New covering codes of radius R, codimension tR and $tR+\frac{R}{2}$, and saturating sets in projective spaces

Alexander A. Davydov*

Institute for Information Transmission Problems (Kharkevich institute), Russian Academy of Sciences

Bol'shoi Karetnyi per. 19, Moscow, 127051, Russian Federation. E-mail: adav@iitp.ru

Stefano Marcugini† and Fernanda Pambianco†

Dipartimento di Matematica e Informatica, Università degli Studi di Perugia,

Via Vanvitelli 1, Perugia, 06123, Italy. E-mail: {stefano.marcugini,fernanda.pambianco}@unipg.it

Abstract. The length function $\ell_q(r,R)$ is the smallest length of a q-ary linear code of codimension r and covering radius R. In this work we obtain new constructive upper bounds on $\ell_q(r,R)$ for all $R\geq 4$, $r=tR$, $t\geq 2$, and also for all even $R\geq 2$, $r=tR+\frac{R}{2}$, $t\geq 1$. The new bounds are provided by infinite families of new covering codes with fixed R and increasing codimension. The new bounds improve upon the known ones.

We propose a general regular construction (called "Line+Ovals") of a minimal ρ-saturating $((\rho+1)q+1)$-set in the projective space $\mathrm{PG}(2\rho+1,q)$ for all $\rho\geq 0$. Such a set corresponds to an $[Rq+1,Rq+1-2R,3]_qR$ locally optimal[1] code of covering radius $R=\rho+1$. Basing on combinatorial properties of these codes regarding to spherical capsules, we give constructions for code codimension lifting and obtain infinite families of new surface-covering[1] codes with codimension $r=tR$, $t\geq 2$.

In addition, we obtain new 1-saturating sets in the projective plane $\mathrm{PG}(2,q^2)$ and, basing on them, construct infinite code families with fixed even radius $R\geq 2$ and codimension $r=tR+\frac{R}{2}$, $t\geq 1$.

Keywords: Covering codes, saturating sets, the length function, upper bounds, projective spaces.

Mathematics Subject Classification (2010). 51E21, 51E22, 94B05

---

*The research of A.A. Davydov was done at IITP RAS and supported by the Russian Government (Contract No 14.W03.31.0019).

†The research of S. Marcugini and F. Pambianco was supported in part by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INDAM) and by University of Perugia, (Project: "Strutture Geometriche, Combinatoria e loro Applicazioni", Base Research Fund 2017).

[1]See the definitions in Sect. 1.

1 Introduction

Let $\mathbb{F}_q$ be the Galois field with q elements, $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Let $\mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$. Denote by $[n, n-r]_q$ a q-ary linear code of length n and codimension (redundancy) r, that is a subspace of $\mathbb{F}_q^n$ of dimension $n-r$.

Let $d(v,c)$ be the Hamming distance between vectors v and c of $\mathbb{F}_q^n$. The *sphere of radius R* with center c in $\mathbb{F}_q^n$ is the set $\{v : v \in \mathbb{F}_q^n,\ d(v,c) \le R\}$. For $0 \le \ell \le R$, a *spherical (R,ℓ)-capsule* with center c in $\mathbb{F}_q^n$ is the set $\{v : v \in \mathbb{F}_q^n,\ \ell \le d(v,c) \le R\}$ [10, Rem. 5], [12, Rem. 2.1], [16, Sect. 2]. An (R,R)*-capsule is the surface of a sphere of radius R.*

Definition 1. A linear $[n, n-r]_q$ code has *covering radius R* and is denoted as an $[n, n-r]_q R$ code if any of the following equivalent properties holds:

(i) The value R is the least integer such that the space $\mathbb{F}_q^n$ is covered by the spheres of radius R centered at the codewords.

(ii) Every column of $\mathbb{F}_q^r$ is equal to a linear combination of at most R columns of a parity check matrix of the code, and R is the smallest value with this property.

An $[n, n-r]_q R$ code of minimum distance d is denoted by $[n, n-r, d]_q R$ code. For an introduction to coverings of Hamming spaces, see [6, 8]. For fixed q, r, and R, the covering quality of an $[n, n-r]_q R$ code is better if its length n is smaller.

Definition 2. [6, 8] *The length function $\ell_q(r,R)$* is the smallest length of a q-ary linear code of codimension r and covering radius R.

It can be shown, see e.g. [2, 16], that if code length n is considerably larger than R (this is the natural case in covering codes investigations) and if q is large enough, then there is a lower bound of the form $\ell_q(r,R) \gtrsim cq^{(r-R)/R}$, where c is independent of q but it is possible that c depends on r and R.

Let t, s, R^* be integers. Let q' be a prime power. Consider the following cases:

$$\textbf{(i)}\ r = tR,\ \text{arbitrary } q.\quad \textbf{(ii)}\ R = sR^*,\ r = tR + s,\ q = (q')^{R^*}.\quad \textbf{(iii)}\ r \ne tR,\ q = (q')^R. \tag{1.1}$$

In [13, 15, 16, 19], for all the cases in (1.1), codes with lengths close (by order) to the bound $\ell_q(r,R) \gtrsim cq^{(r-R)/R}$ are obtained. These lengths are upper bounds on $\ell_q(r,R)$.

The goal of this paper is to improve on the known upper bounds on $\ell_q(r,R)$ in the case (i) of (1.1) for $R \ge 4$ and in the case (ii) of (1.1) for even R with $R^* = 2$.

The following properties of codes are useful for obtaining new bounds.

Definition 3. [14] A linear covering code is called *locally optimal* if one cannot remove any column from its parity check matrix without an increase in covering radius.

Definition 4. [10], [12, Sect. 2], [16, Sect, 2] Let $0 \le \ell \le R$. An $[n, n-r]_q R$ code is called an (R,ℓ)*-object* and is denoted by $[n, n-r]_q R, \ell$ code if any of the following equivalent conditions holds:

(i) The space $\mathbb{F}_q^n$ is covered by the spherical (R,ℓ)-capsules centered at the codewords.

(ii) Every column of the space $\mathbb{F}_q^r$ (including the zero column) is equal to a linear combination with *nonzero coefficients* of at least ℓ and at most R distinct columns of a parity-check matrix of the code.

(iii) Every coset of the code (including the code itself) contains a weight w word of the space $\mathbb{F}_q^n$ such that $\ell \le w \le R$.

Definition 5. An $[n,n-r]_q R,R$ code is called *surface-covering code* of radius R.

Note that the space $\mathbb{F}_q^n$ is covered by *the surfaces* of the spheres of radius R centered at the codewords of an $[n,n-r]_q R,R$ surface-covering code.

Codes with radius $R=2,3$ and codimension $r=tR$ have been widely investigated, see [11–16, 18–20] and the references therein. At the same time, codes with $R\ge 4$, $r=tR$, have not been extensively studied. The main known results for codes with $R\ge 4$, $r=tR$, are available in [15, 16, 19] and collected in Proposition 1.

Proposition 1. [15], [16, Ths. 6.1,6.2, eqs. 6.1,6.2], [19] *The following constructive upper bounds on the length function hold:*

$$\ell_q(r,R) \le Rq^{(r-R)/R} + \left\lceil \frac{R}{3} \right\rceil q^{(r-2R)/R} + \delta_q(r,R),\ R\ge 4,\ r=tR,\ t\ge 2, \tag{1.2}$$

where $\delta_q(r,R)=0$ *if* $q\ge 4, r=2R$, *or* $q=16, q\ge 23, r=3R$, *or* $q\ge 7, q\ne 9, r\ge 5R, r\ne 6R$. *Also,* $\delta_q(r,R)=(2R \bmod 3)\cdot(q^{(r-3R)/R}+1)$ *if* $q\ge 7, q\ne 9, r=4R,6R$.

The main known results for codes with even covering radius $R\ge 2$ and codimension $r=tR+\frac{R}{2}$ are available in [13, 15, 16] and collected in Proposition 2.

Proposition 2. [13, Ex. 6, eq. (33)], [15], [16, Sects. 4.4, 7] *Let q' be a prime power. Let the covering radius $R\ge 2$ be even. Let the code codimension be $r=tR+\frac{R}{2}$ with integer t. The following constructive upper bounds on the length function hold:*

$$\ell_q(r,R) \le \frac{R}{2}\left(3-\frac{1}{\sqrt{q}}\right) q^{\frac{r-R}{R}} + \frac{R}{2}\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor,\ q=(q')^2\ge 16,\ t\ge 1; \tag{1.3}$$

$$\ell_q(r,R) \le R\left(1+\frac{1}{\sqrt[4]{q}}+\frac{1}{\sqrt{q}}\right) q^{\frac{r-R}{R}} + \frac{R}{2}\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor,\ q=(q')^4,\ t\ge 1; \tag{1.4}$$

$$\ell_q(r,R) \le R\left(1+\frac{1}{\sqrt[6]{q}}+\frac{1}{\sqrt[3]{q}}+\frac{1}{\sqrt{q}}\right) q^{(r-R)/R} + R\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor,\ q=(q')^6, \tag{1.5}$$

$q'\le 73$ *prime*, $t\ge 1$, $t\ne 4,6$.

