \setminus \{A_\Upsilon^\infty\}$. Recall that $\Sigma_0 \cup \bigcup\limits_{u=1}^{\Upsilon} \pi_u \subset \Sigma_\Upsilon$. So, we should consider points of the form

$$B = (\overline{x_0 \ldots x_{2\Gamma-2}}x_{2\Gamma-1}\overline{x_{2\Gamma}x_{2\Gamma+1}}) \in \Sigma_\Gamma \setminus (\Sigma_\Upsilon \cup \pi_\Gamma). \tag{3.14}$$

We show that B in (3.14) is a linear combination of at most $\Gamma + 1$ points of $\mathcal{S}_\Gamma$.

1)Let $(x_0 \ldots x_{2\Gamma-1}) \in \mathcal{M}_\Upsilon(A_\Upsilon^\infty)$.

By the hypothesis and by (3.13), $(x_0 \ldots x_{2\Upsilon-1}) \notin \mathcal{M}_{\Upsilon-1}(A_{\Upsilon-1}^\infty)$. Therefore, $(x_0 \ldots x_{2\Upsilon-1}0000)$ is $(\Upsilon-1)$-covered by $\mathcal{S}_{\Upsilon-1} \setminus \{A_{\Upsilon-1}^\infty\}$. Now by (3.14), we have

$$B = (x_0 \ldots x_{2\Upsilon-1}0\widehat{x}_{2\Gamma-1}\overline{x_{2\Gamma}x_{2\Gamma+1}}) = (x_0 \ldots x_{2\Upsilon-1}0000) + (0\ldots0\widehat{x}_{2\Gamma-1}\overline{x_{2\Gamma}x_{2\Gamma+1}}), \tag{3.15}$$

where $(0\ldots0\widehat{x}_{2\Gamma-1}\overline{x_{2\Gamma}x_{2\Gamma+1}}) \in \pi_\Gamma \setminus \{A_\Gamma^0, A_\Gamma^\infty\}$ is 1-covered by $\mathcal{C}_\Gamma^*$, see Lemma 3.4. Thus, B in (3.15) is a linear combination of at most $(\Upsilon - 1 + 1) + (1 + 1) = \Gamma + 1$ points.

2) Let $(x_0 \ldots x_{2\Gamma-1}) \notin \mathcal{M}_\Upsilon(A_\Upsilon^\infty)$.

By the hypothesis, $(x_0 \ldots x_{2\Gamma-1}00)$ is Υ-covered by $S_\Upsilon \setminus \{A_\Upsilon^\infty\}$. We can write

$$B = (x_0 \ldots x_{2\Gamma-1}0\widehat{x}_{2\Gamma+1}) = (x_0 \ldots x_{2\Gamma-1}00) + \widehat{x}_{2\Gamma+1}A_\Gamma^\infty; \tag{3.16}$$

$$B = (x_0 \ldots x_{2\Gamma-1}\widehat{x}_{2\Gamma}0) = (x_0 \ldots x_{2\Gamma-1}00) + \widehat{x}_{2\Gamma}T_\Gamma; \tag{3.17}$$

$$B = (x_0 \ldots x_{2\Gamma-1}\widehat{x}_{2\Gamma}\widehat{x}_{2\Gamma+1}) = (x_0 \ldots x_{2\Gamma-2}z00) + \frac{\widehat{x}_{2\Gamma}^2}{\widehat{x}_{2\Gamma+1}}(0\ldots01yy^2), \tag{3.18}$$

$$z = x_{2\Gamma-1} - \frac{\widehat{x}_{2\Gamma}^2}{\widehat{x}_{2\Gamma+1}},\ y = \frac{\widehat{x}_{2\Gamma+1}}{\widehat{x}_{2\Gamma}}.$$

In (3.16), (3.17), B is a linear combination of at most $(\Upsilon + 1) + 1 = \Gamma + 1$ points. If $(x_0 \ldots x_{2\Gamma-2}z) \notin \mathcal{M}_\Upsilon(A_\Upsilon^\infty)$, then the representation (3.18) is the needed linear combination. If $(x_0 \ldots x_{2\Gamma-2}z) \in \mathcal{M}_\Upsilon(A_\Upsilon^\infty)$ while $(x_0 \ldots x_{2\Gamma-1}) \notin \mathcal{M}_\Upsilon(A_\Upsilon^\infty)$, then the only possible situation is $(x_0 \ldots x_{2\Upsilon-1}) \notin \mathcal{M}_{\Upsilon-1}(A_{\Upsilon-1}^\infty)$ with $(x_{2\Gamma-2}x_{2\Gamma-1}) = (00)$, see (3.13). In this case,

$$B = (x_0 \ldots x_{2\Upsilon-1}00\widehat{x}_{2\Gamma}\widehat{x}_{2\Gamma+1}) = (x_0 \ldots x_{2\Upsilon-1}0000) + (0\ldots0\widehat{x}_{2\Gamma}\widehat{x}_{2\Gamma+1}), \tag{3.19}$$

where $(x_0 \dots x_{2\Upsilon-1}0000)$ is $(\Upsilon - 1)$-covered by $S_{\Upsilon-1} \setminus \{A^\infty_{\Upsilon-1}\}$ and $(0 \dots 0\widehat{x}_4\widehat{x}_{2\Gamma-1}) \in \pi_\Gamma \setminus \{A^0_\Gamma, A^\infty_\Gamma\}$ is 1-covered by $\mathcal{C}^*_\Gamma \cup \{T_\Gamma\}$, see Lemma 3.4. Thus, B in (3.19) is a linear combination of at most $(\Upsilon - 1 + 1) + (1 + 1) = \Gamma + 1$ points.

From (3.15)–(3.19) we see that all the points of $\mathcal{S}_\Gamma \setminus \mathcal{S}_\Upsilon$ are Γ-essential. Also, we take into account that $\mathcal{S}_\Upsilon$ is a *minimal* Υ-saturating set.

(ii) The assertion (3.13) follows from (3.16). For some points in (3.16) we could avoid use of A^∞_Γ. This explains the sign "$\subset$" in (3.13).

(iii) We have, cf. (3.16), $B = (x_0 \dots x_{2\Gamma-1}0\widehat{x}_{2\Gamma+1}) = (x_0 \dots x_{2\Gamma-2}z00)+(0 \dots 010\widehat{x}_{2\Gamma+1})$, where $z = x_{2\Gamma-1} - 1$ and $(0 \dots 010\widehat{x}_{2\Gamma+1}) \in \pi_\Gamma \setminus \{A^0_\Gamma, A^\infty_\Gamma\}$ is 1-covered by $\mathcal{C}^*_\Gamma$, see Lemma 3.4. This representation of B is the needed linear combination of at most $(\Upsilon + 1) + (1 + 1) = \Gamma + 2$ points if $(x_0 \dots x_{2\Gamma-2}z) \notin \mathcal{M}_\Upsilon(A^\infty_\Upsilon)$ whence $(x_0 \dots x_{2\Gamma-2}z00)$ is Υ-covered by $\mathcal{S}_\Upsilon \setminus A^\infty_\Upsilon$.

