Truth-preserving operations on sums of Kripke frames

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Abstract

The operation of sum of a family $(\mathsf{F}_i \mid i \text{ in } \mathsf{I})$ of Kripke frames indexed by elements of another frame I provides a natural way to construct expressive polymodal logics with good semantic and algorithmic properties. This operation has had several important applications over the last decade: it was used by L. Beklemishev in the context of polymodal provability logic; two ways of combining modal logics, the *refinement of modal logics* introduced by S. Babenyshev and V. Rybakov, and the *lexicographic product of modal logics* proposed by Ph. Balbiani, can be defined in terms of sums of frames. This paper provides some general truth-preserving tools for operating with sums of Kripke frames, and then applies them to study properties of resulting modal logics, in particular, to investigate the finite model property.

Keywords: combinations of modal logics, sum of Kripke frames, finite model property, universal modality, polymodal provability logic, refinement of modal logics, lexicographic product of modal logics

1 Introduction

This paper contributes to the area of combining modal logics [9,12].

Given a family $(\mathsf{F}_i \mid i \text{ in } \mathsf{I})$ of frames indexed by elements of another frame I (of the same signature), the *sum of the frames* F_i*'s over* I is obtained from their disjoint union by connecting elements of i-th and j-th distinct components according to the relations in I (this operation is a particular case of *generalized sum of models* introduced by S. Shelah in [15]). Given a class $\mathcal{F}$ of frames-summands and a class $\mathcal{I}$ of frames-indices, we consider the logic of the class $\sum_{\mathcal{I}} \mathcal{F}$ of all possible sums of F_i's in $\mathcal{F}$ over I in $\mathcal{I}$. In a particular case when $\mathcal{F}$ is the class $\mathrm{Fr}\, L_1$ of all the frames of a logic L_1, and $\mathcal{I}$ is $\mathrm{Fr}\, L_2$ for another logic L_2, we obtain a natural operation on Kripke-complete logics.

Over the last decade, sums of Kripke frames have had several important applications in modal logic. In [6], L. Beklemishev used (iterated) sums over

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Noetherian orders to construct models of the polymodal provability logic (this was probably the first application of sums in the context of polymodal logics). Then in [14] it was noted that sums can be a useful tool for studying computational complexity of modal satisfiability problems. At the same time in [1], S. Babenyshev and V. Rybakov considered an operation on frames and logics called *refinement*, and showed that under a very general condition this operation preserves the finite model property and decidability; refinements of frames can be considered as special instances of sums. The *lexicographic product of modal logics*, introduced by Ph. Balbiani in [2] (and then considered in [3,5,4]), is another example of an operation that can be defined via sums of frames.

This paper presents several general tools for studying modal logics of sums. Section 3 provides some basic observations on how sums interact with operations of p-morphism, generated subframe, and disjoint union. In Section 4 we address the following question: given a class of sums $\sum_{\mathcal{I}} \mathcal{F}$, when can we replace $\mathcal{F}$ with some other class of frames $\mathcal{F}'$, preserving the logic of sums? In particular, if the logic of summands $\mathcal{F}$ has the finite model property, can we replace $\mathcal{F}$ with a class of finite frames? Theorem 4.11 provides the following answer: if $\mathcal{F}$ and $\mathcal{F}'$ have the same logic in the language enriched by the universal modality (such classes are said to be *interchangeable*), then the logics of sums $\sum_{\mathcal{I}} \mathcal{F}$ and $\sum_{\mathcal{I}} \mathcal{F}'$ are equal; moreover, these classes of sums are interchangeable again, thus we have $\mathrm{Log} \sum_{\mathcal{J}} (\sum_{\mathcal{I}} \mathcal{F}) = \mathrm{Log} \sum_{\mathcal{J}} (\sum_{\mathcal{I}} \mathcal{F}')$ for any other class of frames-indices $\mathcal{J}$, and so on. Then we apply this theorem and show that the finite model property of the logic $\mathrm{Log}\,\mathcal{F}$ of summands transfers to logics of (iterated) sums over Noetherian orders. Finally, we consider several applications to refinements and lexicographic products.

2 Preliminaries

We assume the reader is familiar with the basic notions of modal logics [7,8,9].

Let A be a set (an alphabet of indices for modalities).

The set ML_A of *modal formulas over A* (or *A-formulas*, for short) is built from a countable set of *variables* $\mathrm{PV} = \{p_0, p_1, \ldots\}$ using Boolean connectives $\bot, \to$ and unary connectives $\Diamond_a$, $a \in A$ (*modalities*). The connectives $\vee, \wedge, \neg, \top, \Box_a$ are defined as abbreviations in the standard way, in particular $\Box_a \varphi$ is $\neg\Diamond_a\neg\varphi$.

An (A-)frame is a structure $\mathsf{F} = (W, (R_a)_{a\in A})$, where $W \neq \varnothing$ and $R_a \subseteq W \times W$ for $a \in A$. A *model on* F is a pair $\mathsf{M} = (\mathsf{F}, \theta)$, where $\theta : \mathrm{PV} \to 2^W$. We write $\mathrm{dom}(\mathsf{F})$ for W, which is called the *domain* of F. For u, v in F, u is *a-accessible from b in* F if uR_av. We write $u \in \mathsf{F}$ for $u \in \mathrm{dom}(\mathsf{F})$. Likewise for models. For $u \in W$, $V \subseteq W$, we put $R_a(u) = \{v \mid uR_av\}$, $R_a[V] = \cup_{v\in V} R_a(v)$.

The *truth relation* $\mathsf{M}, w \models \varphi$ is defined in the usual way, in particular $\mathsf{M}, w \models \Diamond_a\varphi$ means that $\mathsf{M}, v \models \varphi$ for some v in $R_a(w)$. A formula φ is *satisfiable in a model* M if $\mathsf{M}, w \models \varphi$ for some w in M. For a class $\mathcal{F}$ of frames, let $\mathrm{Mod}\,\mathcal{F}$ be the class of all models (F, θ) with $\mathsf{F} \in \mathcal{F}$. A formula is *satisfiable in a frame* F (*in a class $\mathcal{F}$ of frames*) if it is satisfiable in some model on F (in some model in $\mathrm{Mod}\,\mathcal{F}$). φ is valid in a frame F (in a class $\mathcal{F}$ of frames) if $\neg\varphi$ is not satisfiable in F (in $\mathcal{F}$). Validity of a set of formulas means validity of every formula in this set.

A (*propositional normal modal*) *logic* is a set L of formulas that contains all classical tautologies, the axioms $\neg\Diamond_a\bot$ and $\Diamond_a(p_0 \vee p_1) \to \Diamond_a p_0 \vee \Diamond_a p_1$ for each a in A, and is closed under the rules of modus ponens, substitution and monotonicity (if $\varphi \to \psi \in L$, then $\Diamond_a\varphi \to \Diamond_a\psi \in L$, for each a in A). In particular, the set $\mathrm{Log}\,\mathcal{F}$ of all formulas valid in $\mathcal{F}$ is a logic; it is called the *logic of* $\mathcal{F}$; such logics are called *Kripke complete.* A logic has the *finite model property* if it is the logic of a class of finite frames (a frame is finite, if its domain is). Let $\mathrm{Fr}\,L$ and $\mathrm{Fr}_f\,L$ be the classes of all frames and all finite frames validating L respectively.

The notions of p-morphism, generated subframe or submodel are defined in the standard way, see e.g. [9, Section 1.4]. We write $\mathsf{F} \twoheadrightarrow \mathsf{G}$, if G is a p-morphic image of F. The notation $\mathsf{F} \cong \mathsf{G}$ means that F and G are isomorphic. We write $\mathsf{F}[w]$ for the subframe of F generated by the singleton $\{w\}$; such frames are called *cones in* F.

The cardinality of a set V is denoted by $|V|$. Natural numbers are considered as finite ordinals. Given a sequence $\boldsymbol{v} = (v_0, v_1, \ldots)$, we write $\boldsymbol{v}(i)$ for v_i.

3 Sums

We fix $N \leq \omega$ for the alphabet and consider the language ML_N.

Consider a non-empty family $(\mathsf{F}_i)_{i\in I}$ of N-frames $\mathsf{F}_i = (W_i, (R_{i,a})_{a\in N})$. The *disjoint union* of these frames is the N-frame $\bigsqcup_{i\in I}\mathsf{F}_i = (\bigsqcup_{i\in I} W_i, (R_a)_{a\in N})$, where $\bigsqcup_{i\in I} W_i = \bigcup_{i\in I}(\{i\} \times W_i)$ is the *disjoint union of sets* W_i, and

$$(i, w)R_a(j, v) \quad \text{iff} \quad i = j \,\&\, wR_{i,a}v.$$

Suppose that I is the domain of another N-frame $\mathsf{I} = (I, (S_a)_{a\in N})$.

Definition 3.1 The *sum of the family* $(\mathsf{F}_i)_{i\in\mathsf{I}}$ *of* N*-frames over the* N*-frame* I is the N-frame $\sum_{i\in\mathsf{I}}\mathsf{F}_i = (\bigsqcup_{i\in I} W_i, (R^{\Sigma}_a)_{a\in N})$, where

$$(i, w)R^{\Sigma}_a(j, v) \quad \text{iff} \quad i = j \,\&\, wR_{i,a}v \text{ or } i \neq j \,\&\, iS_a j.$$

The sum of models $\sum_{i\in\mathsf{I}}(\mathsf{F}_i, \theta_i)$ is the model $(\sum_{i\in\mathsf{I}}\mathsf{F}_i, \theta)$, where $(i, w) \in \theta(p)$ iff $w \in \theta_i(p)$.

For classes $\mathcal{I}$, $\mathcal{F}$ of N-frames, let $\sum_{\mathcal{I}}\mathcal{F}$ be the class of all sums $\sum_{i\in\mathsf{I}}\mathsf{F}_i$ such that $\mathsf{I} \in \mathcal{I}$ and $\mathsf{F}_i \in \mathcal{F}$ for every i in I.

Remark 3.2 We do not require that S_a's are partial orders or even transitive relations.

The relations R^{Σ}_a are independent of reflexivity of the relations S_a: if $\mathsf{I}' = (I, (S'_a)_{a\in N})$, where S'_a is the reflexive closure of S_a for each $a \in N$, then $\sum_{i\in\mathsf{I}}\mathsf{F}_i = \sum_{i\in\mathsf{I}'}\mathsf{F}_i$.