Problem 1. *Improve on the known bounds on the length function $\ell_q(r,R)$ collected in*

(i) *Proposition 1 where $R\ge 4$, $r=tR$, $t\ge 2$;*

(ii) *Proposition 2 where $R\ge 2$, $r=tR+\frac{R}{2}$, $t\ge 1$.*

Effective methods to obtain upper bounds on $\ell_q(r,R)$ are connected with *saturating sets in projective spaces*. Let $\mathrm{PG}(N,q)$ be the N-dimensional projective space over the field $\mathbb{F}_q$; see [24–26] for an introduction to the projective spaces and [21, 23, 25, 29, 30] for connections between coding theory and Galois geometries.

Definition 6. A point set $S \subseteq \mathrm{PG}(N,q)$ is ρ-*saturating* if any of the following equivalent properties holds:

(i) For any point A of $\mathrm{PG}(N,q) \setminus S$ there exist $\rho+1$ points in S generating a subspace of $\mathrm{PG}(N,q)$ containing A, and ρ is the smallest value with this property.

(ii) Every point $A \in \mathrm{PG}(N,q)$ (in homogeneous coordinates) can be written as a linear combination of at most $\rho+1$ points of S, and ρ is the smallest value with this property (cf. Definition 1(ii)).

Definition 7. A ρ-saturating set in $\mathrm{PG}(N,q)$ is *minimal* if it does not contain a smaller ρ-saturating set in $\mathrm{PG}(N,q)$.

Saturating sets are considered in [2, 6, 7, 12, 14–19, 21, 23, 27, 29, 30, 33]. In the literature, saturating sets are also called "saturated sets", "spanning sets", "dense sets".

Let $s_q(N,\rho)$ be *the smallest size of a ρ-saturating set* in $\mathrm{PG}(N,q)$.

If a column of an $r \times n$ parity check matrix of an $[n,n-r]_qR$ code is treated as a point (in homogeneous coordinates) of $\mathrm{PG}(r-1,q)$ then this parity check matrix defines an $(R-1)$-saturating n-set in $\mathrm{PG}(r-1,q)$ [7, 12, 16, 18, 21, 23, 27, 29, 30]. There is a *one-to-one correspondence between $[n,n-r]_qR$ codes and $(R-1)$-saturating n-sets in* $\mathrm{PG}(r-1,q)$. Therefore, $\ell_q(r,R) = s_q(r-1,R-1)$. If the $[n,n-r]_qR$ code is locally optimal then the corresponding $(R-1)$-saturating n-set is minimal.

The results of Proposition 1 are based on the so-called direct sum [16, Sect. 4.2] of codes with radius $R=2,3$ which use the following geometrical constructions:

- "oval plus line" [7, p. 104], [11, Th. 3.1], [12, Th. 5.1]; the construction gives a 1-saturating $(2q+1)$-set in $\mathrm{PG}(3,q)$ corresponding to a $[2q+1, 2q+1-4, 3]_q2$ code with $r=4=2R$;
- "two ovals plus line" [18, Sect. 4]; the construction gives a 2-saturating $(3q+1)$-set in $\mathrm{PG}(5,q)$ that corresponds to a $[3q+1, 3q+1-6, 3]_q3$ code with $r=6=2R$.

Problem 2. *For all $\rho \geq 3$, obtain a construction of a ρ-saturating $((\rho+1)q+1)$-set in* $\mathrm{PG}(2\rho+1,q)$ *that corresponds to an $[Rq+1, Rq+1-2R]_qR$ code with $R=\rho+1$; thereby prove that $s_q(2\rho+1,\rho) \leq (\rho+1)q+1$ and $\ell_q(2R,R) \leq Rq+1$.*

Note that for $n < Rq+1$, no examples of $[n,n-2R]_qR$ codes seem to be known. Moreover, in [16, Prop. 4.2], it is proved that $\ell_4(4,2) = s_4(3,1) = 2 \cdot 4 + 1$.

Problem 3. [16, Sects. 4, 5] *Determine whether $\ell_q(2R,R) = Rq+1$.*

The results of Proposition 2 are based on 1-saturating sets in the plane $\mathrm{PG}(2,q^2)$.

Problem 4. *In the projective plane* $\mathrm{PG}(2,q)$ *with q square, construct new* 1*-saturating sets with sizes smaller than the known ones.*

The paper is organized as follows. In Sect. 2, we summarize the main results of the paper. In Sect. 3, we propose a construction "Line+Ovals" for ρ-saturating sets in $\mathrm{PG}(2\rho+1,q)$ and codes of codimension $2R$. This solves Problem 2. In Sect. 4, we give two constructions for code codimension lifting. In Sect. 5, we use the codes of Sect. 3 as starting ones for the constructions of Sect. 4 and obtain new infinite code families with fixed radius $R \geq 4$ and codimension tR, $t \geq 2$. This solves Problem 1(i) for the most part. In Sect. 6, using the recent known results on double blocking sets, we obtain new 1-saturating sets in $\mathrm{PG}(2,q^2)$ that solves in part Problem 4. Then starting from these sets, we obtain new infinite code families with fixed even radii $R \geq 2$ and codimension $tR+\frac{R}{2}$, $t \geq 1$. This solves in part Problem 1(ii).

2 The main results

The main results of this paper are as follows:

• Problem 2 is solved, see Sect. 3 where minimal ρ-saturating $((\rho+1)q+1)$-sets in $\mathrm{PG}(2\rho+1,q)$ are constructed. The minimality of these sets gives credence that Problem 3 can be solved.

• Problem 1(i) is solved for the most part, see Sects. 4 and 5. New constructive upper bounds based on Theorems 3, 4, 7, 8 are collected in Theorem 1.

Theorem 1. *For the length function* $\ell_q(r,R)$ *and for the smallest size* $s_q(r-1,R-1)$ *of an* $(R-1)$*-saturating set in* $\mathrm{PG}(r-1,q)$ *the following constructive bounds hold:*

$$\ell_q(r,R) = s_q(r-1,R-1) \leq Rq^{(r-R)/R} + q^{(r-2R)/R} + \Delta_q(r,R),\ r = tR,$$

where for $m_1 = \lceil \log_q(R+1) \rceil + 1$ *we have*

(i) $\Delta_q(r,R) = 0$ *if* $t=2,\ q=4$ *and* $q \geq 7,\ R \geq 4$;

(ii) $\Delta_q(r,R) = 0$ *if* $t=2,\ q=5,\ R=4,5$;

(iii) $\Delta_q(r,R) = 0$ *if* $t \geq \lceil \log_q R \rceil + 3,\ q \geq 7$ *odd*, $R \geq 4$;

(iv) $\Delta_q(r,R) = \sum_{j=2}^{t} q^{(r-jR)/R}$ *if* $m_1+2 < t < 3m_1+2,\ q \geq 8$ *even*, $R \geq 4$;

(v) $\Delta_q(r,R) = \sum_{j=2}^{m_1+2} q^{(r-jR)/R}$ *if* $t = m_1+2$ *and* $t \geq 3m_1+2,\ q \geq 8$ *even*, $R \geq 4$.

The new bounds of Theorem 1 are better than the known ones of Proposition 1 where the coefficient for $q^{(r-2R)/R}$ is $\lceil \frac{R}{3} \rceil$ whereas in Theorem 1 it is equal to 1 or 2.

• Problem 4 is solved in part, see Sect. 6. We use the following notation:

$$\phi(q) \text{ is the order of the largest proper subfield of } \mathbb{F}_q; \tag{2.1}$$

$$f_q(r,R)=\begin{cases} 0 & \text{if} \quad r\neq \frac{9R}{2},\frac{13R}{2} \\ q^{(r-3R)/R-0.5}+q^{(r-4R)/R-0.5} & \text{if} \quad r=\frac{9R}{2},\frac{13R}{2} \end{cases}. \tag{2.2}$$

By Proposition 9(v),(vi), in PG$(2,q)$, $q=p^{2h}$, $h\geq 2$, there are 1-saturating n-sets with

$$n=2\sqrt{q}+2\frac{\sqrt{q}-1}{\phi(\sqrt{q})-1},\ p\geq 3 \text{ prime};\ \ n=2\sqrt{q}+2\frac{\sqrt{q}}{p}+2,\ p\geq 7 \text{ prime}.$$

These new 1-saturating sets have smaller sizes than the known ones, see Remark 3.

• Problem 1(ii) is solved in part. New bounds based on Theorem 10 are as follows.