But if $(x_0 \dots x_{2\Gamma-2}z) \in \mathcal{M}_\Upsilon(A^\infty_\Upsilon)$, then by (3.13), $(x_0 \dots x_{2\Upsilon-1}0000) \notin \mathcal{M}_{\Upsilon-1}(A^\infty_{\Upsilon-1})$, and we have, cf. (3.19), $(x_0 \dots x_{2\Upsilon-1}000\widehat{x}_{2\Gamma+1}) = (x_0 \dots x_{2\Upsilon-1}0000) + \widehat{x}_{2\Gamma+1}(0 \dots 01)$, where $(x_0 \dots x_{2\Upsilon-1}0000)$ is $(\Upsilon - 1)$-covered by $\mathcal{S}_{\Upsilon-1} \setminus \{A^\infty_{\Upsilon-1}\}$ and $(0 \dots 01) = A^\infty_\Gamma \in \pi_\Gamma$ is 2-covered by $\mathcal{C}^*_\Gamma$, see Lemma 3.5. □

By computer search for $q = 5$ we have proved the following proposition.

Proposition 3.9. *Let $q = 5$. Let $0 \le \rho \le 4$. Let $\Sigma_\rho = \mathrm{PG}(2\rho+1, 5)$. Let the $(5\rho+1)$-set $\mathcal{S}_\rho \subset \Sigma_\rho$ be as in* (3.1), (3.2). *Then $\mathcal{S}_\rho$ is a minimal ρ-saturating set in Σ_ρ.*

3.4 Codes of covering radius R and codimension $2R$

In the coding theory language, the results of this section give the following theorem.

Theorem 3.10. *Let $\widehat{V}_\rho$ be the code such that the columns of its parity check matrix are the points (in the homogeneous coordinates) of the ρ-saturating $((\rho + 1)q + 1)$-set $\mathcal{S}_\rho$ of Construction S* (3.1), (3.2).

(i) *Let $q = 4$ or $q \ge 7$. Then for all $R \ge 1$, the code $\widehat{V}_\rho$ is a $[Rq+1, Rq+1-2R, 3]_qR$ locally optimal code of covering radius $R = \rho + 1$.*

(ii) *Let $q = 5$. Then for $1 \le R \le 5$, the code $\widehat{V}_\rho$ is a $[5R+1, 5R+1-2R, 3]_5R$ locally optimal code of covering radius $R = \rho + 1$.*

Proof. We use Theorem 3.8 and Proposition 3.9. The code $\widehat{V}_\rho$ is locally optimal as the corresponding ρ-saturating set $\mathcal{S}_\rho$ is minimal. Minimum distance $d = 3$ is due to $\mathcal{L}^*_0$. □

Conjecture 3.11. **(i)** *Let $q = 5$. Let $\Sigma_\rho = \mathrm{PG}(2\rho+1, 5)$. Let the $(5\rho+1)$-set $\mathcal{S}_\rho \subset \Sigma_\rho$ be as in* (3.1), (3.2). *Then for all $\rho \ge 0$ it holds that $\mathcal{S}_\rho$ is a minimal ρ-saturating set in Σ_ρ.*

(ii) *Let $q = 5$. Let $\widehat{V}_\rho$ be as in Theorem* 3.10. *Then for all $R \ge 1$, the code $\widehat{V}_\rho$ is a $[5R + 1, 5R + 1 - 2R, 3]_5R$ locally optimal code with radius $R = \rho + 1$.*

4 The q^m-concatenating constructions for code codimension lifting

The q^m-concatenating constructions are proposed in [9] and are developed in [10–12, 14, 17, 18], see also [5], [7, Sec. 5.4] and the references in these works. By using a starting code as a "seed", a q^m-concatenating construction yields an infinite family of new codes with a fixed covering radius, growing codimension and with almost the same covering density.

We give versions of the q^m-concatenating constructions convenient for our goals. Several other versions of such constructions can be found in [9–12, 14, 17, 18] and the references therein. In Construction QM_1 below, we use a surface-covering code as a starting one, whereas for Construction QM_2 we need to start with an $[n, n-r]_q R, \ell$ code, $\ell = R-1$. Resulting codes of both the constructions are surface-covering.

Construction $\mathbf{QM_1}$. Let columns $\mathbf{h}_j$ belong to $\mathbb{F}_q^{r_0}$ and let $\mathbf{H}_0 = [\mathbf{h}_1\mathbf{h}_2 \ldots \mathbf{h}_{n_0}]$ be a parity check matrix of an $[n_0, n_0 - r_0]_q R, R$ *starting* surface-covering code V_0 with $R \geq 2$. Let $m \geq 1$ be an integer such that $q^m \geq n_0 - 1$. To each column $\mathbf{h}_j$ we associate an element $\beta_j \in \mathbb{F}_{q^m} \cup \{*\}$ so that $\beta_i \neq \beta_j$ if $i \neq j$. Let a new code V be the $[n, n-(r_0+Rm)]_q R_V, \ell_V$ code with $n = q^m n_0$ and parity check matrix of the form

$$\mathbf{H}_V = [\mathbf{B}_1 \ \mathbf{B}_2 \ \ldots \ \mathbf{B}_{n_0}], \tag{4.1}$$

$$\mathbf{B}_j = \begin{bmatrix} \mathbf{h}_j & \mathbf{h}_j & \cdots & \mathbf{h}_j \\ \xi_1 & \xi_2 & \cdots & \xi_{q^m} \\ \beta_j\xi_1 & \beta_j\xi_2 & \cdots & \beta_j\xi_{q^m} \\ \beta_j^2\xi_1 & \beta_j^2\xi_2 & \cdots & \beta_j^2\xi_{q^m} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_j^{R-1}\xi_1 & \beta_j^{R-1}\xi_2 & \cdots & \beta_j^{R-1}\xi_{q^m} \end{bmatrix} \text{ if } \beta_j \in \mathbb{F}_{q^m}, \quad \mathbf{B}_j = \begin{bmatrix} \mathbf{h}_j & \mathbf{h}_j & \cdots & \mathbf{h}_j \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \xi_1 & \xi_2 & \cdots & \xi_{q^m} \end{bmatrix} \text{ if } \beta_j = *, \tag{4.2}$$

where $\mathbf{B}_j$ is an $(r_0 + Rm) \times q^m$ matrix, 0 is the zero element of $\mathbb{F}_{q^m}$, ξ_u is an element of $\mathbb{F}_{q^m}$, $\{\xi_1, \xi_2, \ldots, \xi_{q^m}\} = \mathbb{F}_{q^m}$. An element of $\mathbb{F}_{q^m}$ written in $\mathbf{B}_j$ denotes an m-dimensional q-ary column vector that is a q-ary representation of this element.

