Homogeneous Sasaki and Vaisman manifolds of unimodular Lie groups

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Abstract

In this paper we develop some basic strategies to classify homogeneous locally conformally Kähler and Sasaki manifolds. In particular, we make a complete classification of simply connected homogeneous Sasaki and Vaisman manifolds of unimodular Lie groups, up to isomorphisms.

Introduction

In our previous papers [7], [8], [1] we have discussed a basic framework of the structure of homogeneous locally conformally Kähler manifolds, and classified completely those of compact Lie group, up to holomorphic isometry, while showing that all of them are of Vaisman type. In this paper we extend our study to those of unimodular Lie group.

We recall that a *locally conformally Kähler manifold*, or shortly an *l.c.K. manifold*, is a Hermitian manifold (M, g, J), where g is a Hermitian metric with a compatible integrable complex structure J, which satisfies the condition

$$d\Omega = \Omega \wedge \theta$$

for its associated fundamental 2-form Ω and a closed 1-form θ, called the *Lee form*. M is of *Vaisman type* if the Lee form θ is parallel with respect to g, or equivalently if the *Lee*

2010 *Mathematics Subject Classification*. Primary 32M10, 53A30; Secondary 53B35.

Key words and phrases. locally conformally Kähler structure, Sasaki structures, Vaisman structures, homogenous Kähler manifolds, unimodular Lie groups.

field ξ given by $g^{-1}\theta$ is Killing with respect to g. We mean by a *homogeneous locally conformally Kähler manifold* a homogeneous Hermitian manifold (M, g, J) satisfying the above condition for its associated fundamental form Ω; in particular the Lee form θ is also invariant. We can express M as G/H in an effective form, where G is a connected Lie group acting transitively on (M, g, J), and H is a closed subgroup of G. $M = G/H$ is said to be of *unimodular type* if G can be taken as a unimodular Lie group.

A homogeneous l.c.K. manifold of compact Lie group is nothing but a compact homogeneous l.c.K. manifold; and we have already shown in [8] a holomorphic structure theorem asserting that it has a holomorphic fiber bundle over a flag manifold with fiber a 1-dimensional complex torus, and a metric structure theorem asserting that all of them are of Vaisman type. Note that we have an extended version of the above metric theorem for homogeneous l.c.K. manifolds in [1], showing a sufficient condition for being of Vaisman type, that is, the normalizer of the isotropy subgroup H in G is compact, while showing an example of a non-Vaisman l.c.K. structures on a reductive Lie group.

For 4-dimensional case, we know that the only compact homogeneous l.c.K. manifold is a Hopf manifold of homogeneous type, and as an application of the classification of unimodular l.c.K. Lie groups with and without lattices, we also have a complete classification of 4-dimensional compact locally homogeneous l.c.K. manifolds [7].

We say that two homogeneous Hermitian manifolds are *isomorphic* if they are holomorphically isometric. As an essential tool in the classification of homogeneous Hermitian manifolds G/H of unimodular type up to isomorphism, we apply *modification* of G/H into G'/H' (see Section 1 for definition), where they are isomorphic and both are of unimodular type.

As main results of the paper we classify unimodular Sasaki and Vaisman Lie groups, and more generally, simply connected homogeneous Sasaki and Vaisman manifolds of unimodular Lie group, up to isomorpshims.

Theorem 1. *A simply connected Vaisman unimodular Lie group is, up to modifications, isomorphic (as Vaisman Lie group) to one of the following:*

$$\mathbf{R} \times N, \mathbf{R} \times SU(2), \mathbf{R} \times \widetilde{SL}(2, \mathbf{R}),$$

where N *is a real Heisenberg Lie group and* $\widetilde{SL}(2, \mathbf{R})$ *is the universal covering Lie group of* $SL(2, \mathbf{R})$.

Theorem 2. *A simply connected homogeneous Vaisman manifold M of unimodular Lie group is isomorphic to $\mathbf{R} \times M_1$, where M_1 is a simply connected homogeneous Sasaki manifold of unimodular Lie group, which is a quantization of a simply connected homogeneous Kähler manifold M_2 of reductive Lie group. As a complex manifold M is a holomorphic principal bundle over a simply connected homogeneous Kähler manifold M_2 with fiber $\mathbf{C}^1$ or $\mathbf{C}^*$.*

In the above statement we mean by a *quantization* of a homogeneous Kähler manifold M_2, a principal bundle M_1 over M_2 with fiber $\mathbf{R}$ or S^1 satisfying $d\psi = \omega$ for a contact form ψ on M_1 and the Kähler form ω on M_2.

A basic idea of the proofs is to show first that, up to modifications, a simply connected homogeneous Vaisman manifold of unimodular Lie group can be assumed to have the form $M = G/H$, where G is a simply connected unimodular Lie group of the form $G = \mathbf{R} \times G_1$ and H is a connected compact subgroup of G_1; and our previous results in [8], [1] yields

Proposition 1. *Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G, H respectively. Then the pair $\{\mathfrak{g}, \mathfrak{h}\}$ is of the following form.*

$$\mathfrak{g} = \mathbf{R} \times \mathfrak{g}_1,$$

where $\mathfrak{g}_1 = \ker\theta \supset \mathfrak{h}$, and $\mathfrak{g}_1$ is a central extension of $\mathfrak{g}_2$:

$$0 \to \mathbf{R} \to \mathfrak{g}_1 \to \mathfrak{g}_2 \to 0.$$

The Lee field ξ and the Reeb field $\eta = J\xi$ generate $Z(\mathfrak{g})$; and the l.c.K. form Ω can be written as

$$\Omega = -\theta \wedge \psi + d\psi,$$

where ψ is the Reeb form defining a contact form on the homogeneous Sasaki manifold G_1/H. Let $\mathfrak{k} = \pi(\mathfrak{h})$ for the projection $\pi : \mathfrak{g}_1 \to \mathfrak{g}_2$. Then the pair $\{\mathfrak{g}_2, \mathfrak{k}\}$ defines a homogeneous Kähler manifold with the Kähler form $d\psi|\mathfrak{g}_2$.