Theorem 2. *Let $R\geq 2$ be even. Let p be prime, $q=p^{2\eta}$, $\eta\geq 2$, $r=tR+\frac{R}{2}$, $t\geq 1$. The following constructive upper bounds on the length function hold:*

$$\textbf{(i)}\ \ell_q(r,R)\leq R\left(1+\frac{\sqrt{q}-1}{\sqrt{q}(\phi(\sqrt{q})-1)}\right)q^{\frac{r-R}{R}}+R\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor+\frac{R}{2}f_q(r,R),\ p\geq 3;$$

$$\textbf{(ii)}\ \ell_q(r,R)\leq R\left(1+\frac{1}{p}+\frac{1}{\sqrt{q}}\right)q^{(r-R)/R}+R\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor+\frac{R}{2}f_q(r,R),\ p\geq 7.$$

If $\sqrt{q}=p^{\eta}$ with $\eta\geq 3$ odd, the new bounds of Theorem 2 are better than the known ones of Proposition 2. If e.g. $q=p^6$, $\eta=3$, then the bound of Theorem 2(ii) is by $Rq^{(r-R)/R-1/3}$ smaller than the known one of (1.5). Also, the new bound holds for all $p\geq 7$ whereas in (1.5) $p\leq 73$. Moreover, if $\eta\geq 5$ odd, the known bounds (1.3) have the main term $\frac{3}{2}Rq^{(r-R)/R}$ whereas for the new bounds it is $Rq^{(r-R)/R}$.

3 Construction "Line+Ovals" for ρ-saturating sets in PG$(2\rho+1,q)$ and codes of codimension $2R$

Notation. Throughout the paper we denote by x_i, $i=0,1,\dots,N$, homogeneous coordinates of points of $PG(N,q)$. In the other words, a point $(x_0x_1\dots x_N)\in$ PG(N,q). The leftmost nonzero coordinate is equal to 1. In general, by default, $x_i\in\mathbb{F}_q$. If $x_i\in\mathbb{F}_q^*$, we denote it as $\widehat{x_i}$. If $(x_i\dots x_{i+m})\neq(0\dots 0)$, we denote it as $\overline{x_i\dots x_{i+m}}$. Also, we write explicit values 0,1 for some coordinates or denote coordinates by the letters a,a_j that are elements of $\mathbb{F}_q$.

3.1 The construction

Let $\mathbb{F}_q=\{a_1=0,a_2,\dots,a_q\}$ be the Galois field of order q. Let $\mathbb{F}_q^*=\mathbb{F}_q\setminus\{0\}=\{a_2,\dots,a_q\}$. Denote $\Sigma_\rho=$ PG$(2\rho+1,q)$. Let Σ_u be the $(2u+1)$-dimensional projective subspace of Σ_ρ such that

$$\Sigma_u = \{(\underbrace{x_0x_1\ldots x_{2u+1}}_{2u+2}\underbrace{0\ldots0}_{2\rho-2u}) : x_i \in \mathbb{F}_q\} \subseteq \Sigma_\rho,\ u = 0,1,\ldots,\rho.$$

In Σ_u, let π_u be the plane such that

$$\pi_u = \{(\underbrace{0\ldots0}_{2u-1}x_{2u-1}x_{2u}x_{2u+1}\underbrace{0\ldots0}_{2\rho-2u}) : x_i \in \mathbb{F}_q\} \subset \Sigma_u,\ u = 1,2,\ldots,\rho.$$

In π_u, let A_u^0 and A_u^∞ be the points of the form

$$A_u^0 = (\underbrace{0\ldots0}_{2u-1}100\underbrace{0\ldots0}_{2\rho-2u}) \in \pi_u,\ \ A_u^\infty = (\underbrace{0\ldots0}_{2u-1}001\underbrace{0\ldots0}_{2\rho-2u}) \in \pi_u,\ u = 1,2,\ldots,\rho.$$

In π_u, let C_u and C_u^* be the conic and the truncated one, respectively, of the form

$$C_u = C_u^* \cup \{A_u^0, A_u^\infty\},\ \ C_u^* = \{(\underbrace{0\ldots0}_{2u-1}1aa^2\underbrace{0\ldots0}_{2\rho-2u}) : a \in \mathbb{F}_q^*\},\ u = 1,2,\ldots,\rho.$$

Let T_u be the nucleus of C_u, if q is even, or the intersection of the tangents to C_u in the points A_u^0 and A_u^∞, if q is odd, so that $T_u = (\underbrace{0\ldots0}_{2u-1}010\underbrace{0\ldots0}_{2\rho-2u}) \in \pi_u,\ u = 1,2,\ldots,\rho$.

In Σ_0, let A_0^0 and A_0^∞ be the points of the form $A_0^0 = (10\underbrace{0\ldots0}_{2\rho})$, $A_0^\infty = (01\underbrace{0\ldots0}_{2\rho})$. Also, let L_0 and L_0^* be the line and the truncated one, respectively, such that

$$L_0 = L_0^* \cup \{A_0^0, A_0^\infty\} \subset \Sigma_0,\ L_0^* = \{(1a\underbrace{0\ldots0}_{2\rho}) : a \in \mathbb{F}_q^*\} \subset \Sigma_0.$$

Note that by Definition 6, a 0-saturating set in $PG(N,q)$ is the whole space.

Construction S. ("Line+Ovals") Let $\rho \geq 0$. Let $S_\rho = \{P_1, P_2, \ldots, P_{(\rho+1)q+1}\}$ be a point $((\rho+1)q+1)$-subset of $\Sigma_\rho = \mathrm{PG}(2\rho+1,q)$. Let P_j be the j-th point of S_ρ. We construct S_ρ as follows:

$$S_0 = \{A_0^0\} \cup L_0^* \cup \{A_0^\infty\} = \{P_1, P_2, \ldots, P_{q+1}\} = \Sigma_0 = \mathrm{PG}(1,q); \tag{3.1}$$

$$S_\rho = \{A_0^0\} \cup L_0^* \cup \bigcup_{u=1}^{\rho} (C_u^* \cup \{T_u\}) \cup \{A_\rho^\infty\} = \{P_1, P_2, \ldots, P_{(\rho+1)q+1}\} \subset \Sigma_\rho \text{ if } \rho \geq 1.$$

$$P_1 = (10\underbrace{0\ldots0}_{2\rho}) = A_0^0;\ \ P_j = (1a_j\underbrace{0\ldots0}_{2\rho}),\ a_j \in \mathbb{F}_q^*,\ \ j = 2,3,\ldots,q. \tag{3.2}$$

$$P_{uq+j-1} = (\underbrace{0\ldots0}_{2u-1}1a_ja_j^2\underbrace{0\ldots0}_{2\rho-2u}),\ a_j \in \mathbb{F}_q^*,\ \ u = 1,2,\ldots,\rho,\ j = 2,3,\ldots,q. \tag{3.3}$$

$$P_{(u+1)q} = (\underbrace{0\dots0}_{2u-1}0100\underbrace{0\dots0}_{2\rho-2u}) = T_u,\ u=1,2,\dots,\rho;\ \ P_{(\rho+1)q+1} = A_\rho^\infty. \tag{3.4}$$

Also, the set S_ρ can be represented in the matrix form $\widehat{\mathbf{H}}_\rho$, where every column is a point in homogeneous coordinates. We have

$$S_\rho = \widehat{\mathbf{H}}_\rho \tag{3.5}$$

$$= \left[\begin{array}{c||c||c|c||c|c||c||c|c||c|c||c}
1 & 1\dots1 & 0\dots0 & 0 & 0\dots0 & 0 & \dots & 0\ \dots\ 0 & 0 & 0\dots0 & 0 & 0\\
0 & a_2\dots a_q & 1\dots1 & 0 & 0\dots0 & 0 & \dots & 0\ \dots\ 0 & 0 & 0\dots0 & 0 & 0\\
0 & 0\dots0 & a_2\dots a_q & 1 & 0\dots0 & 0 & \dots & 0\ \dots\ 0 & 0 & 0\dots0 & 0 & 0\\
0 & 0\dots0 & a_2^2\dots a_q^2 & 0 & 1\dots1 & 0 & \dots & 0\ \dots\ 0 & 0 & 0\dots0 & 0 & 0\\
0 & 0\dots0 & 0\dots0 & 0 & a_2\dots a_q & 1 & \dots & 0\ \dots\ 0 & 0 & 0\dots0 & 0 & 0\\
0 & 0\dots0 & 0\dots0 & 0 & a_2^2\dots a_q^2 & 0 & \dots & 0\ \dots\ 0 & 0 & 0\dots0 & 0 & 0\\
 & \dots & \dots & & \dots & & \dots & \dots & & \dots & & \\
0 & 0\dots0 & 0\dots0 & 0 & 0\dots0 & 0 & \dots & 1\ \dots\ 1 & 0 & 0\dots0 & 0 & 0\\
0 & 0\dots0 & 0\dots0 & 0 & 0\dots0 & 0 & \dots & a_2\ \dots\ a_q & 1 & 0\dots0 & 0 & 0\\
0 & 0\dots0 & 0\dots0 & 0 & 0\dots0 & 0 & \dots & a_2^2\ \dots\ a_q^2 & 0 & 1\dots1 & 0 & 0\\
0 & 0\dots0 & 0\dots0 & 0 & 0\dots0 & 0 & \dots & 0\ \dots\ 0 & 0 & a_2\dots a_q & 1 & 0\\
0 & 0\dots0 & 0\dots0 & 0 & 0\dots0 & 0 & \dots & 0\ \dots\ 0 & 0 & a_2^2\dots a_q^2 & 0 & 1\\
- & - - - & - - - & - & - - - & - & - & - - - & - & - - - & - & -\\
A_0^0 & L_0^* & C_1^* & T_1 & C_2^* & T_2 & \dots & C_{\rho-1}^* & T_{\rho-1} & C_\rho^* & T_\rho & A_\rho^\infty
\end{array}\right].$$

Remark 1. The sets S_1 and S_2 of Construction S are, respectively, the 1-saturating set in $\mathrm{PG}(3,q)$ of the construction "oval plus line" [7, p. 104], [11, Th. 3.1], [12, Th. 5.1] and the 2-saturating set in $\mathrm{PG}(5,q)$ of the construction "two ovals plus line" [18, Sect. 4].