We denote $\mathbf{b}_j(\xi_u) = (\mathbf{h}_j, \xi_u, \beta_j\xi_u, \beta_j^2\xi_u, \ldots, \beta_j^{R-1}\xi_u)$ the u-th column of $\mathbf{B}_j$ with $\beta_j \in \mathbb{F}_{q^m}$. If $\beta_j = *$, we have $\mathbf{b}_j(\xi_u) = (\mathbf{h}_j, 0, \ldots, 0, \xi_u)$.

Theorem 4.1. *In Construction QM_1, the new code V with the parity check matrix* (4.1), (4.2) *is an $[n, n-(r_0+Rm), 3]_q R, R$ surface-covering code with covering radius R and length $n = q^m n_0$. Moreover, if the starting code V_0 is locally optimal (non-shortening), then the new code V is locally optimal too.*

Proof. The length of the code V directly follows from the construction.

The minimum distance d is equal to 3 since for any pair of columns $\mathbf{b}_j(\xi_{u_1}), \mathbf{b}_j(\xi_{u_2})$ of $\mathbf{B}_j$, a 3-rd one can be found such that the column triple corresponds to a codeword of weight 3. Take $a, b, c \in \mathbb{F}_q^*$ with $a+b+c=0$. Put $\xi_{u_3} = (-a\xi_{u_1} - b\xi_{u_2})/c$. Then for all j we have

$$a\mathbf{b}_j(\xi_{u_1}) + b\mathbf{b}_j(\xi_{u_2}) + c\mathbf{b}_j(\xi_{u_3}) = \mathbf{0}, \tag{4.3}$$

where $\mathbf{0}$ is the zero $(r_0 + Rm)$-positional column.

We show that covering radius R_V of V is equal to R.

Consider an arbitrary column $\mathbf{t} = (\mathbf{fs}) \in \mathbb{F}_q^{r_0+Rm}$ with $\mathbf{f} \in \mathbb{F}_q^{r_0}$, $\mathbf{s} \in \mathbb{F}_q^{Rm}$, $\mathbf{s} = (s_1, s_2, \ldots, s_{Rm})$, $s_i \in \mathbb{F}_q$. We partition $\mathbf{s}$ by m-vectors so that $\mathbf{s} = (S_0, S_1, \ldots, S_{R-1})$, $S_v = (s_{vm+1}, s_{vm+2}, \ldots, s_{vm+m})$, $v = 0, 1, \ldots, R-1$. We treat S_v as an element of $\mathbb{F}_{q^m}$.

Since V_0 is an $[n_0, n_0 - r_0]_q R, R$ code, there exists a linear combination of the form

$$\mathbf{f} = \sum_{k=1}^{R} c_k \mathbf{h}_{j_k}, \ c_k \in \mathbb{F}_q^* \text{ for all } k, \tag{4.4}$$

see Definition 1.4. Now we can represent $\mathbf{t}$ as a linear combination (with nonzero coefficients) of R distinct columns of $\mathbf{H}_V$. We have, see (4.2),

$$\mathbf{t} = \sum_{k=1}^{R} c_k \mathbf{b}_{j_k}(x_k), \ c_k \in \mathbb{F}_q^* \text{ and } x_k \in \mathbb{F}_{q^m} \text{ for all } k, \tag{4.5}$$

where values of x_k are obtained from the linear system with nonzero determinant. If for j_k in (4.4) we have $\beta_{j_k} \in \mathbb{F}_{q^m}$ for all k, then the system has the form

$$\sum_{k=1}^{R} c_k \beta_{j_k}^v x_k = S_v, \ v = 0, 1, \ldots, R-1. \tag{4.6}$$

As usual, we put $0^0 = 1$. If in (4.4) we have, for example, $\beta_{j_R} = *$, then the system is as follows:

$$\sum_{k=1}^{R-1} c_k \beta_{j_k}^v x_k = S_v, \ v = 0, 1, \ldots, R-2; \ \sum_{k=1}^{R-1} c_k \beta_{j_k}^{R-1} x_k + c_R x_R = S_{R-1}. \tag{4.7}$$

If V_0 is a locally optimal code, then every column $\mathbf{h}_j$ of $\mathbf{H}_0$ takes part in a representation of the form (4.4). If we remove $\mathbf{b}_{j_k}(\xi_u)$ from $\mathbf{B}_{j_k}$ then there is $(s_1, s_2, \ldots, s_{Rm})$ such that the system (4.6) or (4.7) gives $x_k = \xi_u$. As a result, for some $\mathbf{t}$ the representation (4.5) becomes impossible. So, all columns of $\mathbf{H}_V$ are essential and the code V is locally optimal. □

Construction QM$_2$. Let $\theta_{m,q} = \frac{q^{m+1}-1}{q-1}$. Let columns $\mathbf{h}_j$ belong to $\mathbb{F}_q^{r_0}$ and let $\mathbf{H}_0 = [\mathbf{h}_1 \mathbf{h}_2 \ldots \mathbf{h}_{n_0}]$ be a parity check matrix of an $[n_0, n_0 - r_0]_q R, \ell_0$ *starting* code V_0 with $\ell_0 = R-1$, $R \geq 2$. Let $m \geq 1$ be an integer such that $q^m \geq n_0$. To each column $\mathbf{h}_j$

we associate an element $\beta_j \in \mathbb{F}_{q^m}$ so that $\beta_i \neq \beta_j$ if $i \neq j$. Let a new code V be the $[n, n-(r_0+Rm)]_q R_V, \ell_V$ code with $n = q^m n_0 + \theta_{m,q}$ and parity check matrix of the form

$$\mathbf{H}_V = [\mathbf{C}\ \mathbf{B}_1\ \mathbf{B}_2\ \ldots\ \mathbf{B}_{n_0}], \tag{4.8}$$

where $\mathbf{B}_j$ is an $(r_0 + Rm) \times q^m$ matrix as in (4.2), $\mathbf{C}$ is an $(r_0 + Rm) \times \theta_{m,q}$ matrix,

$$\mathbf{C} = \begin{bmatrix} \mathbf{0}_{r_0+(R-1)m} \\ \mathbf{W}_m \end{bmatrix}, \tag{4.9}$$

$\mathbf{0}_{r_0+(R-1)m}$ is the zero $(r_0+(R-1)m)\times\theta_{m,q}$ matrix, $\mathbf{W}_m$ is a parity check $m\times\theta_{m,q}$ matrix of the $[\theta_{m,q}, \theta_{m,q} - m, 3]_q 1$ Hamming code.