We further observe, applying some basic results from the field of homogeneous Kähler manifolds that the homogeneous Kähler manifold associated to $\{\mathfrak{g}_2, \mathfrak{k}\}$ is of reductive type. Hence we can reduce the classification problem of homogeneous Vaisman manifolds of unimodular type to that of homogeneous Sasaki manifolds of the same type, which are quantizations of homogeneous Kähler manifolds of reductive Lie group.

We already know that a simply connected homogeneous Kähler manifold of reductive Lie group is a Kählerian product of $\mathbf{C}^k$ and a homogeneous Kähler manifold of semi-simple Lie group (which has a structure of a holomorphic fiber bundle over a symmetric domain with fiber a flag manifold).

Conversely, starting from a simply connected homogeneous Kähler manifold of reductive Lie group, we may construct its quantization which is to be a simply connected homogeneous Sasaki manifold and then take a product of $\mathbf{R}$, making it a simply connected homogeneous Vaisman manifold of unimodular type. Here the quantization must be the one induced from a central extension of a Kähler algebra $(\mathfrak{g}_2, \mathfrak{k})$ of reductive Lie algebra as in the above proposition. We assert that a simply connected homogeneous Kähler manifold of reductive Lie group is $\mathbf{R}^1$-quantizable to a simply connected homogeneous Sasaki manifold if and only if it is a product of $\mathbf{C}^k$ and a symmetric domain, which is exactly the case when it contains no flag manifolds; and it is S^1-quantizable in all other cases.

1 Preliminaries

A *contact metric structure* $\{\psi, \eta, \widetilde{J}, g\}$ on M^{2n+1} is a contact structure $\psi\,, \psi \wedge (d\psi)^n \neq 0$ with the *Reeb field* $\eta\,, i(\eta)\psi = 1, i(\eta)d\psi = 0$, a $(1,1)$-tensor $\widetilde{J}\,, \widetilde{J}^2 = -I + \psi \otimes \eta$ and a Riemannian metric $g\,, g(X,Y) = \psi(X)\psi(Y) + d\,\psi(X, \widetilde{J}Y)$. A *Sasaki structure* on M^{2n+1} is a contact metric structure $\{\psi, \eta, \widetilde{J}, g\}$ satisfying $\mathcal{L}_\eta g = 0$ (Killing field) and the integrability of $J = \widetilde{J}|\mathcal{D}$ on $\mathcal{D} = \ker\psi$ (CR-structure). The automorphism group $\mathcal{A}(M)$ of a Sasaki manifold M is the set of all diffeomorphisms ψ with $\psi^*\eta = \eta, J\psi_* = \psi_* J, \psi_*\mathcal{D} \subset \mathcal{D}$. M is a *homogeneous Sasaki manifold*, if $\mathcal{A}(M)$ acts transitively on M. Note that $\mathcal{A}(M)$ is a closed Lie subgroup of the isometry group $\mathcal{I}(M)$ of M; and it is compact if M is compact.

For any Sasaki manifold N, its *Kähler cone* $C(N)$ is defined as $C(N) = \mathbf{R}_+ \times N$ with the Kähler form $\omega = rdr \wedge \psi + \frac{r^2}{2} d\psi$, where a compatible complex structure $\widehat{J}$ is defined by $\widehat{J}\eta = \frac{1}{r}\partial_r$ and $\widehat{J}|\mathcal{D} = J$. Note that a contact metric manifold N^{2n+1} with $\{\psi, \eta, \widetilde{J}\}$ is Sasaki if and only if the Kähler cone $C(N)$ with $(\omega, \widetilde{J})$ is Kählerian.

For any Sasaki manifold N with contact form ψ, we can define an l.c.K. form $\Omega = \frac{2}{r^2}\omega = \frac{2}{r} dr \wedge \psi + d\psi$; or taking $t = -2\log r$, $\Omega = -dt \wedge \psi + d\psi$ on $M = \mathbf{R} \times N$ or $S^1 \times N$, which is of Vaisman type. We can define a family of complex structures J

compatible with Ω by

$$J\,\partial_t = b\,\partial_t + (1+b^2)\,\eta, J\eta = -\partial_t - b\,\eta,$$

where $b \in \mathbf{R}$ and the Lee field is $J\eta$. Conversely, any simply connected complete Vaisman manifold is of the form $\mathbf{R} \times N$ with an l.c.K. structure as above, where N is a simply connected complete Sasaki manifold.

Let $M = G/H$ be a homogeneous space of a connected Lie group G with closed subgroup H. Then the tangent bundle of M is given as a G-bundle $G \times_H \mathfrak{g}/\mathfrak{h}$ over $M = G/H$ with fiber $\mathfrak{g}/\mathfrak{h}$, where the action of H on the fiber is given by $\mathrm{Ad}(x)\,(x \in H)$. A vector field on M is a section of this bundle; and a p-form on M is a section of G-bundle $G \times_H \wedge^p(\mathfrak{g}/\mathfrak{h})^*$, where the action of H on the fiber is given by $\mathrm{Ad}(x)^*\,(x \in H)$. An invariant vector field (respectively p-form), the one which is invariant by the left action of G, is canonically identified with an element of $(\mathfrak{g}/\mathfrak{h})^H$ (respectively $(\wedge^p(\mathfrak{g}/\mathfrak{h})^*)^H$), which is the set of elements of $\mathfrak{g}/\mathfrak{h}$ (respectively $\wedge^p(\mathfrak{g}/\mathfrak{h})^*$) invariant by the adjoint action of H. A complex structure J on M is likewise considered as an element J of $\mathrm{Aut}(\mathfrak{g}/\mathfrak{h})$ such that $J^2 = -1$ and $\mathrm{Ad}(x)J = J\mathrm{Ad}(x)\,(x \in H)$. Note that we may also consider an invariant p-form as an element of $\wedge^p\mathfrak{g}^*$ vanishing on $\mathfrak{h}$ and invariant by the action $\mathrm{Ad}(x)^*\,(x \in H)$.