We shall be mainly interested in the polymodal case. For a simple illustration of the definition let us first consider the following unimodal examples.

Let $\mathsf{F} = (W, R)$ be a preorder. The *(irreflexive) skeleton* of F is the strict partial order $\mathrm{sk}\mathsf{F} = (\overline{W}, <_R)$, where $\overline{W}$ is the quotient set of W by the equivalence $R \cap R^{-1}$, and for $C, D \in \overline{W}$, $C <_R D$ iff $C \neq D$ and $\exists w \in C\, \exists v \in D\; wRv$. Elements of $\overline{W}$ are called *clusters in* F. Then F is isomorphic to the sum $\sum_{C \in \mathrm{sk}\mathsf{F}} (C, C \times C)$ of its clusters over its skeleton.

For another example suppose that $\mathsf{F} = (W, R)$ satisfies the property of *weak transitivity* $xRzRy \Rightarrow xRy \vee x = y$. Then F is isomorphic to a sum $\sum_{i \in \mathsf{I}} \mathsf{F}_i$, where I is a partial order and in every F_i we have $xR_iy \vee x = y$.

The propositions below show how sums interact with p-morphisms, generated subframes, and disjoint unions.

The following fact is immediate from Definition 3.1.

Proposition 3.3 *If J is a generated subframe of I, then $\sum_{i \in \mathsf{J}} \mathsf{F}_i$ is a generated subframe of $\sum_{i \in \mathsf{I}} \mathsf{F}_i$.*

Proposition 3.4 *Consider N-frames I, J, and two families of N-frames $(\mathsf{F}_i)_{i \in \mathsf{I}}$, $(\mathsf{G}_j)_{j \in \mathsf{J}}$. Assume that all the relations in J are irreflexive.*

(i) *If $f : \mathsf{I} \twoheadrightarrow \mathsf{J}$ and $\mathsf{F}_i \twoheadrightarrow \mathsf{G}_{f(i)}$ for all i in I, then $\sum_{i \in \mathsf{I}} \mathsf{F}_i \twoheadrightarrow \sum_{j \in \mathsf{J}} \mathsf{G}_j$.*

(ii) *If $\mathsf{I} = \mathsf{J}$ and $\mathsf{F}_i \twoheadrightarrow \mathsf{G}_i$ for all i in I, then $\sum_{i \in \mathsf{I}} \mathsf{F}_i \twoheadrightarrow \sum_{i \in \mathsf{I}} \mathsf{G}_i$.*

(iii) *If $f : \mathsf{I} \twoheadrightarrow \mathsf{J}$, then $\sum_{i \in \mathsf{I}} \mathsf{G}_{f(i)} \twoheadrightarrow \sum_{j \in \mathsf{J}} \mathsf{G}_j$.*

Proof. (i) The required p-morphism is defined as $g(i, w) = (f(i), g_i(w))$, where $g_i : \mathsf{F}_i \twoheadrightarrow \mathsf{G}_{f(i)}$ for each i in I. (ii) and (iii) are special cases of (i): in (ii), f is the identity map on I; in (iii), $\mathsf{F}_i = \mathsf{G}_{f(i)}$ for each i in I. □

Lemma 3.5 *Consider an N-frame I, a family $(\mathsf{J}_i)_{i \in \mathsf{I}}$ of N-frames, and a family $(\mathsf{F}_{ij})_{i \in \mathsf{I}, j \in \mathsf{J}_i}$ of N-frames. Then*

$$\sum_{i \in \mathsf{I}} \sum_{j \in \mathsf{J}_i} \mathsf{F}_{ij} \quad \cong \quad \sum_{(i,j) \in \sum_{k \in \mathsf{I}} \mathsf{J}_k} \mathsf{F}_{ij}.$$

The proof of this lemma is straightforward from the definition; the detailed verification is given in Appendix.

Let $(\varnothing)_N$ denote the sequence of length N in which every element is the empty set. Disjoint unions are special cases of sums: if I is a frame with empty relations $(I, (\varnothing)_N)$, then $\bigsqcup_{i \in I} \mathsf{F}_i = \sum_{i \in \mathsf{I}} \mathsf{F}_i$.

Proposition 3.6 *For a non-empty set I, a family $(\mathsf{J}_i)_{i \in \mathsf{I}}$ of N-frames, and a family $(\mathsf{F}_{ij})_{i \in I, j \in \mathsf{J}_i}$ of N-frames,*

$$\bigsqcup_{i \in I} \sum_{j \in \mathsf{J}_i} \mathsf{F}_{ij} \quad \cong \quad \sum_{(i,j) \in \bigsqcup_{k \in I} \mathsf{J}_k} \mathsf{F}_{ij}.$$

Proof. This is a special case of Lemma 3.5 in which $\mathsf{I} = (I, (\varnothing)_N)$. □

Proposition 3.7 *For an N-frame I, a family $(J_i)_{i \in \mathsf{I}}$ of non-empty sets, and a family $(\mathsf{F}_{ij})_{i \in \mathsf{I}, j \in J_i}$ of N-frames,*

$$\sum_{i \in \mathsf{I}} \bigsqcup_{j \in J_i} \mathsf{F}_{ij} \quad \cong \quad \sum_{(i,j) \in \sum_{k \in \mathsf{I}} (J_k, (\varnothing)_N)} \mathsf{F}_{ij}.$$

Proof. Follows from Lemma 3.5: let J_i be $(J_i, (\varnothing)_N)$. □

4 Replacing summands

In this section we introduce the notion of *interchangeable classes of frames* and prove the following: if $\mathcal{F}$ and $\mathcal{G}$ are interchangeable, then they have the same logic, and, for any class $\mathcal{I}$ of frames of the same signature, the classes $\sum_{\mathcal{I}} \mathcal{F}$ and $\sum_{\mathcal{I}} \mathcal{G}$ are interchangeable again. Then we show that classes are interchangeable iff they have the same logic in the language enriched by the universal modality.

4.1 Interchangeable classes

Definition 4.1 A sequence $\mathbf{\Gamma} = (\Gamma_a)_{a\in N}$, where Γ_a are sets of N-formulas, is called a *condition* (in the language ML_N).

Consider a model $\mathsf{M} = (W, (R_a)_{a\in N}, \theta)$, w in M. By induction on the length of an N-formula φ, we define the relation $\mathsf{M}, w \models_{\mathbf{\Gamma}} \varphi$ (*"under the condition* $\mathbf{\Gamma}$*, φ is true at w in* M*"*): as usual, $\mathsf{M}, w \not\models_{\mathbf{\Gamma}} \bot$, $\mathsf{M}, w \models_{\mathbf{\Gamma}} p$ iff $\mathsf{M}, w \models p$ for a variable p, $\mathsf{M}, w \models_{\mathbf{\Gamma}} \varphi \to \psi$ iff $\mathsf{M}, w \not\models_{\mathbf{\Gamma}} \varphi$ or $\mathsf{M}, w \models_{\mathbf{\Gamma}} \psi$; for $a \in N$,

$$\mathsf{M}, w \models_{\mathbf{\Gamma}} \Diamond_a \varphi \quad \text{iff} \quad \varphi \in \Gamma_a \text{ or } \exists v \in R_a(w) \ \ \mathsf{M}, v \models_{\mathbf{\Gamma}} \varphi.$$

In particular, if all Γ_a are empty, then we have the standard notion of truth in a Kripke model:

$$\mathsf{M}, w \models_{(\varnothing)_N} \varphi \quad \text{iff} \quad \mathsf{M}, w \models \varphi.$$

Let $\mathrm{sub}(\varphi)$ be the set of all subformulas of φ, and let $\mathrm{sub}(\varphi; \mathsf{M}, \mathbf{\Gamma})$ be the set $\{\psi \in \mathrm{sub}(\varphi) \mid \mathsf{M}, v \models_{\mathbf{\Gamma}} \psi \text{ for some } v\}$. In particular, $\mathrm{sub}(\varphi; \mathsf{M}, (\varnothing)_N)$ is the set of all subformulas of φ satisfiable in M. Models M and M' are said to be $(\varphi, \mathbf{\Gamma})$*-equivalent* if $\mathrm{sub}(\varphi; \mathsf{M}, \mathbf{\Gamma}) = \mathrm{sub}(\varphi; \mathsf{M}', \mathbf{\Gamma})$.

A triple $(\varphi, \Phi, \mathbf{\Gamma})$, where $\Phi \subseteq \mathrm{sub}(\varphi)$, is called a *tie*. A tie $(\varphi, \Phi, \mathbf{\Gamma})$ is *satisfiable* in a frame F (in a class $\mathcal{F}$ of frames) if there exists a model M on F (in $\mathrm{Mod}\,\mathcal{F}$) such that $\Phi = \mathrm{sub}(\varphi; \mathsf{M}, \mathbf{\Gamma})$.

We put $\mathcal{F} \preccurlyeq_\varphi \mathcal{G}$ if every tie of form $(\varphi, \Phi, \mathbf{\Gamma})$, which is satisfiable in $\mathcal{F}$, is satisfiable in $\mathcal{G}$. (Equivalently, $\mathcal{F} \preccurlyeq_\varphi \mathcal{G}$ if for every condition $\mathbf{\Gamma}$ and every model $\mathsf{M} \in \mathrm{Mod}\,\mathcal{F}$, there exists a model $\mathsf{M}' \in \mathrm{Mod}\,\mathcal{G}$ such that M and M' are $(\varphi, \mathbf{\Gamma})$-equivalent.)

If $\mathcal{F} \preccurlyeq_\varphi \mathcal{G}$ and $\mathcal{G} \preccurlyeq_\varphi \mathcal{F}$, then we put $\mathcal{F} \equiv_\varphi \mathcal{G}$. We put $\mathcal{F} \preccurlyeq \mathcal{G}$ if $\mathcal{F} \preccurlyeq_\varphi \mathcal{G}$ for all N-formulas φ. The classes $\mathcal{F}$ and $\mathcal{G}$ are *interchangeable*, denoted $\mathcal{F} \equiv \mathcal{G}$, if $\mathcal{F} \preccurlyeq \mathcal{G}$ and $\mathcal{G} \preccurlyeq \mathcal{F}$.