Theorem 4.2. *In Construction QM$_2$, the new code V with the parity check matrix* (4.8), (4.9), (4.2) *is an $[n, n-(r_0+Rm), 3]_q R, R$ surface-covering code with covering radius R and length $n = q^m n_0 + \frac{q^{m+1}-1}{q-1}$. Moreover, if the starting code V_0 is locally optimal (non-shortening), then the new code V is locally optimal too.*

Proof. The length of the code V directly follows from the construction.

The minimum distance is equal to 3 as the Hamming code is a code with $d = 3$. Also we can use (4.3) from the proof of Theorem 4.1.

We show that covering radius R_V of V is equal to R.

Consider an arbitrary column $\mathbf{t} = (\mathbf{fs}) \in \mathbb{F}_q^{r_0+Rm}$ with $\mathbf{f} \in \mathbb{F}_q^{r_0}$, $\mathbf{s} \in \mathbb{F}_q^{Rm}$, $\mathbf{s} = (s_1, s_2, \ldots, s_{Rm})$, $s_i \in \mathbb{F}_q$. We partition $\mathbf{s}$ by m-vectors so that $\mathbf{s} = (S_0, S_1, \ldots, S_{R-1})$, $S_v = (s_{vm+1}, s_{vm+2}, \ldots, s_{vm+m})$, $v = 0, 1, \ldots, R-1$. We treat S_v as an element of $\mathbb{F}_{q^m}$.

Since V_0 is an $[n_0, n_0 - r_0]_q R, \ell_0$ code with $\ell_0 = R-1$, there exists a linear combination of $\varphi(\mathbf{f})$ distinct columns of $\mathbf{H}_0$ of the form

$$\mathbf{f} = \sum_{k=1}^{\varphi(\mathbf{f})} c_k \mathbf{h}_{j_k},\ c_k \in \mathbb{F}_q^* \text{ for all } k, \varphi(\mathbf{f}) \in \{R-1, R\},$$

see Definition 1.4. If $\varphi(\mathbf{f}) = R$ we act similarly to the proof of Theorem 4.1.

Let $\varphi(\mathbf{f}) = R - 1$. We represent $\mathbf{t}$ as a linear combination (with nonzero coefficients) of at most R distinct columns of $\mathbf{H}_V$. We have, see (4.2), (4.9),

$$\mathbf{t} = \eta\mathbf{c} + \sum_{k=1}^{R-1} c_k \mathbf{b}_{j_k}(x_k),\ c_k \in \mathbb{F}_q^* \text{ and } x_k \in \mathbb{F}_{q^m} \text{ for all } k,\ \eta \in \mathbb{F}_q, \tag{4.10}$$

where $\mathbf{c}$ is a column of $\mathbf{C}$ and $\eta = 0$ means that the summand $\eta\mathbf{c}$ is absent. Also, in (4.10), values of x_k are obtained from the linear system

$$\sum_{k=1}^{R-1} c_k \beta_{j_k}^v x_k = S_v,\ v = 0, 1, \ldots, R-2,$$

with nonzero determinant. Finally, in (4.10), $\mathbf{c} = (\mathbf{0}\mathbf{w})$ where $\mathbf{0}$ is the zero $(r_0+(R-1)m)$-positional column and $\mathbf{w}$ is a column of $\mathbf{W}_m$ that satisfies the equality

$$\eta\mathbf{w} + \sum_{k=1}^{R-1} c_k\beta_{j_k}^{R-1}x_k = S_{R-1}. \tag{4.11}$$

In (4.11), if $\sum_{k=1}^{R-1} c_k\beta_{j_k}^{R-1}x_k = S_{R-1}$ we have $\eta = 0$. If $\sum_{k=1}^{R-1} c_k\beta_{j_k}^{R-1}x_k \neq S_{R-1}$, the needed column $\eta\mathbf{w}$ always exists as the Hamming code has covering radius 1.

Now we show that V is an $[n, n-(r_0+Rm), 3]_qR, R$ code, i.e. $\ell_V = R$. The critical situation is when in (4.10) and (4.11) $\eta = 0$, i.e. the summand $\eta\mathbf{c}$ is absent. We use the approach of the proof of Theorem 4.1 regarding (4.3). In (4.3) we put $j = j_1, \xi_{u_1} = x_1, a = -c_1$ with j_1, x_1, c_1 taken from (4.10). Then

$$\begin{aligned}\mathbf{t} &= -c_1\mathbf{b}_{j_1}(x_1) + b\mathbf{b}j_1(\xi_{u_2}) + c\mathbf{b}_{j_1}(\xi_{u_3}) + \sum_{k=1}^{R-1} c_k\mathbf{b}_{j_k}(x_k)\\ &= b\mathbf{b}j_1(\xi_{u_2}) + c\mathbf{b}_{j_1}(\xi_{u_3}) + \sum_{k=2}^{R-1} c_k\mathbf{b}_{j_k}(x_k).\end{aligned}$$

Thus, we always can represent $\mathbf{t} \in \mathbb{F}_q^{r_0+Rm}$ as a linear combination with nonzero coefficients of exactly R columns of $\mathbf{H}_V$.

By above, if we remove any column of $\mathbf{H}_V$, some representation of $\mathbf{t}$ becomes impossible. So, all columns of $\mathbf{H}_V$ are essential and the code V is locally optimal. □

5 New infinite code families with fixed radius $R \geq 4$ and growing codimension tR

In the minimal ρ-saturating set of Construction S (3.1), (3.2), we consider a point P_j (in the homogeneous coordinates) as a column $\mathbf{h}_j$ of the parity check matrix $\widehat{\mathbf{H}}_\rho$ that defines the $[qR+1, qR+1-2R, 3]_qR, \ell$ locally optimal code $\widehat{V}_\rho$ of covering radius $R = \rho + 1$.