Let $\mathfrak{g}$ be a Lie algebra with a Hermitian structure (h, J), and $\mathrm{Der}(\mathfrak{g})$ the derivation algebra of $\mathfrak{g}$. For a map of Lie algebras

$$\phi : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$$

satisfying the condition

$$h(\phi(X)Y, Z) + h(Y, \phi(X)(Z)) = 0,\ \ J\phi(X) = \phi(X)J$$

for any $X, Y, Z \in \mathfrak{g}$, we define the Lie algebra $\mathfrak{g}_\phi$ by setting

$$\mathfrak{g}_\phi = \mathfrak{g} \rtimes \bar{\mathfrak{g}},$$

on which the new Lie brackets are defined by

$$[(X, \phi(X')), (Y, \phi(Y'))]_\phi = ([X, Y] + \phi(X')Y - \phi(Y')X, [\phi(X'), \phi(Y')]),$$

where $\bar{\mathfrak{g}} = \operatorname{im}\phi$, extending the metric on $\mathfrak{g}_\phi$ as $h(\bar{\mathfrak{g}}, \mathfrak{g}_\phi) = 0$, and the complex structure as $J(\bar{\mathfrak{g}}) = 0$. We define a *modification* $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ as

$$\hat{\mathfrak{g}} = \mathfrak{g}_\phi / \bar{\mathfrak{g}},$$

which is isomorphic to $\mathfrak{g}$ as Hermitian vector spaces.

We define a new Lie algebra $\mathfrak{g}'$ on $\mathfrak{g}$, called also a *modification* of $\mathfrak{g}$, by

$$[X, Y]' = [X, Y] + \phi(X)Y - \phi(Y)X,$$

under the additional conditions:

$$\phi([\mathfrak{g}, \mathfrak{g}]) = 0, [\phi(X), \phi(Y)] = 0, \phi(\phi(X)Y) = 0$$

for any $X, Y \in \mathfrak{g}$. It is easy to check the bracket $[X, Y]'$ actually defines a new Lie algebra structure on $\mathfrak{g}$. We can see that we have an isomorphism

$$\mathfrak{g}' \cong \mathfrak{g} \rtimes \bar{\mathfrak{g}} / \bar{\mathfrak{g}} = \hat{\mathfrak{g}}$$

mapping $X \in \mathfrak{g}'$ to $(X, \phi(X)) \in \mathfrak{g} \rtimes \bar{\mathfrak{g}}$, and to its projection; and thus we get an isomorphism

$$\mathfrak{g}' \cong \mathfrak{g}$$

preserving the Hermitian structure (h, J).

Example 1. Let $\mathfrak{g}'$ be a Lie algebra with a basis $\{X, Y, Z, W\}$ for which the bracket multiplication is defined by

$$[X, Y] = -Z, [W, X] = -Y, [W, Y] = X,$$

and other brackets vanish. A complex structure J on $\mathfrak{g}'$ is defined by

$$JX = -Y, JY = X, JZ = -W, JW = Z$$

A Hermitian metric h is defined such that X, Y, Z, W is an orthogonal basis. Let $\mathfrak{n}$ be the Heisenberg Lie algebra with a basis $\{X, Y, Z\}$ for which the bracket multiplication is defined by

$$[X, Y] = -Z,$$

and other brackets vanish. We see that $\mathfrak{g}'$ is a modification of $\mathfrak{g} = \mathfrak{n} \times \mathbf{R}$. A linear map $\phi : \mathfrak{g} \to \mathrm{Dev}\,(\mathfrak{g})$ is defined as

$$\phi(X) = \phi(Y) = \phi(Z) = 0, \phi(W) = \mathrm{ad}_W,$$

where ad_W is defined by

$$\mathrm{ad}_W(X) = -Y, \mathrm{ad}_W(Y) = X, \mathrm{ad}_W(Z) = 0, \mathrm{ad}_W(W) = 0.$$

It is clear that ad_W is skew-symmetric with respect to h and compatible with J. Hence, setting $\bar{\mathfrak{g}} =< \mathrm{ad}_W >$, we get a modification $\hat{\mathfrak{g}}$ of $\mathfrak{g}$:

$$\hat{\mathfrak{g}} = \mathfrak{g} \rtimes \bar{\mathfrak{g}}/\bar{\mathfrak{g}}.$$

Since $W \not\in [\mathfrak{g}, \mathfrak{g}]$ and clearly satisfying the additional conditions, it can be identified with $\mathfrak{g}'$ through the map $\psi : \mathfrak{g}' \to \mathfrak{g} \rtimes \bar{\mathfrak{g}} \to \hat{\mathfrak{g}}$ defined by $\psi(X) = \mathrm{pr}(X, \phi(X))$.