Proposition 4.2

(i) *If $\mathcal{F} \preccurlyeq_\varphi \mathcal{G}$ and φ is satisfiable in $\mathcal{F}$, then φ is satisfiable in $\mathcal{G}$.*

(ii) *If $\mathcal{F} \equiv \mathcal{G}$, then* $\mathrm{Log}\,\mathcal{F} = \mathrm{Log}\,\mathcal{G}$.

Proof. Follows from the following observation: if $\mathcal{C}$ is a class of N-frames, then φ is satisfiable in $\mathcal{C}$ iff there exists $\Phi \subseteq \mathrm{sub}(\varphi)$ such that $\varphi \in \Phi$ and the tie $(\varphi, \Phi, (\varnothing)_N)$ is satisfiable in $\mathcal{C}$. □

Theorem 4.3 *Let $\mathcal{I}$, $\mathcal{F}$, $\mathcal{G}$ be classes of N-frames.*

(i) *For every N-formula φ, if $\mathcal{F} \preccurlyeq_\varphi \mathcal{G}$, then $\sum_\mathcal{I} \mathcal{F} \preccurlyeq_\varphi \sum_\mathcal{I} \mathcal{G}$.*

(ii) *If $\mathcal{F} \equiv \mathcal{G}$, then $\sum_\mathcal{I} \mathcal{F} \equiv \sum_\mathcal{I} \mathcal{G}$.*

The proof is based on Lemmas 4.5 and 4.6 below. In what follows, $\mathbf{\Gamma}$ is a condition, φ is a formula, $\mathsf{M} = (W, (R_a)_{a\in N}, \theta)$ is a model.

Definition 4.4 Let V be a set of elements of M. Given φ and $\mathbf{\Gamma}$, let $\mathbf{\Delta}$ be the condition defined as follows: for $a \in N$,

$$\mathbf{\Delta}(a) = \mathbf{\Gamma}(a) \cup \{\chi \in \mathrm{sub}(\varphi) \mid \exists w \in R_a[V]\backslash V \ \mathsf{M}, w \models_\mathbf{\Gamma} \chi\}.$$

$\mathbf{\Delta}$ is called the *external condition of V in M with respect to φ and $\mathbf{\Gamma}$*.

We write $\mathsf{M}{\restriction}V$ for the restriction of M to V, i.e., $\mathsf{M}{\restriction}V = (V, (R_a{\restriction}V)_{a\in N}, \theta')$, where $R_a{\restriction}V = R_a \cap (V \times V)$, and $\theta'(p) = \theta(p) \cap V$ for $p \in \mathrm{PV}$.

Lemma 4.5 *Consider a sum of models $\mathsf{M} = \sum_\mathsf{I} \mathsf{M}_i$, i in I, and the set $V = \{i\} \times \mathrm{dom}(\mathsf{M}_i)$. If $\mathbf{\Delta}$ is the external condition of V in M with respect to some given φ, $\mathbf{\Gamma}$, then for all v in M_i, χ in $\mathrm{sub}(\varphi)$,*

$$\mathsf{M}, (i, v) \models_\mathbf{\Gamma} \chi \quad \textit{iff} \quad \mathsf{M}_i, v \models_\mathbf{\Delta} \chi. \tag{1}$$

Proof. By induction on the length of χ. Consider the case $\chi = \Diamond_a\psi$.

Suppose that $\psi \in \mathbf{\Gamma}(a)$. Then $\psi \in \mathbf{\Delta}(a)$, and both sides of (1) are true.

Suppose now that $\psi \notin \mathbf{\Gamma}(a)$.

Assume that $\mathsf{M}, (i, v) \models_\mathbf{\Gamma} \Diamond_a\psi$. Then we have $\mathsf{M}, (i, u) \models_\mathbf{\Gamma} \psi$ for some pair (j, u) such that $(i, v)R_a(j, u)$. If $i = j$, then by induction hypothesis, $\mathsf{M}_i, u \models_\mathbf{\Delta} \psi$; since u is a-accessible from v in M_i, we have $\mathsf{M}_i, v \models_\mathbf{\Delta} \Diamond_a\psi$. If $i \neq j$, then $\psi \in \mathbf{\Delta}(a)$, and we have $\mathsf{M}_i, v \models_\mathbf{\Delta} \Diamond_a\psi$ again.

Conversely, let $\mathsf{M}_i, v \models_\mathbf{\Delta} \Diamond_a\psi$. There are two cases. First, suppose $\mathsf{M}_i, u \models_\mathbf{\Delta} \psi$ for some u, which is a-accessible from v in M_i. Then $\mathsf{M}, (i, u) \models_\mathbf{\Gamma} \psi$ by induction hypothesis, and so $\mathsf{M}, (i, v) \models_\mathbf{\Gamma} \Diamond_a\psi$. Second, suppose $\psi \in \mathbf{\Delta}(a)$. Then since $\psi \notin \mathbf{\Gamma}(a)$, it follows that $\mathbf{\Gamma}(a) \neq \mathbf{\Delta}(a)$. By the definition of $\mathbf{\Delta}$, we have $\mathsf{M}, (j, u) \models_\mathbf{\Gamma} \psi$ for some pair (j, u) in $R_a[V]\backslash V$. It follows that j is a-accessible from i in I, so $(i, v)R_a(j, u)$. Hence $\mathsf{M}, v \models_\mathbf{\Gamma} \Diamond_a\psi$. □

Lemma 4.6 *Consider $\varphi, \mathbf{\Gamma}$, a frame I, and two sums of models $\mathsf{M} = \sum_\mathsf{I} \mathsf{M}_i$, $\mathsf{M}' = \sum_\mathsf{I} \mathsf{M}'_i$. For i in I, let $\mathbf{\Delta}_i$ be the external condition of the set $\{i\} \times \mathrm{dom}(\mathsf{M}_i)$ in M with respect to φ and $\mathbf{\Gamma}$. If the models M_i and M'_i are $(\varphi, \mathbf{\Delta}_i)$-equivalent for each i in I, then the sums M and M' are $(\varphi, \mathbf{\Gamma})$-equivalent.*

Proof. We show that for all i in I, w in M'_i, and χ in $\mathrm{sub}(\varphi)$,

$$\mathsf{M}', (i, w) \models_\mathbf{\Gamma} \chi \quad \text{iff} \quad \mathsf{M}'_i, w \models_{\mathbf{\Delta}_i} \chi. \tag{2}$$

By induction on the length of χ. The only non-trivial case is $\chi = \Diamond_a\psi$.

If $\psi \in \mathbf{\Gamma}(a)$, then $\psi \in \mathbf{\Delta}_i(a)$, and both sides of (2) are true.

Suppose that $\psi \notin \mathbf{\Gamma}(a)$.

Let $\mathsf{M}', (i,w) \models_{\mathbf{\Gamma}} \Diamond_a\psi$. Then $\mathsf{M}', (k,u) \models_{\mathbf{\Gamma}} \psi$ for some pair (k,u) which is a-accessible from (i,w) in M'. By induction hypothesis, $\mathsf{M}'_k, u \models_{\mathbf{\Delta}_k} \psi$. There are two cases: $k = i$ and $k \in S_a(i)\backslash\{i\}$, where S_a is the a-th relation in I. If $k = i$, then u is a-accessible from w in M'_i, and the right-hand side of (2) follows by Definition 4.1. Now let $k \in S_a(i)\backslash\{i\}$. We have $\psi \in \mathrm{sub}(\varphi; \mathsf{M}'_k, \mathbf{\Delta}_k)$, and since M'_k and M_k are $(\varphi, \mathbf{\Delta}_k)$-equivalent, we have $\psi \in \mathrm{sub}(\varphi; \mathsf{M}_k, \mathbf{\Delta}_k)$. It follows that $\mathsf{M}_k, u' \models_{\mathbf{\Delta}_k} \psi$ for some u' in M_k. By Lemma 4.5, $\mathsf{M}, (k,u') \models_{\mathbf{\Gamma}} \psi$. Hence $\psi \in \mathbf{\Delta}_i(a)$, and we have $\mathsf{M}'_i, w \models_{\mathbf{\Delta}_i} \Diamond_a\psi$, as required.

Conversely, let $\mathsf{M}'_i, w \models_{\mathbf{\Delta}_i} \Diamond_a\psi$. If $\mathsf{M}'_i, u \models_{\mathbf{\Delta}_i} \psi$ for some u, which is a-accessible from w in M', then the left-hand side of (2) follows from induction hypothesis. Suppose $\psi \in \mathbf{\Delta}_i(a)$. Since $\psi \notin \mathbf{\Gamma}(a)$, by the definition of $\mathbf{\Delta}_i$ we have $\mathsf{M}, (k,u) \models_{\mathbf{\Gamma}} \psi$ for some $k \in S_a(i)\backslash\{i\}$, $u \in \mathrm{dom}(\mathsf{M}_k)$. By Lemma 4.5, $\mathsf{M}_k, u \models_{\mathbf{\Delta}_k} \psi$. The models M_k and M'_k are $(\varphi, \mathbf{\Delta}_k)$-equivalent, so $\mathsf{M}'_k, u' \models_{\mathbf{\Delta}_k} \psi$ for some u' in M'_k. By induction hypothesis, $\mathsf{M}', (k,u') \models_{\mathbf{\Gamma}} \psi$. Then since $k \in S_a(i)\backslash\{i\}$, it follows that $\mathsf{M}', (i,w) \models_{\mathbf{\Gamma}} \Diamond_a\psi$.

Thus, (2) is proved. It remains only to observe that

$$\mathrm{sub}(\varphi; \mathsf{M}, \mathbf{\Gamma}) \;=\; \bigcup_{i\in\mathsf{I}} \mathrm{sub}(\varphi; \mathsf{M}_i, \mathbf{\Delta}_i) \;=\; \bigcup_{i\in\mathsf{I}} \mathrm{sub}(\varphi; \mathsf{M}'_i, \mathbf{\Delta}_i) \;=\; \mathrm{sub}(\varphi; \mathsf{M}', \mathbf{\Gamma}).$$

Indeed, the first equality holds by Lemma 4.5, the third — by (2), and the second one holds because M_i and M'_i are $(\varphi, \mathbf{\Delta}_i)$-equivalent for all i in I. □

Proof of Theorem 4.3. The first statement follows from Lemma 4.6: for $\mathsf{I} \in \mathcal{I}$, a sum $\sum_{\mathsf{I}} \mathsf{M}_i$ of models in $\mathrm{Mod}\,\mathcal{F}$, and a tie $(\varphi, \Phi, \mathbf{\Gamma})$, we choose models M'_i in $\mathrm{Mod}\,\mathcal{G}$ in such a way that $\sum_{\mathsf{I}} \mathsf{M}'_i$ is $(\varphi, \mathbf{\Gamma})$-equivalent to the initial sum. The second statement immediately follows from the first. □

It follows that $\mathcal{F} \equiv \mathcal{G}$ implies $\mathrm{Log} \sum_{\mathcal{I}} \mathcal{F} = \mathrm{Log} \sum_{\mathcal{I}} \mathcal{G}$. When $\mathcal{F} \equiv \mathcal{G}$?