3.2 Saturation of Construction S

We say that a point $A \in \mathrm{PG}(N,q)$ is ρ*-covered* by a set $S \subseteq \mathrm{PG}(N,q)$ if A is a linear combination of less than or equal to $\rho+1$ points of S. A subset $G \subset \mathrm{PG}(N,q)$ is ρ*-covered* by S if all points of G are ρ-covered by S.

Definition 8. Let S be a ρ-saturating set in $\mathrm{PG}(N,q)$. A point $A \in S$ is ρ*-essential* if $S \setminus \{A\}$ is no longer a ρ-saturating set. A point $A \in S$ is ρ*-essential* for a set $\widetilde{\mathscr{M}_\rho}(A) \subset \mathrm{PG}(N,q)$ if all points of $\widetilde{\mathscr{M}_\rho}(A)$ are not ρ-covered by $S \setminus \{A\}$. We denote by $\mathscr{M}_\rho(A)$ a set such that $\widetilde{\mathscr{M}_\rho}(A) \subseteq \mathscr{M}_\rho(A) \subset \mathrm{PG}(N,q)$.

The following proposition and lemma are obvious.

Proposition 3. *Let* $q \geq 3$. *Let* $\Sigma_0 = \mathrm{PG}(1,q)$. *Let the set* $S_0 = \{A_0^0\} \cup L_0^* \cup \{A_0^\infty\} \subset \Sigma_0$ *be as in* (3.1)–(3.5). *Then it holds that*

(i) *The* $(q+1)$*-set* S_0 *is a minimal* 0*-saturating set in* Σ_0.

(ii) *The point* A_0^∞ *of* S_0 *is* 0*-essential for the set* $\widetilde{\mathscr{M}_0}(A_0^\infty)$ *such that*

$$\widetilde{\mathscr{M}_0}(A_0^\infty) = \mathscr{M}_0(A_0^\infty) = \{A_0^\infty\} = \{(01)\}. \tag{3.6}$$

(iii) *The q-set* $S_0 \setminus \{A_0^\infty\}$ *is* 1*-saturating in* Σ_0.

Lemma 1. *Let* $q \geq 4$, $\rho \geq 2$. *Then the plane* π_u, $u = 1,\dots,\rho$, *is* 2*-covered by* C_u^*. *Also, the point* $A_u^\infty = A_{u+1}^0$, $u = 1,\dots,\rho-1$, *is* 2*-covered by* C_u^* *as well as by* C_{u+1}^*.

Lemma 2. *Let* $q = 4$ *or* $q \geq 7$. *Then all points of* $\pi_u \setminus \{A_u^0, A_u^\infty\}$ *are* 1*-covered by* $C_u^* \cup \{T_u\}$, $u = 1,\dots,\rho$. *Also, all points of* $\pi_\rho \setminus \{A_\rho^0\}$ *are* 1*-covered by* $C_\rho^* \cup \{T_\rho, A_\rho^\infty\}$.

Proof. If q is even, every point of a plane outside of a hyperoval $C_u \cup \{T_u\}$ lies on $(q+2)/2$ its bisecants. If q is odd, every point of a plane outside of a conic C_u lies on at least $(q-1)/2$ its bisecants. At most two of these bisecants will be removed if one removes A_u^0 and A_u^∞ from C_u. Thus, for $q = 4$ and $q \geq 7$, every point of $\pi_u \setminus \{A_u^0, A_u^\infty\}$ lies on at least one bisecant of $C_u^* \cup \{T_u\}$. The same holds for $\pi_\rho \setminus \{A_\rho^0\}$. □

Proposition 4. *Let* $q = 4$ *or* $q \geq 7$. *Let* $\Sigma_1 = \mathrm{PG}(3,q)$. *Let the set* $S_1 = \{A_0^0\} \cup L_0^* \cup C_1^* \cup \{T_1, A_1^\infty\} \subset \Sigma_1$ *be as in* (3.1)–(3.5). *Let* $\mathscr{M}_0(A_0^\infty)$ *be as in* (3.6). *Then it holds that*

(i) *The* $(2q+1)$*-set* S_1 *is a minimal* 1*-saturating set in* Σ_1.

(ii) *The point* A_1^∞ *of* S_1 *is* 1*-essential for the set* $\widetilde{\mathscr{M}_1}(A_1^\infty)$ *such that*

$$\widetilde{\mathscr{M}_1}(A_1^\infty) = \mathscr{M}_1(A_1^\infty) = \{(x_0 \dots x_3) : (x_0x_1) \notin \mathscr{M}_0(A_0^\infty), (x_2x_3) = (0\widehat{x}_3)\}. \tag{3.7}$$

(iii) *The* 2*q-set* $S_1 \setminus \{A_1^\infty\}$ *is* 2*-saturating in* Σ_1.

Proof. **(i)** By Proposition 3(iii) and Lemma 2, Σ_0 and π_1 are 1-covered by $\{A_0^0\} \cup L_0^* \cup C_1^* \cup \{T_1, A_1^\infty\}$. Hence, we should consider points of the form

$$B = (\widehat{x}_0 x_1 \overline{x_2 x_3}) = (1 x_1 \overline{x_2 x_3}) \in \Sigma_1 \setminus (\Sigma_0 \cup \pi_1). \tag{3.8}$$

We show that B in (3.8) is a linear combination of at most 2 points of S_1.

1) Let $(x_0x_1) \in \mathscr{M}_0(A_0^\infty)$. By (3.8), we have no such points B.

2) Let $(x_0x_1) \notin \mathscr{M}_0(A_0^\infty)$. By the hypothesis, $(x_0x_1 00)$ is 0-covered by $S_0 \setminus \{A_0^\infty\}$, i.e. $(x_0x_1 00) = (1x_1 00) \in \{A_0^0\} \cup L_0^*$. For B of (3.8), we have

$$\begin{aligned} B &= (x_0x_1 0\widehat{x}_3) = (x_0x_1 00) + \widehat{x}_3(0001) = (x_0x_1 00) + \widehat{x}_3 A_1^\infty; \\ B &= (x_0x_1\widehat{x}_2 0) = (x_0x_1 00) + \widehat{x}_2(0010) = (x_0x_1 00) + \widehat{x}_2 T_1; \\ B &= (x_0x_1\widehat{x}_2\widehat{x}_3) = (x_0 z 00) + \frac{\widehat{x}_2^2}{\widehat{x}_3}(01yy^2),\ z = x_1 - \frac{\widehat{x}_2^2}{\widehat{x}_3},\ y = \frac{\widehat{x}_3}{\widehat{x}_2}. \end{aligned} \tag{3.9}$$

Note that $(x_0z00)=(1z00)$ is 0-covered by $S_0\setminus\{A_0^\infty\}$ for any z.

From (3.9), we see that all points of S_1 are 1-essential.

(ii) The assertion follows from (3.9).

(iii) We have, cf. (3.9), $(1x_10\widehat{x}_3)=(1z00)+(010\widehat{x}_3)$, where $z=x_1-1$ and $(010\widehat{x}_3)\in\pi_1\setminus\{A_1^0,A_1^\infty\}$ is 1-covered by $C_1^*\cup\{T_1\}$, see Lemma 2. □

Proposition 5. *Let $q=4$ or $q\geq 7$. Let $\Sigma_2=\mathrm{PG}(5,q)$. Let the set $S_2=\{A_0^0\}\cup L_0^*\cup C_1^*\cup\{T_1\}\cup C_2^*\cup\{T_2,A_2^\infty\}\subset\Sigma_2$ be as in (3.1)–(3.5). Let $\mathscr{M}_1(A_1^\infty)$ be as in (3.7). Then it holds that*

(i) *The $(3q+1)$-set S_2 is a minimal 2-saturating set in Σ_2.*

(ii) *The point A_2^∞ of S_2 is 2-essential for the set $\widetilde{\mathscr{M}_2}(A_2^\infty)$ such that*

$$\widetilde{\mathscr{M}_2}(A_2^\infty)\subset\mathscr{M}_2(A_2^\infty)=\{(x_0\ldots x_5):(x_0\ldots x_3)\notin\mathscr{M}_1(A_1^\infty),\ (x_4x_5)=(0\widehat{x}_5)\}. \tag{3.10}$$

(iii) *The $3q$-set $S_2\setminus\{A_2^\infty\}$ is 3-saturating in Σ_2.*

Proof. **(i)** By Propositions 3 and 4 and Lemmas 1 and 2, it holds that Σ_0 is 1-covered by $\{A_0^0\}\cup L_0^*$; π_1 and π_2 are 2-covered by C_1^* and C_2^*, respectively; $\pi_2\setminus\{A_2^0\}$ is 1-covered by $C_2^*\cup\{T_2,A_2^\infty\}$; Σ_1 is 2-covered by $S_1\setminus\{A_1^\infty\}$. Recall that $\Sigma_0\cup\pi_1\subset\Sigma_1$. So, we should consider points of the form

$$B=(\overline{x_0x_1x_2}x_3\overline{x_4x_5})\in\Sigma_2\setminus(\Sigma_1\cup\pi_2). \tag{3.11}$$

We show that B in (3.11) is a linear combination of at most 3 points of S_2.