We can define a modification of a pair $(\mathfrak{g}, \mathfrak{h})$ of a Hermitian Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ under the additional condition:

$$\phi(\mathfrak{h}) = 0, \phi(X)(\mathfrak{h}) = 0$$

for any $X \in \mathfrak{g}$. We get a modification $(\mathfrak{g}', \mathfrak{h}')$ of $(\mathfrak{g}, \mathfrak{h})$ as

$$\mathfrak{g}' = \mathfrak{g} \rtimes \bar{\mathfrak{g}},\ \mathfrak{h}' = \mathfrak{h} \times \bar{\mathfrak{g}}.$$

Lemma 1. *Let $M = G/H$ be a homogeneous Vasiman manifold, we can modify, if necessary, $\mathfrak{g}/\mathfrak{h}$ into $\mathfrak{g}'/\mathfrak{h}' \cong \mathfrak{g}/\mathfrak{h}$ with $\dim Z(\mathfrak{g}') = 2$ and $\dim \mathfrak{h}' = \dim \mathfrak{h} + 1$.*

Proof. In fact, the set of invariant vector fields can be identified with $(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}$; and since the Lee field ξ and Reeb field $\eta = J\xi$ are invariant they belong to this set. Since ξ and η are Killing and compatible with the complex structure J, they define ad_ξ and ad_η in $\mathrm{Der}(\mathfrak{g})$, which commute with each other and are compatible with J. They are also $\mathrm{ad}\,h$-invariant for $h \in \mathfrak{h}$.

Let $\bar{\mathfrak{g}} =< \mathrm{ad}_\xi >$, and $\hat{\mathfrak{g}} = \mathfrak{g} \rtimes \bar{\mathfrak{g}}, \hat{\mathfrak{h}} = \mathfrak{h} \times \bar{\mathfrak{g}}$. We have $\mathfrak{g}/\mathfrak{h} = \hat{\mathfrak{g}}/\hat{\mathfrak{h}}$, where $\hat{\mathfrak{g}}$ has a central element $\zeta = \xi - \mathrm{ad}_\xi$ in $\hat{\mathfrak{g}}$ which is identified with $\xi \,(\mathrm{mod}\,\hat{\mathfrak{h}})$. Since $\xi \not\in [\mathfrak{g}, \mathfrak{g}]$, we have $\hat{\mathfrak{g}}/\bar{\mathfrak{g}} \cong \mathfrak{g}'$ and $\hat{\mathfrak{h}}/\bar{\mathfrak{g}} \cong \mathfrak{h}' = \mathfrak{h}$ through the map $X \to (X, \phi(X))$. Hence we have

$$\mathfrak{g}/\mathfrak{h} = \mathfrak{g}'/\mathfrak{h}' = \mathfrak{g}'/\mathfrak{h}$$

with $\xi \in Z(\mathfrak{g}')$.

Similarly, we can modify $\mathfrak{g}'/\mathfrak{h}'$ into $\mathfrak{g}''/\mathfrak{h}''$ with $\xi, \eta \in Z(\mathfrak{g}'')$. Note that in case ξ or η is already in $Z(\mathfrak{g})$, ad_ξ or ad_η is trivial; and thus $\mathfrak{g}' = \mathfrak{g} \ \times \bar{\mathfrak{g}}, \mathfrak{h}' = \mathfrak{h} \times \bar{\mathfrak{g}}$ without any modification on $\mathfrak{g}$ and $\mathfrak{h}$. Since for homogeneous Vaisman manifolds $\mathfrak{g}''/\mathfrak{h}''$ the dimension of the center is not greater than 2, the Lee and the Reeb fields generate the center of $\mathfrak{g}''$. Since $\mathfrak{h}' = \mathfrak{h}$, $\dim \mathfrak{h}'' = \dim \mathfrak{h} + 1$. □

We review some basic and historical results on a classification of homogeneous Kähler manifolds (due to Dorfmeister, Nakajima, Vinberg, Gindikin, Piatetskii-Shapiro, Matsushima, Borel, Hano, Shima; see [2], [5], [6], [13] and references therein).

Theorem. *A homogeneous complex Kähler manifold is a holomorphic fiber bundle over a homogeneous bounded domain with fiber a product of a locally flat complex manifold and a flag manifold. In particular, due to Grauert-Oka principle [9], it is biholomorphic to the product of these complex manifolds.*

Let $M = G/K$ be a homogeneous Kähler manifold, where K is a closed subgroup of a simply connected Lie group G. Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of G, K respectively. Then, we can consider a Kähler structure on G/K as a pair (J, ω) of a complex structure $J \in \mathrm{End}(\mathfrak{g})$ and a skew symmetric bilinear form ω on $\mathfrak{g}$, satisfying the following condition:

(i) $J\,\mathfrak{k} \subset \mathfrak{k}$, $J^2 = -I \pmod{\mathfrak{k}}$

(ii) $\mathrm{ad}_X J = J\,\mathrm{ad}_X \pmod{\mathfrak{k}}$ for $X \in \mathfrak{k}$

(iii) $[JX, JY] = [X, Y] + J\,[JX, Y] + J\,[X, JY] \pmod{\mathfrak{k}}$

(iv) $\omega(\mathfrak{k}, \mathfrak{g}) = 0,\ \omega(JX, JY) = \omega(X, , Y)$

(v) $\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0$

(vi) $\omega(JX, X) \neq 0$ for X $\notin \mathfrak{k}$

A *Kähler algebra* $(\mathfrak{g}, \mathfrak{k}, J, \omega)$ is a Lie algebra $\mathfrak{g}$ with subalgebra $\mathfrak{k}$, $J \in \mathrm{End}(\mathfrak{g})$ and a skew symmetric bilinear form ω on $\mathfrak{g}$, satisfying the above condition. A *Kähler algebra* $(\mathfrak{g}, \mathfrak{k}, J, \omega)$ is *effective* if $\mathfrak{k}$ includes no non-trivial ideals of $\mathfrak{g}$. A *J-algebra* is a Kähler algebra $(\mathfrak{g}, \mathfrak{k}, J, \omega)$ with a linear form ρ such that $d\rho = \omega$. Note that the condition $d\rho = \omega$ is often referred to as *non-degenerate*; for a Kähler algebra of effective form, it is actually equivalent to non-degeneracy of the Ricci curvature form $\mathfrak{r}$ of the Kähler structure (due to Nakajima [11]).