4.2 Criterion of interchangeability

We shall show that classes of frames are interchangeable iff they have the same logic in the language endowed with the universal modality.

Given a condition $\mathbf{\Gamma}$, by induction on the length of φ we define $[\varphi]^{\mathbf{\Gamma}}$: $[\bot]^{\mathbf{\Gamma}} = \bot$, $[p]^{\mathbf{\Gamma}} = p$ for variables, $[\varphi_1 \to \varphi_2]^{\mathbf{\Gamma}} = [\varphi_1]^{\mathbf{\Gamma}} \to [\varphi_2]^{\mathbf{\Gamma}}$,

$$[\Diamond_a\varphi]^{\mathbf{\Gamma}} = \begin{cases} \top, & \text{if } \varphi \in \mathbf{\Gamma}(a), \\ \Diamond_a[\varphi]^{\mathbf{\Gamma}} & \text{otherwise.} \end{cases}$$

Lemma 4.7 $\mathsf{M}, w \models_{\mathbf{\Gamma}} \varphi$ *iff* $\mathsf{M}, w \models [\varphi]^{\mathbf{\Gamma}}$.

Proof. By induction on the length of φ. Consider the case $\varphi = \Diamond_a\psi$.

Suppose that $\psi \in \mathbf{\Gamma}(a)$. In this case, we have $[\Diamond_a\psi]^{\mathbf{\Gamma}} = \top$; by Definition 4.1, $\mathsf{M}, w \models [\Diamond_a\psi]^{\mathbf{\Gamma}}$ for all w in M.

Now suppose that $\psi \notin \mathbf{\Gamma}$. In this case $\mathsf{M}, w \models_{\mathbf{\Gamma}} \Diamond_a\psi$ means that $\mathsf{M}, v \models_{\mathbf{\Gamma}} \psi$ for some $v \in R_a(w)$, which is equivalent to $\mathsf{M}, w \models \Diamond_a[\psi]^{\mathbf{\Gamma}}$ by induction hypothesis. It remains to observe that in this case $\Diamond_a[\psi]^{\mathbf{\Gamma}} = [\Diamond_a\psi]^{\mathbf{\Gamma}}$. □

We fix some $u \notin N$ and consider the alphabet $N' = N \cup \{u\}$. For an N-frame $\mathsf{G} = (W, (R_a)_{a\in N})$, let G^u be the N'-frame $(W, (R_a)_{a\in N'})$, where $R_u = W \times W$;

likewise for models. For a class $\mathcal{F}$ of N-frames, $\mathcal{F}^u = \{\mathsf{F}^u \mid \mathsf{F} \in \mathcal{F}\}$. For a tie $(\varphi, \Psi, \mathbf{\Gamma})$, where φ is an N-formula, put

$$\delta(\varphi, \Psi, \mathbf{\Gamma}) = \bigwedge_{\psi \in \Psi} \Diamond_u [\psi]^{\mathbf{\Gamma}} \wedge \bigwedge_{\psi \in \mathrm{sub}(\varphi) \setminus \Psi} \neg \Diamond_u [\psi]^{\mathbf{\Gamma}} \tag{3}$$

Lemma 4.8 $(\varphi, \Psi, \mathbf{\Gamma})$ *is satisfiable in* $\mathcal{F}$ *iff* $\delta(\varphi, \Psi, \mathbf{\Gamma})$ *is satisfiable in* $\mathcal{F}^u$*.*

Proof. By Lemma 4.7, for any model M we have: $\Psi = \mathrm{sub}(\varphi; \mathsf{M}, \mathbf{\Gamma})$ iff the formula $\delta(\varphi, \Psi, \mathbf{\Gamma})$ is true (at any point) in the model M^u. □

Lemma 4.9 *If* $\mathcal{F} \preccurlyeq \mathcal{G}$ *and* α *is satisfiable in* $\mathcal{F}^u$*, then* α *is satisfiable in* $\mathcal{G}^u$*.*

Proof. Let $\mathcal{C}$ be the class of all N-frames. By [10, Theorem 3.7], there exists an N'-formula $\alpha' = \Box_u \chi \wedge \psi \wedge \bigwedge_{i<l} \Diamond_u \psi_i$ such that χ, ψ, ψ_i $(i < l)$ are N-formulas, and $\alpha \leftrightarrow \alpha'$ is valid in $\mathcal{C}^u$. Assume that α is satisfiable in M^u for some $\mathsf{M} \in \mathrm{Mod}\,\mathcal{F}$. Consider an N-formula φ containing $\neg\chi$, ψ, and all ψ_i as subformulas. Put $\Psi = \mathrm{sub}(\varphi; \mathsf{M}, (\varnothing)_N)$. Then ψ, ψ_i $(i < l)$ are in Ψ, and $\neg\chi \notin \Psi$. Since $\mathcal{F} \preccurlyeq \mathcal{G}$, for some $\mathsf{M}' \in \mathrm{Mod}\,\mathcal{G}$ we have $\Psi = \mathrm{sub}(\varphi; \mathsf{M}', (\varnothing)_N)$. It follows that α' is true at some point in M'^u. Thus α is satisfiable in $\mathcal{G}^u$. □

From Lemmas 4.8 and 4.9 we obtain the following simple characterization of interchangeable classes.

Proposition 4.10 $\mathcal{F} \equiv \mathcal{G}$ *iff* $\mathrm{Log}\,\mathcal{F}^u = \mathrm{Log}\,\mathcal{G}^u$*.*

Now from Theorem 4.3 and Proposition 4.10 we obtain the main result of this section:

Theorem 4.11 *Let* $\mathcal{I}$*,* $\mathcal{F}$*,* $\mathcal{G}$ *be classes on* N*-frames. If* $\mathrm{Log}\,\mathcal{F}^u = \mathrm{Log}\,\mathcal{G}^u$*, then* $\mathrm{Log}(\sum_{\mathcal{I}} \mathcal{F})^u = \mathrm{Log}(\sum_{\mathcal{I}} \mathcal{G})^u$*, and in particular* $\mathrm{Log} \sum_{\mathcal{I}} \mathcal{F} = \mathrm{Log} \sum_{\mathcal{I}} \mathcal{G}$*.*

The rest of this section provides some more tools for interchangeable classes.

Proposition 4.12 *If* $\mathcal{F} \equiv \mathcal{G}$*, then* $\mathcal{F}^u \equiv \mathcal{G}^u$*.*

Proof. If $\mathrm{Log}\,\mathcal{F}^u = \mathrm{Log}\,\mathcal{G}^u$, then trivially $\mathrm{Log}\left((\mathcal{F}^u)^u\right) = \mathrm{Log}\left((\mathcal{G}^u)^u\right)$ (another universal relation does nothing). Now we use Proposition 4.10. □

Proposition 4.13 *For frames* F, G*, if* $\mathsf{F} \twoheadrightarrow \mathsf{G}$*, then any tie that is satisfiable in* G *is satisfiable in* F*.*

Proof. This follows from Lemma 4.8, because $\mathsf{F} \twoheadrightarrow \mathsf{G}$ implies $\mathsf{F}^u \twoheadrightarrow \mathsf{G}^u$. □

Definition 4.14 Let $\mathsf{M} = (W, (R_a)_{a \in N}, \theta)$ and $\mathsf{M}' = (W', (R'_a)_{a \in N}, \theta)$ be models such that $W' \subseteq W$, $R'_a \subseteq R_a$ for each $a \in N$, and $\theta'(p) = \theta(p) \cap W'$ for variables. The model M' is called a *selective filtration of* M *with respect to given* φ *and* $\mathbf{\Gamma}$ if for all ψ, $a \in N$ such that $\Diamond_a \psi \in \mathrm{sub}(\varphi)$, and all w in M'

$$\mathsf{M}, w \models_{\mathbf{\Gamma}} \Diamond_a \psi \,\&\, \psi \notin \mathbf{\Gamma}(a) \;\Rightarrow \exists v \, (w R'_a v \,\&\, \mathsf{M}, v \models_{\mathbf{\Gamma}} \psi).$$

Proposition 4.15 *If* M' *is a selective filtration of* M *with respect to* φ *and* $\mathbf{\Gamma}$*, then for all* $\psi \in \mathrm{sub}(\varphi)$*,* w *in* M'*, we have* $\mathsf{M}', w \models_{\mathbf{\Gamma}} \psi$ *iff* $\mathsf{M}, w \models_{\mathbf{\Gamma}} \psi$*.*

In our formulation of selective filtration, it is important that $\Box_a$'s are abbreviations. The proof of Proposition 4.15 is straightforward (see Appendix).

Proposition 4.16 *If* M' *is a generated submodel of* M*, then for every condition* $\mathbf{\Gamma}$*, every formula* φ*, and every* w *in* M'*, we have* $\mathsf{M}', w \models_{\mathbf{\Gamma}} \varphi$ *iff* $\mathsf{M}, w \models_{\mathbf{\Gamma}} \varphi$*.*

Proof. A generated submodel is a selective filtration (with respect to any φ and $\mathbf{\Gamma}$). Now we use Proposition 4.15. □

5 Applications

5.1 Sums over Noetherian orders

Definition 5.1 Consider a unimodal frame $\mathsf{I} = (I, S)$ and a family $(\mathsf{F}_i)_{i \in I}$ of N-frames (or N-models). For $a \in N$, the *a-sum* ${}^{a}\sum_{\mathsf{I}} \mathsf{F}_i$ is the sum $\sum_{\mathsf{I}'} \mathsf{F}_i$, where I' is the N-frame whose domain is I, the a-th relation is S and all the other relations are empty. If $\mathcal{F}$ is a class of N-frames, $\mathcal{I}$ is a class of 1-frames, then ${}^{a}\sum_{\mathcal{I}} \mathcal{F}$ is the class of all sums ${}^{a}\sum_{\mathsf{I}} \mathsf{F}_i$, where $\mathsf{I} \in \mathcal{I}$ and all F_i are in $\mathcal{F}$.