We consider some properties of $\widehat{\mathbf{H}}_\rho$ useful to estimate ℓ. Let $\mathbf{f} \in \mathbb{F}_q^r$. Let $\mathcal{J}(\mathbf{f}) = \{\mathbf{h}_{j_1}, \dots, \mathbf{h}_{j_\beta}\}$ and $\mathcal{I}_w = \{\mathbf{h}_{i_1}, \dots, \mathbf{h}_{i_w}\}$ be sets of distinct columns of $\widehat{\mathbf{H}}_\rho$ such that

$$\mathbf{f} = \sum_{k=1}^{\beta} c_k\mathbf{h}_{j_k},\ \mathbf{h}_{j_k} \in \mathcal{J}(\mathbf{f}) \text{ and } c_k \in \mathbb{F}_q^* \text{ for all } k; \tag{5.1}$$

$$\sum_{k=1}^{w} m_k\mathbf{h}_{i_k} = \mathbf{0},\ \mathbf{h}_{i_k} \in \mathcal{I}_w \text{ and } m_k \in \mathbb{F}_q^* \text{ for all } k,\ \mathbf{0} \in \mathbb{F}_q^r \text{ is the zero column;} \tag{5.2}$$

$$\mathbf{f}=\sum_{k=1}^{\beta}c_k\mathbf{h}_{j_k}+\mu\sum_{k=1}^{w}m_k\mathbf{h}_{i_k},\ \ \mu\in\mathbb{F}_q^*. \tag{5.3}$$

Note that $\mathcal{I}_w$ is a set of columns corresponding to a weight w codeword of $\widehat{V}_\rho$.

In the representation (5.3), the number of distinct columns of $\widehat{\mathbf{H}}_\rho$, say $\beta^{\rm new}$, depends on the intersection $\mathcal{I}_w\cap\mathcal{J}(\mathbf{f})$ and the values of nonzero coefficients c_k, m_k, μ. For example,

$$\beta^{\rm new}=\begin{cases}\beta+w & \text{if } \ \mathcal{I}_w\cap\mathcal{J}(\mathbf{f})=\emptyset\\ \beta+w-1 & \text{if } \ |\mathcal{I}_w\cap\mathcal{J}(\mathbf{f})|=1,\ \mathbf{h}_{j_\beta}=\mathbf{h}_{i_w},\ c_\beta+\mu m_w\neq 0\\ \beta+w-2 & \text{if } \ |\mathcal{I}_w\cap\mathcal{J}(\mathbf{f})|=1,\ \mathbf{h}_{j_\beta}=\mathbf{h}_{i_w},\ c_\beta+\mu m_w=0\\ \beta+w-2 & \text{if } \ |\mathcal{I}_w\cap\mathcal{J}(\mathbf{f})|=2,\ \mathbf{h}_{j_\beta}=\mathbf{h}_{i_w},\ c_\beta+\mu m_w\neq 0,\\ & \qquad \mathbf{h}_{j_{\beta-1}}=\mathbf{h}_{i_{w-1}},\ c_{\beta-1}+\mu m_{w-1}\neq 0\end{cases}. \tag{5.4}$$

To use (5.3), (5.4), note that submatrices of $\widehat{\mathbf{H}}_\rho$ can be treated as parity check matrices of codes; we call them *component codes* and write in Table 1, where $u=1,\ldots,\rho$, "MDS" notes a minimum distance separable code and "AMDS" says on an Almost MDS code.

Table 1: Components codes corresponding to submatrices of $\widehat{\mathbf{H}}_\rho$ based on (3.1), (3.2)

rows of $\widehat{\mathbf{H}}_\rho$	columns of $\widehat{\mathbf{H}}_\rho$	geometrical object	code parameters	q	code name	code type
1,2	$\mathbf{h}_1\ldots\mathbf{h}_q$	$\{A_0^0\}\cup\mathcal{L}_0^*$	$[q,q-2,3]_q2$	all	$\mathbb{L}_0$	MDS
$2u,2u+1,2u+2$	$\mathbf{h}_{qu+1}\ldots\mathbf{h}_{qu+q-1}$	$\mathcal{C}_u^*$	$[q-1,q-4,4]_q3$	all	$\mathbb{C}_u$	MDS
$2u,2u+1,2u+2$	$\mathbf{h}_{qu+1}\ldots\mathbf{h}_{qu+q}$	$\mathcal{C}_u^*\cup\{T_u\}$	$[q,q-3,4]_q3$	even	$\mathbb{C}_u^T$	MDS
$2u,2u+1,2u+2$	$\mathbf{h}_{qu+1}\ldots\mathbf{h}_{qu+q}$	$\mathcal{C}_u^*\cup\{T_u\}$	$[q,q-3,3]_q3$	odd	$\mathbb{C}_u^T$	AMDS
$2\rho,2\rho+1,2\rho+2$	$\mathbf{h}_{q\rho+1}\ldots\mathbf{h}_{q\rho+q-1}$, $\mathbf{h}_{q\rho+q+1}$	$\mathcal{C}_\rho^*\cup\{A_\rho^\infty\}$	$[q,q-3,4]_q3$	all	$\mathbb{C}_\rho^\infty$	MDS
$2\rho,2\rho+1,2\rho+2$	$\mathbf{h}_{q\rho+1}\ldots\mathbf{h}_{q\rho+q+1}$	$\mathcal{C}_\rho^*\cup\{A_\rho^\infty,T_\rho\}$	$[q+1,q-2,4]_q3$	even	$\mathbb{C}_\rho^{\infty T}$	MDS
$2\rho,2\rho+1,2\rho+2$	$\mathbf{h}_{q\rho+1}\ldots\mathbf{h}_{q\rho+q+1}$	$\mathcal{C}_\rho^*\cup\{A_\rho^\infty,T_\rho\}$	$[q+1,q-2,3]_q3$	odd	$\mathbb{C}_\rho^{\infty T}$	AMDS

Remark 5.1. The weight spectrum of MDS codes is known, see e.g. [29]. In particular, in $[n,n-r,d]_q$ MDS code any d columns of a parity check matrix correspond to a weight d codeword. If q odd, for AMDS component codes $\mathbb{C}_u^T$ and $\mathbb{C}_\rho^{\infty T}$ we note that T_u lies on two tangents to $\mathcal{C}_u$ (in A_u^0, A_u^∞) and on $\frac{q-1}{2}$ bisecants of $\mathcal{C}_u^*$. Every of these bisecants gives rise to a weight 3 codeword. The $(q-1)$-set of points of $\mathcal{C}_u^*$ is partitioned by $\frac{q-1}{2}$ point pairs; every pair forms a bisecant through T_u.

Note that from the proofs of Section 3 it can be seen that for the representation of a column $\mathbf{f}\in\mathbb{F}_q^r$ it is sufficient to use (for every u) at most 3 columns corresponding to $\mathcal{C}_u^*$. Similarly, one can use 2 columns corresponding to $\{A_0^0\}\cup\mathcal{L}_0^*$. Therefore, if $q\geq 7$ we have in $\{A_0^0\} \cup \mathcal{L}_0^*$ and in every $\mathcal{C}_u^*$ several points (columns) that can be used to form sets $\mathcal{I}_w$ useful to increase β and β^{new} in (5.2)–(5.4).