A key idea of the proof [5] for the above theorem is to show, applying modifications if necessary, that there exists an abelian ideal $\mathfrak{a}$ and a J-algebra $\mathfrak{f}$ containing $\mathfrak{k}$ such that

$$\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{f}$$

which is a semi-direct sum, and $\mathfrak{g}$ is *quasi-normal*, that is, $\mathrm{ad}(X)$ has only real eigenvalues for any element $X \in \mathrm{rad}(\mathfrak{g})$, where $\mathrm{rad}(\mathfrak{g})$ is the radical of $\mathfrak{g}$. There also exists a compact J-subalgebra $\mathfrak{q}$ of $\mathfrak{f}$ satisfying $\mathfrak{f} \supset \mathfrak{q} \supset \mathfrak{k}$ for which we can express M as a fiber bundle:

$$P/K \to M = G/K \to G/P$$

where $P = AQ$, setting A, Q be the Lie groups associated to $\mathfrak{a}, \mathfrak{q}$ respectively; and $P/K = A/\Gamma \times Q/K_0$ with $K = K_0\Gamma$ for the connected component K_0 of K and a discrete subgroup Γ of A. The base space G/P defines a homogeneous bounded domain, A/Γ a locally flat complex manifold, Q/K_0 a flag manifold, and the fibration is holomorphic.

2 Vaisman Unimodular Lie algebras

A Lie group G is a homogeneous space with its own transitive action on the left. It is a homogeneous l.c.K. manifold if it admits a left-invariant Hermitian structure (g, J) satisfying

$$d\Omega = \Omega \wedge \theta$$

for its associated fundamental form Ω and a closed 1-form θ (Lee form). Note that θ must be also left-invariant. It is clear that G admits a left-invariant l.c.K. structure if and only if its Lie algebra $\mathfrak{g}$ admits an l.c.K. form Ω. We call $\mathfrak{g}$ with an l.c.K. form Ω an l.c.K. Lie algebra.

We have already obtained in our previous papers [7], [1] a classification of l.c.K. reductive Lie algebras and nilpotent Lie algebras, determining at the same time which l.c.K. structures are of Vaisman type. In this section we determine all Vaisman unimodular Lie algebras, up to modifications.

Theorem 1. *A Vaisman unimodular Lie algebra is, up to modification, isomorphic (as Vaisman Lie algbera) to one of the following:*

$$\mathbf{R} \times \mathfrak{n}, \mathbf{R} \times \mathfrak{s}u(2), \mathbf{R} \times \mathfrak{s}l(2, \mathbf{R}),$$

where $\mathfrak{n}$ *is a Heisenberg Lie algebra. In terms of Lie groups, a simply connected Vaisman unimodular Lie group is, up to modification, isomorphic (as Vaisman Lie group) to one of the following:*

$$\mathbf{R} \times N, \mathbf{R} \times SU(2), \mathbf{R} \times \widetilde{SL}(2, \mathbf{R}).$$

Proof. Let $\mathfrak{g}$ be a Vaisman unimodular Lie algebra of dimension $2k+2$ with an l.c.K. form Ω and Lee form θ. Applying modification, if necessary, we can assume that

$$\mathfrak{g} = \mathbf{R} \times \mathfrak{g}_0,$$

where $\mathfrak{g}_0 = \ker \theta$, and $\mathbf{R}$ is generated by the Lee field ξ. $\mathfrak{g}_0$ is a Sasaki Lie algebra with Reeb field η. Let ψ be the contact form and $\mathfrak{k} =< \eta >$, then $(\mathfrak{g}_0, \mathfrak{k}, J|\mathfrak{g}_0, d\psi)$ defines a Kähler algebra. The Koszul form κ is defined by

$$\kappa(X) = \mathrm{Tr}_{\mathfrak{g}_0/\mathfrak{k}}(\mathrm{ad}\, JX - J\mathrm{ad}\, X).$$

Then, the Ricci curvature form $\mathfrak{r}$ of the Kähler structure is given by

$$\mathfrak{r}\,(X, Y) = -\kappa([X, Y]).$$

Now, in case $\dim Z(\mathfrak{g}_0) = 1$, $Z(\mathfrak{g}_0) = \mathfrak{k}$, and $\mathfrak{g}_0/\mathfrak{k}$ is a unimodular Kähler Lie algebra. Then due to Hano, $\mathfrak{g}_0/\mathfrak{k}$ is meta-abelian and locally flat; and thus, up to modification, isomorphic to $\mathbf{C}^n$. Therefore we get $\mathfrak{g}_0 = \mathfrak{n}$, up to modification. In case $\dim Z(\mathfrak{g}_0) = 0$, we see that the Ricci form κ is non-degenerate. In fact, since $\mathrm{ad}\,(\eta)$ is not trivial, $\mathfrak{k}$ is not an ideal of $\mathfrak{g}_0$.; and thus the Kähler algebra $(\mathfrak{g}_0, \mathfrak{k}, d\psi)$ is in effective form. Since the Kähler algebra $(\mathfrak{g}_0, \mathfrak{k}, d\psi)$ is non-degenerate (that is, it defines a J-algebra) the Ricci form $\mathfrak{r}$ is non-degenerate [11]; it follows (due to Hano [6]) that $\mathfrak{g}_0$ must be semi-simple. Then it is well known that $\mathfrak{g}_0$ must be either $\mathfrak{s}u(2)$ or $\mathfrak{s}l(2, \mathbf{R})$. □

Remark 1. A Vaisman unimodular solvable Lie algebra $\mathfrak{g}$ is, up to modification, isomorphic to $\mathbf{R} \times \mathfrak{n}$ (see Example 2.1. for a non-nilpotent case). Since modification ϕ is a skew-symmetric operation, its eigenvalues are all pure-imaginary; in particular, a Vaisman unimodular completely solvable Lie algebra is isomorphich to $\mathbf{R} \times \mathfrak{n}$ [12].