For $s < \omega$ and a tuple $\boldsymbol{a} = (a_0, \ldots, a_{s-1}) \in N^s$, let ${}^{\boldsymbol{a}}\sum_{\mathcal{I}} \mathcal{F}$ be the class ${}^{a_0}\sum_{\mathcal{I}} \ldots {}^{a_{s-1}}\sum_{\mathcal{I}} \mathcal{F}$ (we put ${}^{\boldsymbol{a}}\sum_{\mathcal{I}} \mathcal{F} = \mathcal{F}$ if $\boldsymbol{a}$ is the empty sequence).

A strict partial order $(I, <)$ is *Noetherian* if it has no infinite ascending chain. Let NPO and PO_f be the classes of all non-empty Noetherian partial orders and all finite non-empty strict partial orders respectively (we say that a partial order is non-empty, if its domain is).

Sums over Noetherian orders play a significant role in the context of provability logics. In [6], L. Beklemishev introduced a system J, a Kripke complete approximation of the well-known polymodal provability logic GLP [11]. Semantically, J was characterised as the logic of frames called *stratified* in [6]. In our notation, this can be formulated as follows: for $N < \omega$, the N-modal fragment of J is the logic of the class ${}^{\boldsymbol{a}_N}\sum_{\mathrm{NPO}} \{\mathsf{S}_N\}$, where S_N is a singleton with N empty relations, and $\boldsymbol{a}_N = (0, \ldots, N-1)$. From [6] it follows that

$$\mathrm{Log} \; {}^{\boldsymbol{a}_N}\sum_{\mathrm{NPO}} \{\mathsf{S}_N\} \;=\; \mathrm{Log} \; {}^{\boldsymbol{a}_N}\sum_{\mathrm{PO}_f} \{\mathsf{S}_N\}. \tag{4}$$

We are going to generalize this fact in the following ways: in (4), we may replace $\{\mathsf{S}_N\}$ with an arbitrary class $\mathcal{F}$ of N-frames; if, moreover, the logic of the class $\mathcal{F}^u$ has the finite model property, then in the right-hand side of the equation we may replace $\mathcal{F}$ with the class of finite frames of its logic.

A strict partial order $(I, <)$ is called a (*transitive irreflexive*) *tree* if it has a least element (the *root*) and for all $i \in I$ the set $\{j \mid j < i\}$ is a finite chain. Let Tr_f and NTr be the classes of all finite trees and Noetherian trees respectively.

Consider a finite tree $\mathsf{I} = (I, <)$. The *branching of* i *in* I, denoted by $br(i, \mathsf{I})$, is the number of immediate successors of i (j *is an immediate successor of* i, if $i < j$ and there is no k such that $i < k < j$); the *branching of* I, denoted by $br(\mathsf{I})$, is $\max\{br(i, \mathsf{I}) \mid i \text{ in } \mathsf{I}\}$. The *height of* I, denoted by $ht(\mathsf{I})$, is $\max\{|V| \mid V \text{ is a chain in } \mathsf{I}\}$. For $n \in \omega$, let $\mathrm{Tr}(n)$ be the class of all finite trees with height and branching $\leq n$: $\mathrm{Tr}(n) = \{\mathsf{I} \in \mathrm{Tr}_f \mid ht(\mathsf{I}) \leq n \,\&\, br(\mathsf{I}) \leq n\}$.

Let $\bigsqcup \mathcal{F}$ be the class of all disjoint unions $\bigsqcup_I \mathsf{F}_i$, where I is a non-empty set and all F_i are in $\mathcal{F}$, and $\bigsqcup_{\leq k}\mathcal{F}$ the class of all such frames where $|I| \leq k$.

Let $\sharp\varphi$ be the number of subformulas of φ.

Theorem 5.2 *Let $\mathcal{F}$ be a class of N-frames, $s < \omega$, $\boldsymbol{a} = (a_0, \dots a_{s-1}) \in N^s$, $\mathrm{Tr}_f \subseteq \mathcal{I}_0, \dots, \mathcal{I}_{s-1} \subseteq \mathrm{NPO}$, $\mathcal{G} = {}^{a_0}\sum_{\mathcal{I}_0} \dots {}^{a_{s-1}}\sum_{\mathcal{I}_{s-1}} \mathcal{F}$.*

(i) *If $s > 0$, then for every φ we have ${}^{\boldsymbol{a}}\sum_{\mathrm{NPO}} \mathcal{F} \equiv_\varphi \bigsqcup_{\leq \sharp\varphi} {}^{\boldsymbol{a}}\sum_{\mathrm{Tr}(\sharp\varphi)} \mathcal{F}$.*

(ii) $\mathrm{Log}\,\mathcal{G} = \mathrm{Log}\ {}^{\boldsymbol{a}}\sum_{\mathrm{Tr}_f} \mathcal{F}$; *moreover, a formula φ is satisfiable in $\mathcal{G}$ iff φ is satisfiable in ${}^{\boldsymbol{a}}\sum_{\mathrm{Tr}(\sharp\varphi)} \mathcal{F}$.*

(iii) *If $\mathrm{Log}\,\mathcal{F}^u$ has the finite model property, then so does $\mathrm{Log}\,\mathcal{G}$:*

$$\mathrm{Log}\,\mathcal{G} = \mathrm{Log}\ {}^{\boldsymbol{a}}\sum_{\mathrm{Tr}_f} \mathrm{Fr}_f\,\mathrm{Log}\,\mathcal{F}.$$

The proof of this theorem is based on the following statements.

Lemma 5.3 *Let $a \in N$. Every frame in ${}^{a}\sum_{\mathrm{NPO}} \bigsqcup \mathcal{F}$ is isomorphic to a frame in ${}^{a}\sum_{\mathrm{NPO}} \mathcal{F}$.*

Proof. By Proposition 3.7, a sum of form ${}^{a}\sum_{i \in \mathsf{I}} \bigsqcup_{j \in J_i} \mathsf{F}_{ij}$ is isomorphic to ${}^{a}\sum_{(i,j) \in \sum_{k \in \mathsf{I}} (J_k, \varnothing)} \mathsf{F}_{ij}$. If I is Noetherian, then the sum $\sum_{k \in \mathsf{I}} (J_k, \varnothing)$ is. □

Proposition 5.4 *Let $(I, <)$ be a Noetherian tree, $\mathcal{V}$ a finite family of subsets of I, $i_0 \in I$. Then there exists $J \subseteq I$ such that $ht(J, <) \leq |\mathcal{V}| + 1$, $br(J, <) \leq |\mathcal{V}|$, i_0 is the root of $(J, <)$, and for all $V \in \mathcal{V}$, $j \in J$ we have*

$$\exists i > j\ i \in V \ \Rightarrow\ \exists i > j\ i \in V \cap J. \tag{5}$$

The proof of this fact is by a standard 'step-by-step' construction, the details are given in Appendix. We shall use it in the following lemma, which is the crucial technical step in the proof of Theorem 5.2.

Lemma 5.5 *Let $a \in N$. Consider a model $\mathsf{M} \in \mathrm{Mod}\ {}^{a}\sum_{\mathrm{NTr}} \mathcal{F}$. For every φ, $\boldsymbol{\Gamma}$, and x in M, there exists a model $\mathsf{M}' \in \mathrm{Mod}\ {}^{a}\sum_{\mathrm{Tr}(\sharp\varphi)} \mathcal{F}$ which contains x and is a selective filtration of M with respect to φ and $\boldsymbol{\Gamma}$.*

Proof. Let $\mathsf{M} = {}^{a}\sum_{i \in \mathsf{I}} \mathsf{M}_i$, where $\mathsf{I} = (I, <)$ is a Noetherian tree. Consider the family $\mathcal{V} = \{P(\alpha) \mid \Diamond_a \alpha \in \mathrm{sub}(\varphi)\}$, where

$$P(\alpha) = \{i \in I \mid \mathsf{M}, (i, w) \models_{\boldsymbol{\Gamma}} \alpha \text{ for some } w\}.$$

Assume that $x = (i_0, w_0)$. By Proposition 5.4, there exists a restriction $\mathsf{J} = (J, <)$ of I such that $\mathsf{J} \in \mathrm{Tr}(|\mathcal{V}| + 1)$, $i_0 \in J$, and for all $j \in J$, $V \in \mathcal{V}$ we have (5).

Put $\mathsf{M}' = {}^a\sum_{i\in\mathsf{J}} \mathsf{M}_i$ and show that M' is the required selective filtration.

Let $b \in N$, $\Diamond_b\alpha \in \mathrm{sub}(\varphi)$, $\alpha \notin \mathbf{\Gamma}(b)$ and $\mathsf{M},(j,w) \models_{\mathbf{\Gamma}} \Diamond_b\alpha$ for some j in J and some w in M_j. Let R_b be the b-th relation in M. Since $\alpha \notin \mathbf{\Gamma}(b)$, we have $\mathsf{M},(k,u) \models_{\mathbf{\Gamma}} \alpha$ for some k in I and u in M_k such that $(j,w)R_b(k,u)$. Our aim is to choose i in J and v in M_i such that $\mathsf{M},(i,v) \models_{\mathbf{\Gamma}} \alpha$ and $(j,w)R_b(i,v)$.

If $j = k$, we can put $i = k$ and $v = u$.