Assume that for a column $\mathbf{f} \in \mathbb{F}_q^r$ we have the representation (5.1) with $1 \leq \beta < R$. Then using weight w codewords of the component codes we can increase β by w, $w-1$, $w-2$, see (5.4). The increase by $w-1$, $w-2$ is possible if some column of $\mathcal{J}(\mathbf{f})$ and $\mathcal{I}_w$ corresponds to the same component code. In particular, the situations with $w=3$, $w-2=1$ can be provided if some column or a column pair of $\mathcal{J}(\mathbf{f})$ and $\mathcal{I}_w$ correspond to the same code $\mathbb{L}_0$ (for all q) or to the same code $\mathbb{C}_u^T$, $\mathbb{C}_\rho^{\infty T}$ (for q odd). There exist columns $\mathbf{f} \in \mathbb{F}_q^r$ such that $\mathbb{L}_0$ is not used for their representation. Therefore, in general, for even q (where MDS codes $\mathbb{C}_u^T$, $\mathbb{C}_\rho^{\infty T}$ have minimum distance $d=4$) we are not able to do $\beta^{\text{new}} = R$ when $\beta = R-1$, see (5.3), (5.4). In the other side, for odd q, AMDS codes $\mathbb{C}_u^T$, $\mathbb{C}_\rho^{\infty T}$ have $d=3$ that allows us to increase β by $w-2=1$. Note also, see Remark 5.1, that for $q \geq 7$ the structure of minimum weight codewords in the component codes provides the situation that some columns of $\mathcal{J}(\mathbf{f})$ and $\mathcal{I}_w$ correspond to the same code.

By above, we have the following lemma.

Lemma 5.2. *Let $q \geq 7$. Let $R \geq 4$. Let an $[n, n-r]_q R, \ell$ code be defined as in Definition 1.4. Let $\widehat{V}_\rho$ be the $[Rq+1, Rq+1-2R, 3]_q R, \ell$ locally optimal code such that the columns of its parity check matrix correspond to points (in the homogeneous coordinates) of the minimal ρ-saturating set of Construction S (3.1), (3.2) with $\rho = R-1$. Then $\ell = R$ if q is odd (i.e. we have a surface-covering code) and $\ell = R-1$ if q is even.*

In Theorems 5.3 and 5.4 we consider $R \geq 4$ since for $R = 1, 2, 3$, several short covering codes with $r = tR$ are given in detail in [11, 13, 14, 16, 17] and the references therein.

Theorem 5.3. *Let $q \geq 7$ be odd. Let t be an integer. Then for all $R \geq 4$ there is an infinite family of $[n, n-r, 3]_q R, R$ locally optimal surface-covering codes with the parameters*

$$n = Rq^{(r-R)/R} + q^{(r-2R)/R}, \ \ r = tR, \ \ t = 2 \text{ and } t \geq \lceil \log_q R \rceil + 3.$$

Proof. We take the $[Rq+1, Rq+1-2R, 3]_q R, R$ code $\widehat{V}_\rho$, see Lemma 5.2, as the starting code V_0 of Construction QM$_1$. By Theorem 4.1, we obtain an $[n, n-r, 3]_q, R, R$ code with $n = (qR+1)q^m$, $r = 2R + mR$. Obviously, $m+1 = \frac{r-R}{R}$. The condition $q^m \geq n_0 - 1$ implies $q^m \geq qR$ whence $m \geq \lceil \log_q R \rceil + 1$. Finally, we put $t = m+2$. □

Theorem 5.4. *Let $q \geq 8$ be even. Let t be an integer. Then for all $R \geq 4$ there are infinite families of $[n, n-r, 3]_q R, R$ locally optimal surface-covering codes with the parameters*

(i) $n = Rq^{(r-R)/R} + 2q^{(r-2R)/R} + \sum_{j=3}^{t} q^{(r-jR)/R}, \ \ r = tR, \ \ m_1 + 2 < t < 3m_1 + 2,$

$m_1 = \lceil \log_q (R+1) \rceil + 1;$

(ii) $n = Rq^{(r-R)/R} + 2q^{(r-2R)/R} + \sum_{j=3}^{m_1+2} q^{(r-jR)/R}, \; r = tR, \; t = m_1 + 2$ *and* $t \geq 3m_1 + 2$.

Proof. **(i)** We take the $[qR+1, qR+1-2R, 3]_q R, \ell$ code $\widehat{V}_\rho$ with $\ell = R-1$, see Lemma 5.2, as the starting code V_0 of Construction QM_2. By Theorem 4.2, we obtain an $[n, n-r, 3]_q, R, R$ code with $n = (qR+1)q^m + \frac{q^{m+1}-1}{q-1}$, $r = 2R+mR$. Obviously, $m-(j-2) = \frac{r-jR}{R}$. The condition $q^m \geq n_0$ implies $q^m \geq qR+1$ whence $m \geq \lceil \log_q(qR+1) \rceil = \lceil \log_q(R+1) \rceil + 1$. The restriction $m < 3m_1$ is introduced as for $m \geq 3m_1$ we have codes of (i) that are better than ones in (ii). For $m = m_1$, codes of (i) and (ii) are the same. Finally, we put $t = m+2$.

(ii) In the relation (i), we put $t = m_1 + 2$ and obtain an $[n_1, n_1 - r_1, 3]_q R, R$ code with $n_1 = (qR+1)q^{m_1} + \frac{q^{m_1+1}-1}{q-1}$, $r_1 = 2R + m_1 R$. We take this code as the starting code V_0 of Construction QM_1. By Theorem 4.1, we obtain an $[n, n-r, 3]_q, R, R$ code with $r = 2R + m_1R + m_2R$, $q^{m_2} \geq n_1$, $n = n_1 q^{m_2} = (qR+1)q^{m_1+m_2} + \sum_{i=0}^{m_1} q^{m_1+m_2-i}$. Obviously, $m_1 + m_2 - i = \frac{r-(i+2)R}{R}$. Since $(R+1)q^{m_1+1} > n_1$, the condition $q^{m_2} \geq n_1$ is satisfied when $q^{m_2} \geq (R+1)q^{m_1+1}$ whence $m_2 \geq \lceil \log_q(R+1) \rceil + m_1 + 1 = 2m_1$. Then we denote $2 + m_1 + m_2$ by t. □

6 New infinite code families with fixed even radius $R \geq 2$ and growing codimension $tR + \frac{R}{2}$

In the projective plane $\mathrm{PG}(2, q)$, a *blocking* (resp. *double blocking*) set S is a set of points such that every line of $\mathrm{PG}(2, q)$ contains at least one (resp. two) points of S.

There is an useful connection between double blocking sets and 1-saturating sets.

Proposition 6.1. [14, Cor. 3.3], [25] *Let q be a square. Any double blocking set in the subplane* $\mathrm{PG}(2, \sqrt{q}\,) \subset \mathrm{PG}(2, q)$ *is a 1-saturating set in the plane* $\mathrm{PG}(2, q)$.