1) Let $(x_0\ldots x_3)\in\mathscr{M}_1(A_1^\infty)$. By the hypothesis and by (3.7), (3.11), we have

$$(x_0x_1)\notin\mathscr{M}_0(A_0^\infty),\ B=(x_0x_10\widehat{x}_3\overline{x_4x_5})=(x_0x_10000)+(000\widehat{x}_3\overline{x_4x_5}),$$

where (x_0x_10000) is 0-covered by $S_0\setminus\{A_0^\infty\}$ and $(000\widehat{x}_3\overline{x_4x_5})\in\pi_2\setminus\{A_2^0,A_2^\infty\}$ is 1-covered by $C_2^*\cup\{T_2\}$, see Lemma 2.

2) Let $(x_0\ldots x_3)\notin\mathscr{M}_1(A_1^\infty)$.

By the hypothesis, $(x_0\ldots x_300)$ is 1-covered by $S_1\setminus\{A_1^\infty\}$. Also,

$$B=(x_0\ldots x_30\widehat{x}_5)=(x_0\ldots x_300)+\widehat{x}_5(000001)=(x_0\ldots x_300)+\widehat{x}_5A_2^\infty; \tag{3.12}$$

$$B=(x_0\ldots x_3\widehat{x}_40)=(x_0\ldots x_300)+\widehat{x}_4(000010)=(x_0\ldots x_300)+\widehat{x}_4T_2; \tag{3.13}$$

$$B=(x_0\ldots x_3\widehat{x}_4\widehat{x}_5)=(x_0x_1x_2z00)+\frac{\widehat{x}_4^2}{\widehat{x}_5}(0001yy^2),\ z=x_3-\frac{\widehat{x}_4^2}{\widehat{x}_5},\ y=\frac{\widehat{x}_5}{\widehat{x}_4}. \tag{3.14}$$

In (3.12), (3.13), B is a linear combination of at most $(1+1)+1=3$ points. If $(x_0x_1x_2z)\notin\mathscr{M}_1(A_1^\infty)$, then the representation (3.14) is the needed linear combination. If $(x_0x_1x_2z)\in\mathscr{M}_1(A_1^\infty)$ whereas $(x_0\ldots x_3)\notin\mathscr{M}_1(A_1^\infty)$, then the only possible case is $(x_0x_1)\notin\mathscr{M}_0(A_0^\infty)$ with $(x_2x_3)=(00)$, see (3.7). In this case,

$$B=(x_0x_100\widehat{x}_4\widehat{x}_5)=(1x_100\widehat{x}_4\widehat{x}_5)=(1x_10000)+(0000\widehat{x}_4\widehat{x}_5), \tag{3.15}$$

where $(1x_1 0000)$ is 0-covered by $\{A_0^0\} \cup L_0^*$ and $(0000\widehat{x}_4\widehat{x}_5) \in \pi_2 \setminus \{A_2^0, A_2^\infty\}$ is 1-covered by $C_2^* \cup \{T_2\}$, see Lemma 2. Thus, B in (3.15) is a linear combination of at most $(0+1)+(1+1) = 3$ points.

From (3.12)–(3.15) we see that all points of $S_2 \setminus S_1$ are 2-essential. Also, we take into account that S_1 is a *minimal* 1-saturating set.

(ii) The assertion follows from (3.12). For some (but not for all) points in (3.12) we could avoid use of A_2^∞; this explains the sign "$\subset$" in (3.10). Let, for example, $B = (001\widehat{x}_3 0\widehat{x}_5) \notin \mathscr{M}_1(A_1^\infty)$. Then $B = (001000) + \widehat{x}_3 \left(00010\frac{\widehat{x}_5}{\widehat{x}_3}\right)$, where $(001000) = T_1$ and $\left(00010\frac{\widehat{x}_5}{\widehat{x}_3}\right) \in \pi_2 \setminus \{A_2^0, A_2^\infty\}$ is 1-covered by $C_2^* \cup \{T_2\}$. But, if $B = (00100\widehat{x}_5) \notin \mathscr{M}_1(A_1^\infty)$, we are not able to avoid A_2^∞.

(iii) We have, cf. (3.12), $B = (x_0 \ldots x_3 0\widehat{x}_5) = (x_0x_1x_2z00) + (00010\widehat{x}_5)$, where $z = x_3 - 1$ and $(00010\widehat{x}_5) \in \pi_2 \setminus \{A_2^0, A_2^\infty\}$ is 1-covered by $C_2^* \cup \{T_2\}$, see Lemma 2. This representation of B is the needed linear combination of at most $(1+1)+(1+1) = 4$ columns if $(x_0x_1x_2z) \notin \mathscr{M}_1(A_1^\infty)$ whence $(x_0x_1x_2z00)$ is 1-covered by $S_1 \setminus \{A_1^\infty\}$.

But if $(x_0x_1x_2z) \in \mathscr{M}_1(A_1^\infty)$, then by (3.7), $(x_0x_1) \notin \mathscr{M}_0(A_0^\infty)$ and we have, similarly to (3.15), $B = (1x_1000\widehat{x}_5) = (1x_10000) + \widehat{x}_5(000001)$, where $(1x_10000)$ is 0-covered by $\{A_0^0\} \cup L_0^*$ and $(000001) = A_2^\infty \in \pi_2$ is 2-covered by C_2^*, see Lemma 1. □

Theorem 3. *Let $q = 4$ or $q \geq 7$. Let $\Upsilon \geq 1$. Let $\Sigma_\rho = \mathrm{PG}(2\rho+1, q)$. Let S_ρ be a point $((\rho+1)q+1)$-subset of Σ_ρ as in Construction S of* (3.1)–(3.5). *Then it holds that*

(i) *The $((\rho+1)q+1)$-set S_ρ is a minimal ρ-saturating set in Σ_ρ, $\rho = 0, 1, \ldots, \Upsilon$.*

(ii) *The point A_ρ^∞ of S_ρ is ρ-essential for the set $\widetilde{\mathscr{M}_\rho}(A_\rho^\infty)$ such that*

$$\begin{aligned}
&\widetilde{\mathscr{M}_0}(A_0^\infty) = \mathscr{M}_0(A_0^\infty) = \{(01)\},\\
&\widetilde{\mathscr{M}_1}(A_1^\infty) = \mathscr{M}_1(A_1^\infty) = \{(x_0 \ldots x_3) : (x_0x_1) \notin \mathscr{M}_0(A_0^\infty), (x_2x_3) = (0\widehat{x}_3)\},\\
&\widetilde{\mathscr{M}_\rho}(A_\rho^\infty) \subset \mathscr{M}_\rho(A_\rho^\infty) = \{(x_0 \ldots x_{2\rho+1}) : (x_0 \ldots x_{2\rho-1}) \notin \mathscr{M}_{\rho-1}(A_{\rho-1}^\infty),\\
&(x_{2\rho}x_{2\rho+1}) = (0\widehat{x}_{2\rho+1})\},\ \rho = 2, 3, \ldots, \Upsilon.
\end{aligned} \tag{3.16}$$

(iii) *The $(\rho+1)q$-set $S_\rho \setminus \{A_\rho^\infty\}$ is $(\rho+1)$-saturating in Σ_ρ, $\rho = 0, 1, \ldots, \Upsilon$.*

Proof. We prove by induction on Υ.

For $\Upsilon = 3$ the theorem is proved in Propositions 3, 4, 5.

Assumption: let the assertions (i)–(iii) *hold for some* $\Upsilon \geq 3$.

We show that under Assumption, the assertions hold for $\Gamma = \Upsilon + 1$.