Assume that $j \neq k$. In this case $a = b$ and $k > j$. Then $k \in P(\alpha)$, and by (5) there exists $i > j$ such that $i \in J$ and $i \in P(\alpha)$. By the definition of $P(\alpha)$, we have $\mathsf{M},(i,v) \models_{\mathbf{\Gamma}} \alpha$ for some v in M_i. Since $i > j$, we have $(j,w)R_a(i,v)$. □

Lemma 5.6 *For $a \in N$, ${}^a\sum_{\mathrm{NPO}} \mathcal{F} \equiv_\varphi \bigsqcup_{\leq \sharp\varphi} {}^a\sum_{\mathrm{Tr}(\sharp\varphi)} \mathcal{F}$.*

Proof. First, we claim that the classes ${}^a\sum_{\mathrm{NPO}} \mathcal{F}$ and $\bigsqcup {}^a\sum_{\mathrm{NTr}} \mathcal{F}$ are interchangeable. By standard unravelling arguments, if a non-empty Noetherian order J has a least element, then it is a p-morhpic image of a Noetherian tree $\mathsf{T}(\mathsf{J})$. Every frame is a p-morphic image of the disjoint union of its cones. Thus, for a non-empty Noetherian order I we have

$$\bigsqcup_{i\in\mathsf{I}} \mathsf{T}(\mathsf{I}[i]) \twoheadrightarrow \bigsqcup_{i\in\mathsf{I}} \mathsf{I}[i] \twoheadrightarrow \mathsf{I};$$

so I is a p-morhpic image of a disjoint union of Noetherian trees. Now by Propositions 3.4 and 4.13 we obtain

$$ {}^a\sum_{\mathrm{NPO}} \mathcal{F} \preccurlyeq {}^a\sum_{\bigsqcup \mathrm{NTr}} \mathcal{F}.$$

Since $\bigsqcup \mathrm{NTr} \subseteq \mathrm{NPO}$, we have

$$ {}^a\sum_{\bigsqcup \mathrm{NTr}} \mathcal{F} \preccurlyeq {}^a\sum_{\mathrm{NPO}} \mathcal{F};$$

it follows that these classes are interchangeable. By Proposition 3.6,

$$ {}^a\sum_{\bigsqcup \mathrm{NTr}} \mathcal{F} \equiv \bigsqcup {}^a\sum_{\mathrm{NTr}} \mathcal{F},$$

which proves the claim.

Trivially,

$$\bigsqcup_{\leq\sharp\varphi} {}^a\sum_{\mathrm{Tr}(\sharp\varphi)} \mathcal{F} \preccurlyeq_\varphi \bigsqcup {}^a\sum_{\mathrm{NTr}} \mathcal{F}.$$

To prove the converse, consider a model $\mathsf{M} = \bigsqcup_{i\in I} \mathsf{M}_i$, where I is a set and all M_i are in Mod ${}^a\sum_{\mathrm{NTr}} \mathcal{F}$. Let $\Psi = \mathrm{sub}(\varphi; \mathsf{M}, \mathbf{\Gamma})$ for a given $\mathbf{\Gamma}$. For each ψ in Ψ we chose some j in I and x_j in M_j such that $\mathsf{M}_j, x_j \models_{\mathbf{\Gamma}} \psi$. Let J be the set of all these j's (if Ψ is empty, let $J = \{j\}$ for some arbitrary $j \in I$, and x_j be an arbitrary element of M_j). By Lemma 5.5, for each $j \in J$ there exists a model $\mathsf{M}'_j \in \mathrm{Mod}\ {}^{\boldsymbol{a}}\sum_{\mathrm{Tr}(\sharp\varphi)} \mathcal{F}$ which contains x_j and is a selective filtration of M_j with respect to φ and $\mathbf{\Gamma}$; it follows that $\mathsf{M}'_j, x_j \models \psi$ by Proposition 4.15. On the other hand, for each $j \in J$, $\mathrm{sub}(\varphi; \mathsf{M}'_j, \mathbf{\Gamma}) \subseteq \mathrm{sub}(\varphi; \mathsf{M}_j, \mathbf{\Gamma})$ by Proposition 4.15, and $\mathrm{sub}(\varphi; \mathsf{M}_j, \mathbf{\Gamma}) \subseteq \Psi$ by Proposition 4.16. It follows that $\Psi = \bigcup_{j\in J} \mathrm{sub}(\varphi; \mathsf{M}'_j, \mathbf{\Gamma})$. By Proposition 4.16 again, we have $\mathrm{sub}(\varphi; \bigsqcup_{j\in J} \mathsf{M}'_j, \mathbf{\Gamma}) = \bigcup_{j\in J} \mathrm{sub}(\varphi; \mathsf{M}'_j, \mathbf{\Gamma})$. Thus $\bigsqcup_{j\in J} \mathsf{M}'_j$ and M are $(\varphi, \mathbf{\Gamma})$-equivalent. □

Proof of Theorem 5.2. (i) By induction on s. The case $s = 1$ is given by Lemma 5.6. For $s > 1$, we put $\boldsymbol{b} = (a_1, \ldots, a_{s-1})$, $\mathcal{G} = {}^{\boldsymbol{b}}\sum_{\mathrm{NPO}} \mathcal{F}$, $\mathcal{H} = {}^{\boldsymbol{b}}\sum_{\mathrm{Tr}(\sharp\varphi)} \mathcal{F}$. We have

$$ {}^{a_0}\sum_{\mathrm{NPO}} \mathcal{G} \equiv_\varphi {}^{a_0}\sum_{\mathrm{NPO}} \bigsqcup\nolimits_{\leq\sharp\varphi} \mathcal{H} \equiv {}^{a_0}\sum_{\mathrm{NPO}} \mathcal{H} \equiv_\varphi \bigsqcup\nolimits_{\leq\sharp\varphi} {}^{a_0}\sum_{\mathrm{Tr}(\sharp\varphi)} \mathcal{H}; $$

the first equivalence holds by induction hypothesis and Theorem 4.3; the next step is immediate from Lemma 5.3; finally, we apply Lemma 5.6 again.

(ii) Since $\mathcal{G}$ contains ${}^{\boldsymbol{a}}\sum_{\mathrm{Tr}_f} \mathcal{F}$, we only have to check that if φ is satisfiable in $\mathcal{G}$, then φ is satisfiable in ${}^{\boldsymbol{a}}\sum_{\mathrm{Tr}(\sharp\varphi)} \mathcal{F}$. The class $\mathcal{G}$ is contained in ${}^{\boldsymbol{a}}\sum_{\mathrm{NPO}} \mathcal{F}$. Now (ii) follows from (i) and Proposition 4.2.

(iii) follows from (ii) and Theorem 4.11. □

5.2 Refinements and lexicographic products

The following construction was introduced in [1] by S. Babenyshev and V. Rybakov.

Definition 5.7 Let $\mathsf{F} = (W, R)$ be a preorder, $\mathrm{sk}F = (\overline{W}, <)$ its skeleton. Consider a family $(\mathsf{F}_C)_{C\in\overline{W}}$ of N-frames such that $\mathrm{dom}(\mathsf{F}_C) = C$ for all $C \in \overline{W}$. The *refinement* of F *by* $(\mathsf{F}_C)_{C\in\overline{W}}$ is the $(1+N)$-frame $(W, R, (R^\rhd_a)_{a\in N})$, where

$$ R^\rhd_a \subseteq \bigcup\nolimits_{C\in\overline{W}} C \times C \quad \text{for all } a \in N, \tag{6} $$

$$ (W, (R^\rhd_a)_{a\in N})\restriction C = \mathsf{F}_C \ \text{ for all } C \in \overline{W}. \tag{7} $$

For a class $\mathcal{I}$ of preorders and a class $\mathcal{G}$ of N-frames let $\mathrm{Ref}(\mathcal{I}, \mathcal{F})$ be the class of all refinements of frames from $\mathcal{I}$ by frames in $\mathcal{F}$. For logics $L_1 \supseteq \mathrm{S4}, L_2$, we put $\mathrm{Ref}(L_1, L_2) = \mathrm{Log}\,\mathrm{Ref}(\mathrm{Fr}\,L_1, \mathrm{Fr}\,L_2)$.

Remark 5.8 In [1], refinements are defined in a more general way — for the cases when F is a K-frame ($K \leq \omega$) with transitive relations.

In [1] it was shown that in many cases the refinement operation preserves the finite model property. In particular, if both L_1 and L_2 *admit filtration* (in the sense of Lemmon and Scott [13]), then $\mathrm{Ref}(L_1, L_2)$ is the logic of the class $\mathrm{Ref}(\mathrm{Fr}_f\,L_1, \mathrm{Fr}_f\,L_2)$. Moreover, from the proof it follows that if L_2 admit filtrations, then $\mathrm{Ref}(L_1, L_2)$ is the logic of $\mathrm{Ref}(\mathrm{Fr}\,L_1, \mathrm{Fr}_f\,L_2)$ ([1, Lemma 3.3]).

We consider refinements of frames as sums and provide another condition for the latter equality.

Let us make the convention that the universal modality comes first in the language and shifts other modalities: for an N-frame $\mathsf{G} = (W, R_0, R_1, \ldots)$, G^u is the $(1+N)$-frame $(W, W \times W, R_0, R_1, \ldots)$.

Proposition 5.9 *If* $\mathsf{F}^{\triangleright}$ *is the refinement of a preorder* F *by the frames* $(\mathsf{F}_C)_{C \in \mathrm{sk}\mathsf{F}}$*, then*

$$\mathsf{F}^{\triangleright} \cong {}^{0}\!\!\sum_{C \in \mathrm{sk}\mathsf{F}} \mathsf{F}_C^u.$$

Proof. The required isomorphism is defined as $w \mapsto (C, w)$, where $w \in C$. $\square$

For a logic L, let L^u be the logic of the class $(\mathrm{Fr}\, L)^u$.

Theorem 5.10 *Let* L_1 *be a unimodal logic containing* S4*. For every logic* L_2 *such that* L_2^u *has the finite model property, we have*

$$\mathrm{Ref}(L_1, L_2) = \mathrm{Log}\,\mathrm{Ref}(\mathrm{Fr}\, L_1, \mathrm{Fr}_f\, L_2).$$

Proof. Suppose that a formula φ is satisfiable in $\mathrm{Ref}(\mathrm{Fr}\, L_1, \mathrm{Fr}\, L_2)$ and show that it is satisfiable in $\mathrm{Ref}(\mathrm{Fr}\, L_1, \mathrm{Fr}_f\, L_2)$. By Proposition 5.9, φ is satisfiable in a model $\mathsf{M} = {}^{0}\!\sum_{C \in \mathrm{sk}\mathsf{F}} \mathsf{M}_C^u$, where $\mathsf{F} \models L_1$ and for every $C \in \mathrm{sk}\mathsf{F}$, M_C is a model on a frame validating L_2. The classes $(\mathrm{Fr}\, L_2)^u$ and $(\mathrm{Fr}_f\, L_2)^u$ have the same logic L_2^u, since it has the finite model property. Hence by Proposition 4.10, $\mathrm{Fr}\, L_2 \equiv \mathrm{Fr}_f\, L_2$. Then by Proposition 4.12, $(\mathrm{Fr}\, L_2)^u \equiv (\mathrm{Fr}_f\, L_2)^u$. We consider the condition $\mathbf{\Gamma} = (\varnothing)_{N+1}$ and use Lemma 4.6 to construct models M'_C $(C \in \mathrm{sk}\mathsf{F})$ such that

- the sums M and $\mathsf{M}' = {}^{0}\!\sum_{C \in \mathrm{sk}\mathsf{F}} {\mathsf{M}'_C}^u$ are $(\varphi, \mathbf{\Gamma})$-equivalent,
- every M'_C is based on a finite frame validating L_2, and
- $\mathsf{M}_C = \mathsf{M}'_C$ whenever C is finite.