In future we use the following results, see also [1–3], [14, Sect. 3.2].

Proposition 6.2. *Let p be prime. Let $\phi(q)$ be as in* (2.1). *The following bounds on the smallest size $\tau_2(2, q)$ of a double blocking set in* $\mathrm{PG}(2, q)$ *hold:*

$$\tau_2(2,q) \leq 2(q + q^{2/3} + q^{1/3} + 1), \quad q = p^{3h}, \; p^h \equiv 2 \bmod 7 \quad [3, \text{Th. } 5.5];$$
$$\tau_2(2,q) \leq 2\left(q + \frac{q-1}{\phi(q)-1}\right), \quad q = p^h, \; h \geq 2, \; p \geq 3 \quad [1, \text{Cor. } 1.9];$$
$$\tau_2(2,q) \leq 2\left(q + \frac{q}{p} + 1\right), \quad q = p^h, \; h \geq 2, \; p \geq 7 \quad [2, \text{Th. } 1.8, \text{Cor. } 4.10].$$

Now we give a list of 1-saturating sets in the projective plane of square order. The sets (iv)–(vi) are new.

Theorem 6.3. *Let q be a square. Let p be prime. Let $\phi(\sqrt{q})$ be as in* (2.1)*. Then in* $\mathrm{PG}(2,q)$ *there are* 1*-saturating sets of the following sizes:*

(i) $3\sqrt{q}-1, \quad q=p^{2h}\ge 4,\ h\ge 1$ [10, Th. 5.2];

(ii) $2\sqrt{q}+2\sqrt[4]{q}+2, \quad q=p^{4h}\ge 16,\ h\ge 1$ [13, Th. 3.3], [14, Th. 3.4], [25];

(iii) $2\sqrt{q}+2\sqrt[3]{q}+2\sqrt[6]{q}+2, \quad q=p^{6},\ p\le 73$ [13, Th. 3.4], [14, Th. 3.5];

(iv) $2\sqrt{q}+2\sqrt[3]{q}+2\sqrt[6]{q}+2, \quad q=p^{6h},\ p^h\equiv 2 \bmod 7$;

(v) $2\sqrt{q}+2\dfrac{\sqrt{q}-1}{\phi(\sqrt{q})-1}, \quad q=p^{2h},\ h\ge 2,\ p\ge 3$;

(vi) $2\sqrt{q}+2\dfrac{\sqrt{q}}{p}+2, \quad q=p^{2h},\ h\ge 2,\ p\ge 7.$

Proof. For (i), a geometric construction is proposed in [10, Th. 5.2]. We describe it in Remark 6.5. The 1-saturating sets of (ii), (iii) are considered in [13, 14, 25]. For (iv)–(vi) we use Propositions 6.1 and 6.2. □

Remark 6.4. In Theorem 6.3, if $\sqrt{q}=p^\eta$ with $\eta\ge 3$ odd, then *the new* 1*-saturating sets* of (iv)–(vi) *have smaller sizes than the known ones* of (i)–(iii). For example, if $q=p^6$, $\eta=3$, then the new size of (vi) is $2\sqrt{q}+2\sqrt[3]{q}+2$, cf. (iii). If $\eta\ge 5$ odd, the known sets have size $3\sqrt{q}-1$ whereas new sizes are $2\sqrt{q}+o(\sqrt{q})$. For example, if $q=p^{30}$, $\eta=15$, then the new size of (iv), (v) is $2\sqrt{q}+2\sqrt[3]{q}+2\sqrt[6]{q}+2$, cf. (i). In general, if $\eta\ge 3$ is prime, then the case (vi) gives smaller sizes than other variants. If η is odd non-prime, then the variant (v) is the best.

The case (iv) gives the same size as (v), if $3|\eta$. Therefore, in future we consider new codes and bounds resulting from Theorem 6.3(v),(vi).

Note also that if $q=p^2$, i.e. $\eta=1$, then the size (i) is the smallest in Theorem 6.3. It is why we pay attention to this case, see Remarks 6.5–6.7 and Problem 5 below.

Remark 6.5. Let a point of $\mathrm{PG}(2,q)$ have the form (x_0,x_1,x_2) where $x_i\in\mathbb{F}_q$, the leftmost nonzero coordinate is equal to 1. Let β be a primitive element of $\mathbb{F}_q$.

In [10, Th. 5.2, eq. (30)], the following construction of a 1-saturating $(3\sqrt{q}-1)$-set $\mathcal{S}$ in $\mathrm{PG}(2,q)$, q square, is proposed:

$$\mathcal{S}=\{(1,0,x_2)|x_2\in\mathbb{F}_{\sqrt{q}}\}\cup\{(1,0,c\beta)|c\in\mathbb{F}^*_{\sqrt{q}}\}\cup\{(0,1,x_2)|x_2\in\mathbb{F}_{\sqrt{q}}\}. \qquad (6.1)$$

We describe this construction in more detail than in [10] using, for the description, the Baer sublines similarly to [4, Prop. 3.2]. In [10], see (6.1), specific Baer sublines are noted. Here we explain the structure and role of these specific sublines. Two Baer subplanes $\mathcal{B}_1$ and $\mathcal{B}_2$ are considered. In the points of $\mathcal{B}_1$, all coordinates $x_i\in\mathbb{F}_{\sqrt{q}}$. Also, $\mathcal{B}_2=\mathcal{B}_1\Phi$ where Φ is the collineation such that $(x_0,x_1,x_2)\Phi=(x_0,x_1\beta,x_2\beta)$. Let $L_i\subset\mathrm{PG}(2,q)$ be the "long" line of equation $x_i=0$. Let $\mathcal{L}_{i,j}=L_i\cap\mathcal{B}_j$ be the Baer subline of L_i

in the Baer subplane $\mathcal{B}_j$. We denote points $A_1 = (0, 0, 1)$, $A_2 = (1, 0, 0)$. Obviously, $\{A_1, A_2\} \subset \mathcal{B}_1 \cap \mathcal{B}_2$.