(i) By Propositions 3, 4, and 5, Lemmas 2 and 1, and Assumption, we have the following: Σ_0 is 1-covered by $\{A_0^0\} \cup L_0^*$; $\pi_1 \setminus \{A_1^\infty\}$, $\pi_u \setminus \{A_u^0, A_u^\infty\}$, $u = 2, 3, \ldots, \Gamma$, are 1-covered by $\{A_0^0\} \cup L_0^* \cup \bigcup_{u=1}^{\Gamma} (C_u^* \cup \{T_u\})$; $\pi_\Gamma \setminus \{A_\Gamma^0\}$ is 1-covered by $C_\Gamma^* \cup \{T_\Gamma, A_\Gamma^\infty\}$; π_1, π_2, …, π_Γ are 2-covered by C_1^*, C_2^*, …, C_Γ^*, respectively; Σ_Υ is Γ-covered by $S_\Upsilon \setminus \{A_\Upsilon^\infty\}$. Recall that $\Sigma_0 \cup \bigcup_{u=1}^{\Upsilon} \pi_u \subset \Sigma_\Upsilon$. So, we should consider points of the form

$$B = (\overline{x_0 \dots x_{2\Gamma-2}} x_{2\Gamma-1} \overline{x_{2\Gamma} x_{2\Gamma+1}}) \in \Sigma_\Gamma \setminus (\Sigma_\Upsilon \cup \pi_\Gamma). \quad (3.17)$$

We show that B in (3.17) is a linear combination of at most $\Gamma+1$ points of S_Γ.

1) Let $(x_0 \dots x_{2\Gamma-1}) \in \mathscr{M}_\Upsilon(A_\Upsilon^\infty)$.

By the hypothesis and by (3.16), $(x_0 \dots x_{2\Upsilon-1}) \notin \mathscr{M}_{\Upsilon-1}(A_{\Upsilon-1}^\infty)$. Therefore, $(x_0 \dots x_{2\Upsilon-1}0000)$ is $(\Upsilon-1)$-covered by $S_{\Upsilon-1} \setminus \{A_{\Upsilon-1}^\infty\}$. Now by (3.17), we have

$$B = (x_0 \dots x_{2\Upsilon-1} 0 \widehat{x}_{2\Gamma-1} \overline{x_{2\Gamma} x_{2\Gamma+1}}) = (x_0 \dots x_{2\Upsilon-1}0000) + (0 \dots 0 \widehat{x}_{2\Gamma-1} \overline{x_{2\Gamma} x_{2\Gamma+1}}), \quad (3.18)$$

where $(0 \dots 0 \widehat{x}_{2\Gamma-1} \overline{x_{2\Gamma} x_{2\Gamma+1}}) \in \pi_\Gamma \setminus \{A_\Gamma^0, A_\Gamma^\infty\}$ is 1-covered by C_Γ^*, see Lemma 2. So, B in (3.18) is a linear combination of at most $(\Upsilon-1+1)+(1+1) = \Gamma+1$ points.

2) Let $(x_0 \dots x_{2\Gamma-1}) \notin \mathscr{M}_\Upsilon(A_\Upsilon^\infty)$.

By the hypothesis, $(x_0 \dots x_{2\Gamma-1}00)$ is Υ-covered by $S_\Upsilon \setminus \{A_\Upsilon^\infty\}$. We can write

$$B = (x_0 \dots x_{2\Gamma-1} 0 \widehat{x}_{2\Gamma+1}) = (x_0 \dots x_{2\Gamma-1}00) + \widehat{x}_{2\Gamma+1} A_\Gamma^\infty; \quad (3.19)$$

$$B = (x_0 \dots x_{2\Gamma-1} \widehat{x}_{2\Gamma} 0) = (x_0 \dots x_{2\Gamma-1}00) + \widehat{x}_{2\Gamma} T_\Gamma; \quad (3.20)$$

$$B = (x_0 \dots x_{2\Gamma-1} \widehat{x}_{2\Gamma} \widehat{x}_{2\Gamma+1}) = (x_0 \dots x_{2\Gamma-2} z 00) + \frac{\widehat{x}_{2\Gamma}^2}{\widehat{x}_{2\Gamma+1}} (0 \dots 01 y y^2), \quad (3.21)$$

$$z = x_{2\Gamma-1} - \frac{\widehat{x}_{2\Gamma}^2}{\widehat{x}_{2\Gamma+1}}, \; y = \frac{\widehat{x}_{2\Gamma+1}}{\widehat{x}_{2\Gamma}}.$$

In (3.19), (3.20), B is a linear combination of at most $(\Upsilon+1)+1 = \Gamma+1$ points. If $(x_0 \dots x_{2\Gamma-2} z) \notin \mathscr{M}_\Upsilon(A_\Upsilon^\infty)$, then the representation (3.21) is the needed linear combination. If $(x_0 \dots x_{2\Gamma-2} z) \in \mathscr{M}_\Upsilon(A_\Upsilon^\infty)$ while $(x_0 \dots x_{2\Gamma-1}) \notin \mathscr{M}_\Upsilon(A_\Upsilon^\infty)$, then the only possibility is $(x_0 \dots x_{2\Upsilon-1}) \notin \mathscr{M}_{\Upsilon-1}(A_{\Upsilon-1}^\infty)$ with $(x_{2\Gamma-2} x_{2\Gamma-1}) = (00)$, see (3.16). In this case,

$$B = (x_0 \dots x_{2\Upsilon-1} 00 \widehat{x}_{2\Gamma} \widehat{x}_{2\Gamma+1}) = (x_0 \dots x_{2\Upsilon-1}0000) + (0 \dots 0 \widehat{x}_{2\Gamma} \widehat{x}_{2\Gamma+1}), \quad (3.22)$$

where $(x_0 \dots x_{2\Upsilon-1}0000)$ is $(\Upsilon-1)$-covered by $S_{\Upsilon-1} \setminus \{A_{\Upsilon-1}^\infty\}$ and $(0 \dots 0 \widehat{x}_4 \widehat{x}_{2\Gamma-1}) \in \pi_\Gamma \setminus \{A_\Gamma^0, A_\Gamma^\infty\}$ is 1-covered by $C_\Gamma^* \cup \{T_\Gamma\}$, see Lemma 2. Thus, B in (3.22) is a linear combination of at most $(\Upsilon-1+1)+(1+1) = \Gamma+1$ points.

From (3.18)–(3.22) we see that all the points of $S_\Gamma \setminus S_\Upsilon$ are Γ-essential. Also, we take into account that S_Υ is a *minimal* Υ-saturating set.

(ii) The assertion (3.16) follows from (3.19). For some (but not for all) points in (3.19) we could avoid use of A_Γ^∞. This explains the sign "$\subset$" in (3.16).

(iii) We have, cf. (3.19), $B = (x_0 \dots x_{2\Gamma-1} 0 \widehat{x}_{2\Gamma+1}) = (x_0 \dots x_{2\Gamma-2} z 00) + (0 \dots 010 \widehat{x}_{2\Gamma+1})$, where $z = x_{2\Gamma-1} - 1$ and $(0 \dots 010 \widehat{x}_{2\Gamma+1}) \in \pi_\Gamma \setminus \{A_\Gamma^0, A_\Gamma^\infty\}$ is 1-covered by C_Γ^*, see Lemma 2. This representation of B is the needed linear combination of at most $(\Upsilon+1)+(1+1) = \Gamma+2$ points if $(x_0 \dots x_{2\Gamma-2} z) \notin \mathscr{M}_\Upsilon(A_\Upsilon^\infty)$ whence $(x_0 \dots x_{2\Gamma-2} z 00)$ is Υ-covered by $S_\Upsilon \setminus A_\Upsilon^\infty$.

But if $(x_0\dots x_{2\Gamma-2}z)\in\mathscr{M}_\Upsilon(A^\infty_\Upsilon)$, then by (3.16), $(x_0\dots x_{2\Upsilon-1}0000)\notin\mathscr{M}_{\Upsilon-1}(A^\infty_{\Upsilon-1})$, and we have, cf. (3.22), $(x_0\dots x_{2\Upsilon-1}000\widehat{x}_{2\Gamma+1})=(x_0\dots x_{2\Upsilon-1}0000)+\widehat{x}_{2\Gamma+1}(0\dots01)$, where $(x_0\dots x_{2\Upsilon-1}0000)$ is $(\Upsilon-1)$-covered by $S_{\Upsilon-1}\setminus\{A^\infty_{\Upsilon-1}\}$ and $(0\dots01)=A^\infty_\Gamma\in\pi_\Gamma$ is 2-covered by C^*_Γ, see Lemma 1. □

By computer search for $q=5$ we have proved the following proposition.

Proposition 6. *Let $q=5$. Let $0\le\rho\le4$. Let $\Sigma_\rho=\mathrm{PG}(2\rho+1,5)$. Let the $(5\rho+1)$-set $S_\rho\subset\Sigma_\rho$ be as in (3.1)–(3.5). Then S_ρ is a minimal ρ-saturating set in Σ_ρ.*

3.3 Codes of covering radius R and codimension $2R$

In the coding theory language, the results of this section give the following theorem.