Thus φ is satisfiable in M'. For $C \in \mathrm{sk}\mathsf{F}$, we put $C' = \mathrm{dom}(\mathsf{M}'_C)$. The frame of M' is the refinement of the preorder $\mathsf{G} = \sum_{C \in \mathrm{sk}\mathsf{F}} (C', C' \times C')$ by the frames of models M'_C. It follows that $\mathsf{F} \twoheadrightarrow \mathsf{G}$ (indeed, the preorder F is isomorphic to $\sum_{C \in \mathrm{sk}\mathsf{F}} (C, C \times C)$, and for each C in $\mathrm{sk}\mathsf{F}$ we have $|\,\mathrm{dom}(\mathsf{M}'_i)| \leq |C|$). It follows that G validates L_1. Thus, the frame of M' is in $\mathrm{Ref}(\mathrm{Fr}\, L_1, \mathrm{Fr}_f\, L_2)$ as required. $\square$

Another sum-based operation is the *lexicographic product of logics*, introduced in [2] by Ph. Balbiani. Fix $N, K < \omega$.

Definition 5.11 Consider frames $\mathsf{I} = (I, (S_a)_{a \in K})$ and $\mathsf{F} = (W, (R_b)_{b \in N})$. The *l-product* $\mathsf{I} \leftthreetimes \mathsf{F}$ is the $(K+N)$-frame $(I \times W, (S_a^{\leftthreetimes})_{a \in K}, (R_b^{\leftthreetimes})_{b \in N})$, where

$$(i, w) S_a^{\leftthreetimes} (j, u) \quad \text{iff} \quad i S_a j,$$
$$(i, w) R_b^{\leftthreetimes} (j, u) \quad \text{iff} \quad i = j \;\&\; w R_b u.$$

For a class $\mathcal{I}$ of K-frames and a class $\mathcal{G}$ of N-frames, the class $\mathcal{I} \rightthreetimes \mathcal{F}$ is the class of all products $\mathsf{I} \rightthreetimes \mathsf{F}$ such that $\mathsf{I} \in \mathcal{I}$ and $\mathsf{F} \in \mathcal{F}$. For logics L_1, L_2, we put $L_1 \rightthreetimes L_2 = \mathrm{Log}\,(\mathrm{Fr}\, L_1 \rightthreetimes \mathrm{Fr}\, L_2)$.

From the definitions we have

Proposition 5.12 $\mathsf{I} \rightthreetimes \mathsf{F} = \sum_{\mathsf{I}'} \mathsf{F}_i$, *where* $\mathsf{I}' = (I, (S_a)_{a\in K}, (\varnothing)_N)$, *and for each* $i \in I$, $\mathsf{F}_i = (W, (S_{i,a})_{a\in K}, (R_b)_{b\in N})$ *with* $S_{i,a} = W \times W$ *if* $iS_a i$, *and* $S_{i,a} = \varnothing$ *otherwise.*

Let QO and QO_f be the classes of all non-empty preorders and finite non-empty preorders respectively.

Theorem 5.13 $\mathrm{S4} \rightthreetimes \mathrm{S4} = \mathrm{Ref}(\mathrm{S4}, \mathrm{S4}) = \mathrm{Log}\ {}^0\!\sum_{\mathrm{Tr}_f} {\mathrm{QO}_f}^u$.

Proof. First, we show that every product $\mathsf{I} \rightthreetimes \mathsf{F}$ of preorders is in $\mathrm{Ref}(\mathrm{QO}, \mathrm{QO})$. Let $\mathsf{I} \rightthreetimes \mathsf{F} = (W, R_0, R_1)$, $\mathsf{H} = (W, R_0)$. Notice that H is a preorder. Then $\mathsf{I} \rightthreetimes \mathsf{F}$ is the refinement of H by the family $(\mathsf{G}_C)_{C\in \mathrm{sk}\mathsf{H}}$, where G_C is the restriction of (W, R_1) to C. Each G_C is a disjoint union of copies of F, thus G_C is a preorder. Hence $\mathsf{I} \rightthreetimes \mathsf{F} \in \mathrm{Ref}(\mathrm{QO}, \mathrm{QO})$.

By the definition, $\mathrm{Ref}(S4, S4) = \mathrm{Log}\,\mathrm{Ref}(\mathrm{QO}, \mathrm{QO})$. It follows that

$$\mathrm{S4} \rightthreetimes \mathrm{S4} \supseteq \mathrm{Ref}(\mathrm{S4}, \mathrm{S4}).$$

Suppose that φ is satisfiable in $\mathrm{Ref}(\mathrm{QO}, \mathrm{QO})$. In [1], it was shown that $\mathrm{Ref}(S4, S4) = \mathrm{Log}\,\mathrm{Ref}(\mathrm{QO}_f, \mathrm{QO}_f)$. Thus φ is satisfiable in $\mathrm{Ref}(\mathrm{QO}_f, \mathrm{QO}_f)$. Hence by Proposition 5.9, φ is satisfiable in ${}^0\!\sum_{\mathrm{PO}_f} {\mathrm{QO}_f}^u$. By Theorem 5.2, φ is satisfiable in ${}^0\!\sum_{\mathrm{Tr}_f} {\mathrm{QO}_f}^u$. Thus

$$\mathrm{Ref}(\mathrm{S4}, \mathrm{S4}) \supseteq \mathrm{Log}\ {}^0\!\sum_{\mathrm{Tr}_f} {\mathrm{QO}_f}^u.$$

In [2], it was shown that $\mathrm{S4} \rightthreetimes \mathrm{S4}$ is the least logic containing the axioms of S4 for $\Diamond_0, \Diamond_1$ and the formulas

$$\Diamond_0\Diamond_1 p \to \Diamond_0 p, \quad \Diamond_1\Diamond_0 p \to \Diamond_0 p, \quad \Diamond_0 p \to \Box_1\Diamond_0 p.$$

They are valid in ${}^0\!\sum_{\mathrm{Tr}_f} {\mathrm{QO}_f}^u$, thus $\mathrm{Log}\ {}^0\!\sum_{\mathrm{Tr}_f} {\mathrm{QO}_f}^u \supseteq \mathrm{S4} \rightthreetimes \mathrm{S4}$. □

As another example, we consider the logic $\mathrm{GL} \rightthreetimes \mathrm{S4}$, where GL is the Gödel-Löb logic. The next theorem shows that $\mathrm{GL} \rightthreetimes \mathrm{S4}$ is approximable by finite products and sums. Let $\mathcal{C}$ be the class of finite frames of form $(C, \varnothing, C \times C)$.

Theorem 5.14 $\mathrm{GL} \rightthreetimes \mathrm{S4} = \mathrm{Log}\,(\mathrm{Tr}_f \rightthreetimes \mathrm{QO}_f) = \mathrm{Log}\ {}^0\!\sum_{\mathrm{Tr}_f} {}^1\!\sum_{\mathrm{Tr}_f} \mathcal{C}$.

Proof. For a frame $\mathsf{F} = (W, R)$ let $\mathsf{F}^{[\varnothing]}$ be the 2-frame $(W, \varnothing, R)$; for a class $\mathcal{F}$ of 1-frames we put $\mathcal{F}^{[\varnothing]} = \{\mathsf{F}^{[\varnothing]} \mid \mathsf{F} \in \mathcal{F}\}$.

Since $\mathrm{NPO} = \mathrm{Fr}\,\mathrm{GL}$, by the definition, $\mathrm{GL} \rightthreetimes \mathrm{S4}$ is the logic of the class $\mathrm{NPO} \rightthreetimes \mathrm{QO}$. By Proposition 5.12, $\mathrm{NPO} \rightthreetimes \mathrm{QO} \subseteq {}^0\sum_{\mathrm{NPO}} \mathrm{QO}^{[\varnothing]}$. It follows that

$$\mathrm{Log}\ {}^0\sum_{\mathrm{NPO}} \mathrm{QO}^{[\varnothing]} \subseteq \mathrm{GL} \rightthreetimes \mathrm{S4}.$$

Consider the class $(\mathrm{QO}^{[\varnothing]})^u = \{(W, W \times W, \varnothing, R) \mid (W,R) \in \mathrm{QO}\}$. It is a standard fact the the logic of this class has the finite model property (e.g., it follows from [10, Theorem 5.9]). By Theorem 5.2 we obtain

$$\mathrm{Log}\ {}^0\sum_{\mathrm{NPO}} \mathrm{QO}^{[\varnothing]} = \mathrm{Log}\ {}^0\sum_{\mathrm{Tr}_f} {\mathrm{QO}_f}^{[\varnothing]}.$$

We shall now prove that

$$\mathrm{Log}\ {}^0\sum_{\mathrm{Tr}_f} {\mathrm{QO}_f}^{[\varnothing]} = \mathrm{Log}\,(\mathrm{Tr}_f \rightthreetimes \mathrm{QO}_f).$$

If φ is satisfiable in $\mathrm{Tr}_f \rightthreetimes \mathrm{QO}_f$, then φ is satisfiable in ${}^0\sum_{\mathrm{Tr}_f} {\mathrm{QO}_f}^{[\varnothing]}$ by Proposition 5.12. Conversely, suppose that φ is satisfiable in a sum ${}^0\sum_{\mathsf{I}} \mathsf{F}_i^{[\varnothing]}$, where I is a finite tree and F_i are finite preorders. Consider the Cartesian product G of the preorders $(\mathsf{F}_i)_{i \in I}$. It is easy to see that $\mathsf{G} \twoheadrightarrow \mathsf{F}_i$ and so $\mathsf{G}^{[\varnothing]} \twoheadrightarrow \mathsf{F}_i^{[\varnothing]}$ for each i in I. Now it follows from Propositions 3.4 and 5.12 that $\mathsf{I} \rightthreetimes \mathsf{G} \twoheadrightarrow {}^0\sum_{\mathsf{I}} \mathsf{F}_i^{[\varnothing]}$. Since G is a finite preorder, φ is satisfiable in $\mathrm{Tr}_f \rightthreetimes \mathrm{QO}_f$.