Remark 2. We have determined all homogeneous l.c.K. structures on $\mathbf{R} \times \mathfrak{n}$ and $\mathbf{R} \times \mathfrak{s}u(2)$, which are all of Vaisman type [7]. We have also determined all homogeneous l.c.K. structures on $\mathbf{R} \times \mathfrak{s}l(2, \mathbf{R})$, some of them are of non-Vaisman type, as we will see in the next section.

3 l.c.K. unimodular Lie groups of non-Vaisman type

In this section we see examples of l.c.K. reductive Lie algebras of non-Vaisman type (which we already discussed in our previous papers [8], [1], illustrating how Vaisman and non-Vaisman structures can defined on $\mathbf{R} \times \mathfrak{s}l(2, \mathbf{R})$.

Example 2. There exists a homogeneous l.c.K. structure on $\mathfrak{g} = \mathbf{R} \times \mathfrak{s}l(2, \mathbf{R})$ which is not of Vaisman type. Take a basis $\{X, Y, Z\}$ for $\mathfrak{s}l(2, \mathbf{R})$ with bracket multiplication defined by

$$[X, Y] = -Z,\ [Z, X] = Y,\ [Z, Y] = -X,$$

and T as a generator of the center $\mathbf{R}$ of $\mathfrak{g}$, where we set

$$X = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ Z = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let t, x, y, z, be the Maurer-Cartan forms corresponding to T, X, Y, Z respectively; then we have

$$dt = 0, dx = z \wedge y, dy = x \wedge z, dz = x \wedge y,$$

and an l.c.K. structure $\Omega = z \wedge w + x \wedge y$ compatible with an integrable homogeneous complex structure J on $\mathfrak{g}$ defined by

$$JY = X, JX = -Y, JT = Z, JZ = -T.$$

We can generalize Ω to an l.c.K. structure of the form

$$\Omega_\psi = \psi \wedge w + d\psi$$

compatible with the above complex structure J on $\mathfrak{g}$, where $\psi = ax + by + cz$ with $a, b, c \in \mathbf{R}$.

We see that the symmetric bilinear form $h_\psi(U, V) = \Omega_\psi(JU, V)$ is represented, with respect to the basis $\{T, X, Y, Z\}$, by the matrix

$$A = \begin{pmatrix} c & -b & a & 0 \\ -b & c & 0 & a \\ a & 0 & c & b \\ 0 & a & b & c \end{pmatrix},$$

which has the characteristic polynomial $\Phi_A(u) = \{(u-c)^2 - (a^2+b^2)\}^2$, and has only positive eigenvalues if and only if $c > 0, c^2 > a^2 + b^2$. The Lee form is $\theta = t$ and the Lee field is

$$\xi = \frac{1}{D}(cT + bX - aY),$$

with $D = c^2 - a^2 - b^2$. We have also

$$h_\psi(\xi, \xi) = \frac{c}{D}.$$

We can see that $h_\psi([\xi, U], V) + h_\psi(U, [\xi, V]) \not\equiv 0$ unless $a = b = 0$. In fact for $U = V = Z$,

$$h_\psi([\xi, Z], Z) + h_\psi(Z, [\xi, Z]) = 2h_\psi([\xi, Z], Z) = -\frac{2}{D}(a^2 + b^2) = 0$$

if and only if $a = b = 0$. Conversely for the case $a = b = 0$, it is easy to check that $h_\psi([\xi, U], V) + h_\psi(U, [\xi, V]) \equiv 0$. Therefore we have shown

For J and Ω_ψ defined above, h_ψ defines a (positive definite) l.c.K. metric if and only if $c > 0, c^2 > a^2 + b^2$. It is of Vaisman type if and only if $c > 0$, $a = b = 0$. And it is of non-Vasiman type if and only if $c > 0$, $c^2 > a^2 + b^2 > 0$.

We see that $\mathfrak{g}$ can be modified into $\mathfrak{g} \cong \mathfrak{g}' / < S >$, where $\mathfrak{g}' = \mathbf{R} \times \mathfrak{g}l(2, \mathbf{R})$ for which the basis consists of X, Y, Z and

$$W = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and we set

$$S = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Since we have $W = Z + S \in \mathfrak{g}l(2, \mathbf{R})$, ad_S defines a skew-symmetric action on $\mathfrak{g}$ and $Z = W$ (mod$S$). Hence we get $\mathfrak{g} \cong \mathfrak{g}' / < S >$ as an l.c.K. algebra with the original l.c.K. form Ω, which is of Vaisman type. Note that $\dim_{\mathbf{R}} Z(\mathfrak{g}') = 2$. We see that for $\mathfrak{g}$

with the l.c.K. form Ω_ψ of non-Vaisman type, ad_S is not compatible with the metric h_ψ. In fact for $U = bX - aY$,

$$h_\psi([S,U],Z) + h_\psi(U,[S,Z]) = h_\psi([Z,U],Z) = a^2 + b^2 = 0$$

if and only if $a = b = 0$. Hence we can not modify $\mathfrak{g}$ with the l.c.K. form Ω_ψ of non-Vaisman type into $\mathfrak{g} \cong \mathfrak{g}'/ < S >$ with a compatible Vaisman structure.