Theorem 4. *Let $\widehat{V}_\rho$ be the code such that the columns of its parity check matrix are the points (in homogeneous coordinates) of the ρ-saturating $((\rho+1)q+1)$-set S_ρ of Construction S by (3.1)–(3.5).*

(i) *Let $q=4$ or $q\ge7$. Then for all $R\ge1$, the code $\widehat{V}_\rho$ is an $[Rq+1,Rq+1-2R,3]_qR$ locally optimal code of covering radius $R=\rho+1$.*

(ii) *Let $q=5$. Then for $1\le R\le5$, the code $\widehat{V}_\rho$ is a $[5R+1,5R+1-2R,3]_5R$ locally optimal code of covering radius $R=\rho+1$.*

Proof. We use Theorem 3 and Proposition 6. The code $\widehat{V}_\rho$ is locally optimal as the corresponding ρ-saturating set S_ρ is minimal. Distance $d=3$ is due to L^*_0. □

Conjecture 1. *Let $q=5$. Let $\widehat{V}_\rho$ be as in Theorem 4. Then for all $R\ge1$, the code $\widehat{V}_\rho$ is a $[5R+1,5R+1-2R,3]_5R$ locally optimal code with radius $R=\rho+1$.*

4 The q^m-concatenating constructions for code codimension lifting

The q^m-concatenating constructions are proposed in [10] and are developed in [11–14, 16, 19, 20], see also [6], [8, Sec. 5.4]. By using a starting code as a "seed", a q^m-concatenating construction yields an infinite family of new codes with a fixed covering radius, increasing codimension, and with almost the same covering density.

We give versions of the q^m-concatenating constructions convenient for our goals. Several other versions of such constructions can be found in [10–14, 16, 19, 20].

Construction QM$_1$. Let columns $\mathbf{h}_j$ belong to $\mathbb{F}_q^{r_0}$ and let $\mathbf{H}_0=[\mathbf{h}_1\mathbf{h}_2\dots\mathbf{h}_{n_0}]$ be a parity check matrix of an $[n_0,n_0-r_0]_qR,R$ *starting* surface-covering code V_0 with $R\ge2$. Let $m\ge1$ be an integer such that $q^m\ge n_0-1$. To each column $\mathbf{h}_j$ we associate an element $\beta_j\in\mathbb{F}_{q^m}\cup\{*\}$ so that $\beta_i \neq \beta_j$ if $i \neq j$. Let a new code V be the $[n, n-(r_0+Rm)]_q R_V, \ell_V$ code with $n = q^m n_0$ and parity check matrix $\mathbf{H}_V$ of the form

$$\mathbf{H}_V = [\mathbf{B}_1\ \mathbf{B}_2\ \ldots\ \mathbf{B}_{n_0}], \tag{4.1}$$

$$\mathbf{B}_j = \begin{bmatrix} \mathbf{h}_j & \mathbf{h}_j & \cdots & \mathbf{h}_j \\ \xi_1 & \xi_2 & \cdots & \xi_{q^m} \\ \beta_j\xi_1 & \beta_j\xi_2 & \cdots & \beta_j\xi_{q^m} \\ \beta_j^2\xi_1 & \beta_j^2\xi_2 & \cdots & \beta_j^2\xi_{q^m} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_j^{R-1}\xi_1 & \beta_j^{R-1}\xi_2 & \cdots & \beta_j^{R-1}\xi_{q^m} \end{bmatrix} \text{ if } \beta_j \in \mathbb{F}_{q^m},\quad \mathbf{B}_j = \begin{bmatrix} \mathbf{h}_j & \mathbf{h}_j \cdots & \mathbf{h}_j \\ 0 & 0 \cdots & 0 \\ \vdots & \vdots\ \vdots & \vdots \\ 0 & 0 \cdots & 0 \\ \xi_1 & \xi_2 \cdots & \xi_{q^m} \end{bmatrix} \text{ if } \beta_j = *, \tag{4.2}$$

where $\mathbf{B}_j$ is an $(r_0+Rm) \times q^m$ matrix, 0 is the zero element of $\mathbb{F}_{q^m}$, ξ_u is an element of $\mathbb{F}_{q^m}$, $\{\xi_1, \xi_2, \ldots, \xi_{q^m}\} = \mathbb{F}_{q^m}$. An element of $\mathbb{F}_{q^m}$ written in $\mathbf{B}_j$ denotes an m-dimensional q-ary column vector that is a q-ary representation of this element.

We denote $\mathbf{b}_j(\xi_u) = (\mathbf{h}_j, \xi_u, \beta_j\xi_u, \beta_j^2\xi_u, \ldots, \beta_j^{R-1}\xi_u)$ the u-th column of $\mathbf{B}_j$ with $\beta_j \in \mathbb{F}_{q^m}$. If $\beta_j = *$, we have $\mathbf{b}_j(\xi_u) = (\mathbf{h}_j, 0, \ldots, 0, \xi_u)$.

Theorem 5. *In Construction QM_1, the new code V with the parity check matrix* (4.1), (4.2) *is an $[n, n-(r_0+Rm), 3]_q R, R$ surface-covering code with radius R and length $n = q^m n_0$. If the starting code V_0 is locally optimal, then V is locally optimal too.*

Proof. The minimum distance d is equal to 3 since for any pair of columns $\mathbf{b}_j(\xi_{u_1})$, $\mathbf{b}_j(\xi_{u_2})$ of $\mathbf{B}_j$, a 3-rd one can be found such that the column triple corresponds to a codeword of weight 3. Take $a, b, c \in \mathbb{F}_q^*$ with $a+b+c=0$. Put $\xi_{u_3} = (-a\xi_{u_1} - b\xi_{u_2})/c$. Let $\mathbf{0}$ be the zero (r_0+Rm)-positional column. Then for all j we have

$$a\mathbf{b}_j(\xi_{u_1}) + b\mathbf{b}_j(\xi_{u_2}) + c\mathbf{b}_j(\xi_{u_3}) = \mathbf{0}. \tag{4.3}$$

The length of the code V directly follows from the construction.

We show that covering radius R_V of V is equal to R.

Consider an arbitrary column $\mathbf{t} = (\mathbf{fs}) \in \mathbb{F}_q^{r_0+Rm}$ with $\mathbf{f} \in \mathbb{F}_q^{r_0}$, $\mathbf{s} \in \mathbb{F}_q^{Rm}$,
$\mathbf{s} = (s_1, s_2, \ldots, s_{Rm})$, $s_i \in \mathbb{F}_q$. We partition $\mathbf{s}$ by m-vectors so that $\mathbf{s} = (S_0, S_1, \ldots, S_{R-1})$, $S_v = (s_{vm+1}, s_{vm+2}, \ldots, s_{vm+m})$, $v = 0, 1, \ldots, R-1$. We treat S_v as an element of $\mathbb{F}_{q^m}$.

Since V_0 is an $[n_0, n_0 - r_0]_q R, R$ code, there exists a linear combination of the form

$$\mathbf{f} = \sum_{k=1}^{R} c_k \mathbf{h}_{j_k},\ c_k \in \mathbb{F}_q^* \text{ for all } k, \tag{4.4}$$

see Definition 4. Now we can represent $\mathbf{t}$ as a linear combination (with nonzero coefficients) of R distinct columns of $\mathbf{H}_V$. We have, see (4.2),

$$\mathbf{t} = \sum_{k=1}^{R} c_k \mathbf{b}_{j_k}(x_k),\ c_k \in \mathbb{F}_q^* \text{ and } x_k \in \mathbb{F}_{q^m} \text{ for all } k, \tag{4.5}$$

where values of x_k are obtained from the linear system with nonzero determinant. If for j_k in (4.4) we have $\beta_{j_k} \in \mathbb{F}_{q^m}$ for all k, then the system has the form

$$\sum_{k=1}^{R} c_k \beta_{j_k}^{v} x_k = S_v,\ v = 0, 1, \ldots, R-1. \tag{4.6}$$

We put $0^0 = 1$. If in (4.4) we have, for example, $\beta_{j_R} = *$, then the system is as follows:

$$\sum_{k=1}^{R-1} c_k \beta_{j_k}^{v} x_k = S_v,\ v = 0, 1, \ldots, R-2; \quad \sum_{k=1}^{R-1} c_k \beta_{j_k}^{R-1} x_k + c_R x_R = S_{R-1}. \tag{4.7}$$

If V_0 is a locally optimal code, then every column $\mathbf{h}_j$ of $\mathbf{H}_0$ takes part in a representation of the form (4.4). If we remove $\mathbf{b}_{j_k}(\xi_u)$ from $\mathbf{B}_{j_k}$ then there is $(s_1, s_2, \ldots, s_{Rm})$ such that the system (4.6) or (4.7) gives $x_k = \xi_u$; for some $\mathbf{t}$ the representation (4.5) becomes impossible. So, all columns of $\mathbf{H}_V$ are essential and V is locally optimal. □

Construction QM$_2$. Let columns $\mathbf{h}_j$ belong to $\mathbb{F}_q^{r_0}$ and let $\mathbf{H}_0 = [\mathbf{h}_1 \mathbf{h}_2 \ldots \mathbf{h}_{n_0}]$ be a parity check matrix of an $[n_0, n_0 - r_0]_q R, \ell_0$ *starting* code V_0 with $\ell_0 = R-1$, $R \geq 2$. Let $m \geq 1$ be an integer such that $q^m \geq n_0$. Let $\theta_{m,q} = \frac{q^{m+1}-1}{q-1}$. To each column $\mathbf{h}_j$ we associate an element $\beta_j \in \mathbb{F}_{q^m}$ so that $\beta_i \neq \beta_j$ if $i \neq j$. Let a new code V be the $[n, n-(r_0+Rm)]_q R_V, \ell_V$ code with $n = q^m n_0 + \theta_{m,q}$ and parity check matrix $\mathbf{H}_V$ of the form

$$\mathbf{H}_V = [\mathbf{C}\ \mathbf{B}_1\ \mathbf{B}_2\ \ldots\ \mathbf{B}_{n_0}], \quad \mathbf{C} = \begin{bmatrix} \mathbf{0}_{r_0+(R-1)m} \\ \mathbf{W}_m \end{bmatrix}, \tag{4.8}$$

where $\mathbf{B}_j$ is an $(r_0+Rm) \times q^m$ matrix as in (4.2), $\mathbf{C}$ is an $(r_0+Rm) \times \theta_{m,q}$ matrix, $\mathbf{0}_{r_0+(R-1)m}$ is the zero $(r_0+(R-1)m) \times \theta_{m,q}$ matrix, $\mathbf{W}_m$ is a parity check $m \times \theta_{m,q}$ matrix of the $[\theta_{m,q}, \theta_{m,q} - m, 3]_q 1$ Hamming code.