We have $\mathcal{L}_{0,1} = \mathcal{L}_{0,2}$, $\mathcal{B}_1 \cap \mathcal{B}_2 = \mathcal{L}_{0,1} \cup \{A_2\}$. Thus, the Baer subplanes $\mathcal{B}_1$ and $\mathcal{B}_2$ have the common Baer subline $\mathcal{L}_{0,1}$ and also the common point A_2 not on $\mathcal{L}_{0,1}$. Also, $\mathcal{L}_{0,1} \cap \mathcal{L}_{1,1} \cap \mathcal{L}_{1,2} = \{A_1\}$. So, we consider three Baer sublines through A_1; one of them $\mathcal{L}_{0,1}$ is common for $\mathcal{B}_1$ and $\mathcal{B}_2$; the other two ($\mathcal{L}_{1,1}$ and $\mathcal{L}_{1,2}$) belong to the same long line L_1 that passes through $A_2 \notin \mathcal{L}_{0,1}$ and $A_1 \in \mathcal{L}_{0,1}$. The needed set consists of these three Baer sublines without their intersection point, i.e. $\mathcal{S} = (\mathcal{L}_{0,1} \cup \mathcal{L}_{1,1} \cup \mathcal{L}_{1,2}) \setminus \{A_1\}$. Since $\mathcal{L}_{1,1} \cap \mathcal{L}_{1,2} = \{A_1, A_2\}$ it holds that $|\mathcal{S}| = 3\sqrt{q} - 1$. Note that if A_1 is not removed from $\mathcal{S}$ then we have no bisecants of $\mathcal{S}$ through A_1.

All points on L_0 and L_1 are 1-covered by $\mathcal{S}$. Consider a point $A = (1, a, b) \notin (L_0 \cup L_1)$ with $a = a_1\beta + a_0 \in \mathbb{F}_q^*$, $b = b_1\beta + b_0 \in \mathbb{F}_q$. (If $a = 0$ then $A \in L_1$.) Let $a_0 \neq 0$. Then $A = (1, 0, (b_1 - a_1 a_0^{-1} b_0)\beta) + a(0, 1, a_0^{-1} b_0)$. Let $a_0 = 0$. Then $a_1 \neq 0$ and $A = (1, 0, b_0) + a(0, 1, a_1^{-1} b_1)$. Thus, A is 1-covered by $\mathcal{S}$. Also, from the above consideration it follows that all points of $\mathcal{S}$ are 1-essential and $\mathcal{S}$ is a *minimal* 1-saturating set.

Remark 6.6. In [30, Ex. B] and [4, Prop. 3.2], constructions of a 1-saturating $3\sqrt{q}$-set in $\mathrm{PG}(2, q)$, q square, are proposed. In [30], the set is minimal; it consists of three non-concurrent Baer sublines in a Baer subplane. In [4], the set is non-minimal; it is similar to one of the construction [10, Th. 5.2], see its description in Remark 6.5. However, in [4], the intersection point of the three Baer sublines is not removed from the 1-saturating set.

Remark 6.7. Let p be prime. To construct a 1-saturating $(3p - 1)$-set in $\mathrm{PG}(2, p^2)$, another way than in [10] is possible. One can apply Proposition 6.1 to a double blocking set in $\mathrm{PG}(2, p)$. However, double blocking $(3p - 1)$-sets in $\mathrm{PG}(2, p)$ are known only for $q = 13, 19, 31, 37, 43$, see [8] and the references therein. Moreover, in $\mathrm{PG}(2, p)$, no double blocking sets of size less than $3p - 1$ are known.

In $\mathrm{PG}(2, p^2)$, p prime, by [14, Tab. 2], we have the following sporadic examples of 1-saturating k-sets with $k < 3p - 1$: $p^2 = 9, k = 6$; $p^2 = 25, k = 12$; $p^2 = 49, k = 18$.

Problem 5. *Develop a general construction of a 1-saturating k-set in $\mathrm{PG}(2, p^2)$, p prime, such that $k < 3p - 1$.*

In [11, Ex. 6], a lift-construction is given. It provides the following result.

Proposition 6.8. [11, Ex. 6], [14, Th. 4.4] *Let an $[n_q, n_q - 3]_q 2$ code exist. Let $n_q < q$ and $q + 1 \geq 2n_q$. Let $f_q(r, 2)$ be as in (2.2). Then there is an infinite family of $[n, n - r]_q 2$ codes with growing odd codimension $r = 2t + 1 \geq 5$ and length $n = n_q q^{(r-3)/2} + 2q^{(r-5)/2} + f_q(r, 2)$.*

Theorem 6.9. *Assume that p is prime, $q = p^{2h}$, $h \geq 2$, and covering radius $R = 2$. Let $\phi(\sqrt{q}\,)$ and $f_q(r, 2)$ be as in (2.1), (2.2). Then there exist infinite families of $[n, n - r]_q 2$ codes with growing odd codimension $r = 2t + 1 \geq 4$, $t \geq 1$, and length*

$$n = \left(2 + 2\frac{\sqrt{q} - 1}{\sqrt{q}(\phi(\sqrt{q}\,) - 1)}\right) q^{(r-2)/2} + 2\lfloor q^{(r-5)/2} \rfloor + f_q(r, 2), \ p \geq 3;$$

$$n=\left(2+\frac{2}{p}+\frac{2}{\sqrt{q}}\right)q^{(r-2)/2}+2\lfloor q^{(r-5)/2}\rfloor+f_q(r,2),\ p\geq 7.$$

Proof. Let n_q be the size of the 1-saturating sets of Theorem 6.3(iii),(iv). We treat every point (in homogeneous coordinates) of the set as a column of an $3\times n_q$ parity check matrix of an $[n_q,n_q-3]_q2$ code. For these codes it can be shown that $n_q<q$ and $q+1\geq 2n_q$. Then we use Proposition 6.8. □

The direct sum construction [14, Sect. 4.2] gives the following lemma.

Lemma 6.10. *Let covering radius $R\geq 2$ be even. Let an $[n'',n''-r'']_q2$ code exist. Then there is an $[\frac{R}{2}n'',\frac{R}{2}n''-\frac{R}{2}r'']_qR$ code.*

Theorem 6.11. *Assume that p is prime, $q=p^{2h}$, $h\geq 2$, $R\geq 2$ even, and code codimension is $r=tR+\frac{R}{2}$ with growing integer $t\geq 1$. Let $\phi(\sqrt{q}\,)$ and $f_q(r,R)$ be as in (2.1), (2.2). Then for all even $R\geq 2$ there are infinite families of $[n,n-r]_qR$ codes with fixed covering radius R, growing codimension $r=tR+\frac{R}{2}$, $t\geq 1$, and length*

$$n=R\left(1+\frac{\sqrt{q}-1}{\sqrt{q}(\phi(\sqrt{q}\,)-1)}\right)q^{(r-R)/R}+R\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor+\frac{R}{2}f_q(r,R),\ p\geq 3;$$
$$n=R\left(1+\frac{1}{p}+\frac{1}{\sqrt{q}}\right)q^{(r-R)/R}+R\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor+\frac{R}{2}f_q(r,R),\ p\geq 7.$$

Proof. We take codes of Theorem 6.9 as the codes $[n'',n''-r'']_q2$ of Lemma 6.10. □

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