4 Homogeneous Vaisman manifolds of unimodular Lie group

In this section we show our main results.

Theorem 2. *A simply connected homogeneous Vaisman manifold M of unimodular Lie group is isomorphic to $\mathbf{R} \times M_1$, where M_1 is a simply connected homogeneous Sasaki manifold of unimodular Lie group, which is a quantization of a simply connected homogeneous Kähler manifold M_2 of reductive Lie group. As a complex manifold M is a holomorphic principal bundle over a simply connected homogeneous Kähler manifold M_2 with fiber $\mathbf{C}^1$ or $\mathbf{C}^*$.*

Hence we have reduced the classification problem of homogeneous Vaisman manifolds of unimodular Lie groups to that of homogeneous Sasaki manifolds. The following theorem may be considered independently as a result on classification of homogeneous Sasaki manifolds of unimodular Lie groups, which extend a known result on compact homogeneous Sasaki manifolds (cf. [3]). Note that a homogeneous Sasaki manifold, and more generally a homogeneous contact manifold is necessarily regular (cf. [4], [8]).

Theorem 3. *A simply connected homogeneous Sasaki manifold M_1 of unimodular Lie group is a quantization of a simply connected homogeneous Kähler manifold M_2 of reductive Lie group; that is, M_1 is a principal bundle over M_2 with fiber $\mathbf{R}$ or S^1 satisfying $d\psi = \omega$ for a contact form ψ on M_1 and the Kähler form ω on M_2.*

The simply connected homogeneous Kähler manifold M_2 is a Kählerian product of $\mathbf{C}^k$, a flag manifold Q/V with a compact semi-simple Lie group Q and a parabolic subgroup V, and a homogeneous Kähler manifold P/U with a non-compact semisimple Lie group P and a closed subgroup U. The homogeneous Kähler manifold P/U has a structure of a holomorphic fiber bundle over a symmetric domain P/L with fiber a flag manifold L/U for a maximal compact subgroup L of P.

Furthermore, M_1 is $\mathbf{R}$-quantization of M_2 if and only if M_2 is a product of $\mathbf{C}^k$ and a symmetric domain P/L with $L = U$, and S^1-quantization of M_2 in all other cases.

We have shown in Lemma 1 that up to modifications, a simply connected homogeneous Vaisman manifold of unimodular Lie group can be assumed to have the form $M = G/H$, where G is a simply connected unimodular Lie group of the form $G = \mathbf{R} \times G_1$ and H is a connected compact subgroup of G_1; and our previous results in [8], [1] yields

Proposition 1. *Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G, H respectively. Then the pair $\{\mathfrak{g}, \mathfrak{h}\}$ is of the following form.*

$$\mathfrak{g} = \mathbf{R} \times \mathfrak{g}_1,$$

where $\mathfrak{g}_1 = \ker \theta \supset \mathfrak{h}$, and $\mathfrak{g}_1$ is a central extension of $\mathfrak{g}_2$:

$$0 \to \mathbf{R} \to \mathfrak{g}_1 \to \mathfrak{g}_2 \to 0.$$

The Lee field ξ and the Reeb field $\eta = J\xi$ generate $Z(\mathfrak{g})$; and the l.c.K. form Ω can be written as

$$\Omega = -\theta \wedge \psi + d\psi,$$

where ψ is the Reeb form defining a contact form on the homogeneous Sasaki manifold G_1/H. Let $\mathfrak{k} = \pi(\mathfrak{h})$ for the projection $\pi : \mathfrak{g}_1 \to \mathfrak{g}_2$. Then the pair $\{\mathfrak{g}_2, \mathfrak{k}\}$ defines a homogeneous Kähler manifold G_2/K with the Kähler form $\omega = d\psi|\mathfrak{g}_2$, where G_1 and K are the Lie groups corresponding to $\mathfrak{g}_1$ and $\mathfrak{k}$ respectively.

We have the following result on homogeneous unimodular Kähler manifolds, which could be of independent interest.

Proposition 2. *A simply connected homogeneous unimodular Kähler manifold $M = G/K$ is, up to isomorphism, of reductive type; that is, the Kähler algebra $\{\mathfrak{g}, \mathfrak{k}\}$ of M has, up to modification, a decomposition*

$$\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{l},$$

where $\mathfrak{a}$ is an abelian Kähler subalgebra, $\mathfrak{l}$ is a semi-simple Kähler subalgebra which contains $\mathfrak{k}$. As a Kähler manifold, M is a product of $\mathbf{C}^k$ and a homogeneous Kähler manifold $N = L/K$ of a semi-simple Lie group L:

$$M = \mathbf{C}^k \times N.$$

Furthermore, N can be decomposed into a Kähler product of flag manifolds and non-compact homogeneous Kähler manifolds each of which is a holomorphic fiber bundle over a symmetric domain with fiber a flag manifold.