Altogether we have proved

$$\mathrm{Log}\,(\mathrm{Tr}_f \rightthreetimes \mathrm{QO}_f) \;=\; \mathrm{Log}\ {}^0\sum_{\mathrm{Tr}_f} {\mathrm{QO}_f}^{[\varnothing]} \;=\; \mathrm{Log}\ {}^0\sum_{\mathrm{NPO}} \mathrm{QO}^{[\varnothing]} \;\subseteq\; \mathrm{GL} \rightthreetimes \mathrm{S4}.$$

It follows that these four logics coincide: indeed, $\mathrm{GL} \rightthreetimes \mathrm{S4}$ is contained in the logic of the class $\mathrm{Tr}_f \rightthreetimes \mathrm{QO}_f$, since this class is contained in $\mathrm{NPO} \rightthreetimes \mathrm{QO}$.

Every finite preorder is (up to isomorphism) the sum of finite frames of form $(C, C \times C)$ over a finite partial order, and vice versa. Thus, the classes ${\mathrm{QO}_f}^{[\varnothing]}$ and ${}^1\sum_{\mathrm{PO}_f} \mathcal{C}$ coincide up to isomorphisms. It follows that $\mathrm{GL} \rightthreetimes \mathrm{S4}$ is the logic of the class ${}^0\sum_{\mathrm{Tr}_f} {}^1\sum_{\mathrm{PO}_f} \mathcal{C}$. Finally, we have

$$\mathrm{Log}\ {}^0\sum_{\mathrm{Tr}_f} {}^1\sum_{\mathrm{PO}_f} \mathcal{C} \;=\; \mathrm{Log}\ {}^0\sum_{\mathrm{Tr}_f} {}^1\sum_{\mathrm{Tr}_f} \mathcal{C}$$

by Theorem 5.2. □

6 Further results

For classes $\mathcal{I}$ and $\mathcal{F}$, let $\mathcal{I}_f = \mathrm{Fr}_f \operatorname{Log} \mathcal{I}$, and $\mathcal{F}_f = \mathrm{Fr}_f \operatorname{Log} \mathcal{F}$. When do we have $\operatorname{Log} \sum_{\mathcal{I}} \mathcal{F} = \operatorname{Log} \sum_{\mathcal{I}_f} \mathcal{F}_f$? Finite summands can be obtained by Theorem 4.11. Theorem 5.2 allows us to obtain finite indices in the case of sums over Noetherian orders. The proof of Theorem 5.2 is based on selective filtration. Another way is to use filtration in the sense of Lemmon and Scott [13]: this approach was successfully used in [1] to obtain the finite model property for refinements in numerous cases. The methods developed in [1] in a combination with Theorem 4.11 suggest the following conjecture: in the case of finitely many modalities, if $\operatorname{Log} \mathcal{F}^u$ has the finite model property, and $\operatorname{Log} \mathcal{I}$ admits filtration, then the classes $\sum_{\mathcal{I}} \mathcal{F}$ and $\sum_{\mathcal{I}_f} \mathcal{F}_f$ are interchangeable.

Theorem 5.2 can be used to obtain complexity results for logics of sums over Noetherian orders, in particular – over finite orders. Let $\operatorname{Sat} \mathcal{F}$ denote the satisfiability problem for $\mathcal{F}$.

Theorem 6.1 *Let $\mathcal{F}$ be a non-empty class of N-frames, $a \in N$, $\mathcal{G} = {}^{a}\sum_{\mathcal{I}} \mathcal{F}$, where $\mathrm{Tr}_f \subseteq \mathcal{I} \subseteq \mathrm{NPO}$. If $\operatorname{Sat} \mathcal{F}^u$ is in* PSPACE*, then $\operatorname{Sat} \mathcal{G}^u$ is* PSPACE*-complete.*

This result generalizes [14, Theorem 35]; the proof will be given in a forthcoming paper. In particular, in view of Theorems 5.13 and 5.14, it follows that the logics $\mathrm{S4} \leftthreetimes \mathrm{S4}$ and $\mathrm{GL} \leftthreetimes \mathrm{S4}$ are PSPACE-complete.

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Appendix

Proof of Lemma 3.5. Let $\mathsf{I} = (I, (S_a)_{a\in N})$. For $i \in I$, let $\mathsf{J}_i = (J_i, (S_{i,a})_{a\in N})$, and for $j \in J_i$, $\mathsf{F}_{ij} = (W_{ij}, (R_{ij,a})_{a\in N})$. Let W be the set of all triples (i, j, w) such that $i \in I$, $j \in J_i$, and $w \in W_{ij}$. By the definition, the domain of $\sum_{i\in\mathsf{I}} \sum_{j\in\mathsf{J}_i} \mathsf{F}_{ij}$ is the set of all the pairs $(i, (j, w))$ such that $(i, j, w) \in W$. Likewise, the domain of $\sum_{(i,j)\in\sum_{k\in\mathsf{I}} \mathsf{J}_k} \mathsf{F}_{ij}$ consists of all $((i, j), w)$ such that $(i, j, w) \in W$.

For $a \in N$, let R'_a, R''_a be respectively the a-th relations in $\sum_{i\in\mathsf{I}} \sum_{j\in\mathsf{J}_i} \mathsf{F}_{ij}$ and $\sum_{(i,j)\in\sum_{k\in\mathsf{I}} \mathsf{J}_k} \mathsf{F}_{ij}$. We claim that for all $(i, j, w), (i', j', w') \in W$, $a \in N$,

$$(i, (j, w))R'_a(i', (j', w')) \quad \text{iff} \quad ((i, j), w)R''_a((i', j'), w'). \tag{A.1}$$

By the definition, $(i, (j, w))R'_a(i', (j', w'))$ iff

$$i \neq i' \;\&\; iS_ai' \quad \text{or} \quad i = i' \;\&\; (j \neq j' \;\&\; jS_{i,a}j' \text{ or } j = j' \;\&\; wR_{ij,a}w'). \tag{A.2}$$

Likewise, $((i, j), w)R''_a((i', j'), w')$ iff

$$(i, j) \neq (i', j') \;\&\; (i \neq i' \;\&\; iS_ai' \text{ or } i = i' \;\&\; jS_{i,a}j') \quad \text{or} \quad (i, j) = (i', j') \;\&\; wR_{ij,a}w'. \tag{A.3}$$

It is straightforward that (A.2) and (A.3) are equivalent. □

Proof of Proposition 4.15. By induction on the length of ψ. Consider the case $\psi = \Diamond_a\chi \in \mathrm{sub}(\varphi)$.

If $\chi \in \mathbf{\Gamma}(a)$, then, by Definition 4.1, $\mathsf{M}', w \models_{\mathbf{\Gamma}} \Diamond_a\chi$ and $\mathsf{M}, w \models_{\mathbf{\Gamma}} \Diamond_a\chi$.

Assume that $\chi \notin \mathbf{\Gamma}(a)$.

If $\mathsf{M}', w \models_{\mathbf{\Gamma}} \Diamond_a \chi$, then for some $v \in R'_a(w)$ we have $\mathsf{M}', v \models_{\mathbf{\Gamma}} \chi$, which is equivalent to $\mathsf{M}, v \models_{\mathbf{\Gamma}} \chi$ by induction hypothesis; since $R'_a \subseteq R_a$, we have $\mathsf{M}, w \models_{\mathbf{\Gamma}} \Diamond_a \chi$.

Conversely, assume that $\mathsf{M}, w \models_{\mathbf{\Gamma}} \Diamond_a \chi$. By Definition 4.14, $\mathsf{M}, v \models_{\mathbf{\Gamma}} \chi$ for some $v \in R'_a(w)$; by induction hypothesis, $\mathsf{M}', v \models_{\mathbf{\Gamma}} \chi$, and so $\mathsf{M}', w \models_{\mathbf{\Gamma}} \Diamond_a \chi$.□

Proof of Proposition 5.4. For $V \subseteq I$, let V' be all maximal elements of V, $\Diamond V = \{j \mid \exists i > j \; i \in V\}$. Since $(I, >)$ is well-founded, we have

$$\Diamond V = \Diamond V'. \tag{A.4}$$

Put $K = \{i_0\} \cup \bigcup\{V' \mid V \in \mathcal{V}\}$, $\mathsf{K} = (K, <)$. The height of K is not greater than $|\mathcal{V}| + 1$: indeed, if $i \in U'$, $j \in V'$, and $i < j$, then $U \neq V$.

Let h be the height of the cone $\mathsf{K}[i_0]$ (the *depth* of i_0 in K). We construct the required $J \subseteq K$ by induction on h.

If $h = 1$, then $\mathsf{K}[i_0] = (\{i_0\}, \varnothing)$, and we put $\mathsf{J} = \mathsf{K}[i_0]$; then (5) is trivial, the branching of J is 0.

Assume that $h > 1$. Consider the family

$$\mathcal{U} = \{U \in \mathcal{V} \mid i_0 \in \Diamond U\}.$$

Let $U \in \mathcal{U}$. By (A.4), we have $i_0 < j$ for some $j \in U' \subseteq K$; the height of K is finite, thus for some immediate successor i_U of i_0 in K we have

$$i_U \in U' \cup \Diamond U'. \tag{A.5}$$

In K, the depth of i_U is less than the depth of i_0. By induction hypothesis, i_U is the root of a tree $(J(i_U), <)$ whose branching is not greater than $|\mathcal{V}|$ and

$$\forall V \in \mathcal{V} \; \forall j \in J(i_U) \; (j \in \Diamond V \;\Rightarrow\; j \in \Diamond V \cap J(i_U)). \tag{A.6}$$

We put $J = \{i_0\} \cup \bigcup\{J(i_U) \mid U \in \mathcal{U}\}$. The branching of i_0 in $(J, <)$ is not greater than the cardinality of $\mathcal{U} \subseteq \mathcal{V}$, thus $br(J, <) \leq |\mathcal{V}|$. Since $J \subseteq K$, $ht(J, <) \leq ht(\mathsf{K}) \leq |\mathcal{V}| + 1$. By (A.5) and (A.6) we have (5). □