Proof. Let $M = G/K$ be a simply connected homogeneous Kähler manifold with unimodular Lie group G and a closed subgroup K of G. We have a decomposition

$$\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{f},$$

where $\mathfrak{a}$ is a maximal abelian J-ideal of $\mathfrak{g}$ isomorphic to $\mathbf{C}^k$ and $\mathfrak{f}$ is a J-subalgebra which contains $\mathfrak{k}$. Moreover, due to [13], $\mathfrak{f}$ decomposes into a product of a solvable J-subalgebra $\mathfrak{s}$, a reductive J-subalgebra $\mathfrak{q}$,

$$\mathfrak{f} = \mathfrak{s} \times \mathfrak{q},$$

where $\mathfrak{q}$ contains $\mathfrak{k}$, and the center of $\mathfrak{q}$ is contained in $\mathfrak{k}$. It is also known that $\mathfrak{s}$ corresponds to a homogeneous domain, and applying the De Rham decomposition of homogeneous Kähler manifolds (cf. [10]) we see that $\mathfrak{s}$ is actually the radical of $\mathfrak{f}$, which is a maximal solvable ideal of $\mathfrak{f}$. We see also that $\mathfrak{a} \rtimes \mathfrak{s}$ is the radical of $\mathfrak{g}$. Since $\mathfrak{g}$ is by assumption a unimodular Lie algebra, so is $\mathfrak{a} \rtimes \mathfrak{s}$. It follows, due to Hano [6] that $\mathfrak{s}$ must be trivial. Since the center of $\mathfrak{q}$ is contained in $\mathfrak{k}$, we may express $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{l},$$

where $\mathfrak{l}$ is the semi-simple part of $\mathfrak{q}$ and $\mathfrak{k}$ is contained in $\mathfrak{l}$. Since M is by assumption simply connected, $\mathfrak{a}$ corresponds to $\mathbf{C}^k$ as a flat Kähler manifold, and thus the action of L (the Lie group corresponding to $\mathfrak{l}$) on $\mathbf{C}^k$ is isometric and holomorphic. Thus as a Kähler manifold M is isomorphic to $\mathbf{C}^k \times L/K$ (cf. [5]), where L/K is a product of homogeneous Kähler manifolds of compact semi-simple Lie groups and homogeneous Kähler manifolds of non-compact semi-simple Lie groups each of which is a holomorphic fiber bundle over a symmetric domain with fiber a flag manifold (c.f. [2]). □

Let $\mathfrak{g}_1$ be the Sasaki algebra with the Reeb field η and the Kähler form ω in Proposition 2. Then, the Lie bracket on $\mathfrak{g}_1$ is the extension of $\mathfrak{g}_2$ given by

$$[X, Y]_{\mathfrak{g}_1} = [X, Y]_{\mathfrak{g}_2} - \omega(X, Y)\eta, \ [\eta, Z]_{\mathfrak{g}_2} = 0$$

for $X, Y, Z \in \mathfrak{g}_2$. Conversely, given a Kähler algebra $\{\mathfrak{g}_2, \mathfrak{k}\}$ with a Kähler form ω we can define a Sasaki Lie algebra $\mathfrak{g}_1$, which is a central extension with a generator η of $\mathbf{R}$ by the above formula. Since η is Killing, $\mathfrak{g}_1$ is unimodular if and only if $\mathfrak{g}_2$ is unimodular. Hence $M_1 = G_1/H$ is of unimodular type if and only if $M_2 = G_2/K$ is of the same type.

Now we study a quantization of a homogeneous Kähler manifold $M_2 = G_2/K$ of reductive type. In case $M_2 = \mathbf{C}^k$, its quantization is the Heisenberg Lie group N, which is a central extension of $\mathbf{R}$ by $\mathbf{C}^k$. In case $M_2 = L/K$ is a flag manifold, where L is a compact semi-simple Lie group, since M_2 is a Hodge manifold it is quantizable to a compact homogeneous Sasaki manifold with fiber S^1. In case L is a non-compact semi-simple Lie group, M_2 is a holomorphic fiber bundle over a symmetric domain L/B with fiber a flag manifold B/K, where B is a maximal compact Lie subgroup of L containing K. Since the flag manifold B/K is a Kähler submanifold of $M_2 = G/K$ and S^1-quatizable, M_2 itself must be S^1-quantizable. In general cases, for two or more homogeneous Kähler manifolds each of which is quantizable, we construct naturally a quantization of their products in the following way. For two Kähler algebras $\mathfrak{g}_2$ and $\mathfrak{g}'_2$ with their central extension $\mathfrak{g}_1$ and $\mathfrak{g}'_1$ respectively, we can define a new central $\mathbf{R}$-extension of $\mathfrak{g}_2 \times \mathfrak{g}'_2$ by taking $\mathbf{R} \times \mathbf{R}/\Delta \cong \mathbf{R}$ with $\Delta = \{(X, -X) | X \in \mathbf{R}\}$::

$$0 \to \mathbf{R} \to \mathfrak{g}_1 \times_\Delta \mathfrak{g}'_1 \to \mathfrak{g}_2 \times \mathfrak{g}'_2 \to 0,$$

where $\mathfrak{g}_1 \times_\Delta \mathfrak{g}'_1 = (\mathfrak{g}_1 \times \mathfrak{g}'_1)/\Delta$, the quotient Lie algebra by the canonical action of Δ on $\mathfrak{g}_1 \times \mathfrak{g}'_1$. Correspondingly, we obtain a quantization $G_1 \times_\Delta G'_1$ of $G_2 \times G'_2$; and in general the quantization $G_1/H \times_\Delta G'_1/H'$ of $G_2/K \times G'_2/K'$. Now, in case M_2 is a product of $\mathbf{C}^k$ and a symmetric domain, since M_2 is contractible, it must be $\mathbf{R}$-quantizable. In all other cases, as we have seen in the above for the case in which L is a non-compact semi-simple Lie group and K is not a maximal compact subgroup, M_2 must be S^1-quatizable.

We have thus obtained the following result, which could be of independent interest.

Proposition 3. *A simply connected homogeneous Kähler manifold of reductive Lie group is $\mathbf{R}^1$-quatizable or S^1-quantizable to a simply connected homogeneous Sasaki manifold, according to whether it is a product of $\mathbf{C}^k$ and a symmetric domain, or not. In other words it is $\mathbf{R}^1$-quatizable exactly when it contains no flag manifolds.*

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