Theorem 6. *In Construction QM$_2$, the new code V with the parity check matrix* (4.8)*,* (4.2) *is an $[n, n-(r_0+Rm), 3]_q R, R$ surface-covering code with covering radius R and length $n = q^m n_0 + \frac{q^{m+1}-1}{q-1}$. Moreover, if the starting code V_0 is locally optimal, then the new code V is locally optimal too.*

Proof. The length of the code V directly follows from the construction.

The minimum distance is equal to 3 as the Hamming code is a code with $d = 3$.

We show that covering radius R_V of V is equal to R.

Consider an arbitrary column $\mathbf{t} = (\mathbf{fs}) \in \mathbb{F}_q^{r_0+Rm}$ with $\mathbf{f} \in \mathbb{F}_q^{r_0}$, $\mathbf{s} \in \mathbb{F}_q^{Rm}$, $\mathbf{s} = (s_1, s_2, \ldots, s_{Rm})$, $s_i \in \mathbb{F}_q$. We partition $\mathbf{s}$ by m-vectors so that $\mathbf{s} = (S_0, S_1, \ldots, S_{R-1})$, $S_v = (s_{vm+1}, s_{vm+2}, \ldots, s_{vm+m})$, $v = 0, 1, \ldots, R-1$. We treat S_v as an element of $\mathbb{F}_{q^m}$.

Since V_0 is an $[n_0, n_0 - r_0]_q R, \ell_0$ code with $\ell_0 = R-1$, there exists a linear combination of $\varphi(\mathbf{f})$ distinct columns of $\mathbf{H}_0$ of the form

$$\mathbf{f} = \sum_{k=1}^{\varphi(\mathbf{f})} c_k \mathbf{h}_{j_k},\ c_k \in \mathbb{F}_q^* \text{ for all } k, \varphi(\mathbf{f}) \in \{R-1, R\},$$

see Definition 4. If $\varphi(\mathbf{f}) = R$ we act similarly to the proof of Theorem 5.

Let $\varphi(\mathbf{f}) = R-1$. We represent $\mathbf{t}$ as a linear combination (with nonzero coefficients) of at most R distinct columns of $\mathbf{H}_V$. We have, see (4.2), (4.8),

$$\mathbf{t} = \eta \mathbf{c} + \sum_{k=1}^{R-1} c_k \mathbf{b}_{j_k}(x_k),\ c_k \in \mathbb{F}_q^* \text{ and } x_k \in \mathbb{F}_{q^m} \text{ for all } k,\ \eta \in \mathbb{F}_q, \tag{4.9}$$

where $\mathbf{c}$ is a column of $\mathbf{C}$ and $\eta = 0$ means that the summand $\eta \mathbf{c}$ is absent. Also, in (4.9), values of x_k are obtained from the linear system

$$\sum_{k=1}^{R-1} c_k \beta_{j_k}^{v} x_k = S_v,\ v = 0, 1, \ldots, R-2,$$

with nonzero determinant. Finally, in (4.9), $\mathbf{c} = (\mathbf{0}\mathbf{w})$ where $\mathbf{0}$ is the zero $(r_0 + (R-1)m)$-positional column and $\mathbf{w}$ is a column of $\mathbf{W}_m$ that satisfies the equality

$$\eta \mathbf{w} + \sum_{k=1}^{R-1} c_k \beta_{j_k}^{R-1} x_k = S_{R-1}. \tag{4.10}$$

In (4.10), if $\sum_{k=1}^{R-1} c_k \beta_{j_k}^{R-1} x_k = S_{R-1}$ we have $\eta = 0$. If $\sum_{k=1}^{R-1} c_k \beta_{j_k}^{R-1} x_k \neq S_{R-1}$, the needed column $\eta \mathbf{w}$ always exists as the Hamming code has covering radius 1.

Now we show that V is an $[n, n-(r_0+Rm), 3]_q R, R$ code, i.e. $\ell_V = R$. The critical case is when in (4.9) and (4.10) $\eta = 0$, i.e. the summand $\eta \mathbf{c}$ is absent. We use the approach of the proof of Theorem 5 regarding (4.3). In (4.3) we put $j = j_1, \xi_{u_1} = x_1, a = -c_1$ with j_1, x_1, c_1 taken from (4.9). Then

$$\mathbf{t} = -c_1 \mathbf{b}_{j_1}(x_1) + b \mathbf{b}_{j_1}(\xi_{u_2}) + c \mathbf{b}_{j_1}(\xi_{u_3}) + \sum_{k=1}^{R-1} c_k \mathbf{b}_{j_k}(x_k) = \sum_{k=2}^{R-1} c_k \mathbf{b}_{j_k}(x_k)$$
$$+ b \mathbf{b}_{j_1}(\xi_{u_2}) + c \mathbf{b}_{j_1}(\xi_{u_3}).$$

Thus, we always can represent $\mathbf{t} \in \mathbb{F}_q^{r_0+Rm}$ as a linear combination with nonzero coefficients of exactly R columns of $\mathbf{H}_V$.

By above, if we remove any column of $\mathbf{H}_V$, some representation of $\mathbf{t}$ becomes impossible. So, all columns of $\mathbf{H}_V$ are essential and the code V is locally optimal. □

5 New infinite code families with fixed radius $R \geq 4$ and increasing codimension tR

In the ρ-saturating set of Construction S (3.1)–(3.5), we consider a point P_j (in homogeneous coordinates) as a column $\mathbf{h}_j$ of the parity check matrix $\widehat{\mathbf{H}}_\rho$ that defines the $[qR+1, qR+1-2R, 3]_q R, \ell$ code $\widehat{V}_\rho$ of covering radius $R = \rho + 1$. To use Constructions QM_1 and QM_2 we show that $\ell = R-1$ if q is even, and $\ell = R$ if q is odd. This means that any column $\mathbf{f}$ of $\mathbb{F}_q^{2R}$ is equal to a linear combination with nonzero coefficients of $R-1$ or R columns of $\widehat{\mathbf{H}}_\rho$ for even q and R columns of $\widehat{\mathbf{H}}_\rho$ for odd q.

We consider some properties of $\widehat{\mathbf{H}}_\rho$ useful to estimate ℓ. Let $\mathbf{f} \in \mathbb{F}_q^{2R}$. Let $J(\mathbf{f}) = \{\mathbf{h}_{j_1}, \dots, \mathbf{h}_{j_\beta}\}$ and $I_w = \{\mathbf{h}_{i_1}, \dots, \mathbf{h}_{i_w}\}$ be sets of distinct columns of $\widehat{\mathbf{H}}_\rho$ such that

$$\mathbf{f} = \sum_{k=1}^{\beta} c_k \mathbf{h}_{j_k},\ \mathbf{h}_{j_k} \in J(\mathbf{f}) \text{ and } c_k \in \mathbb{F}_q^* \text{ for all } k; \tag{5.1}$$

$$\sum_{k=1}^{w} m_k \mathbf{h}_{i_k} = \mathbf{0},\ \mathbf{h}_{i_k} \in I_w \text{ and } m_k \in \mathbb{F}_q^* \text{ for all } k,\ \mathbf{0} \in \mathbb{F}_q^{2R} \text{ is the zero column.} \tag{5.2}$$

By (5.1) and (5.2), we have

$$\mathbf{f} = \sum_{k=1}^{\beta} c_k \mathbf{h}_{j_k} + \mu \sum_{k=1}^{w} m_k \mathbf{h}_{i_k},\ \mu \in \mathbb{F}_q^*. \tag{5.3}$$

Note that I_w is a set of columns corresponding to a w*eight w codeword* of $\widehat{V}_\rho$.

In the representation (5.3), the number of distinct columns of $\widehat{\mathbf{H}}_\rho$, say β^{new}, depends on the intersection $I_w \cap J(\mathbf{f})$ and the values of nonzero coefficients c_k, m_k, μ, for example,

$$\beta^{\text{new}} = \begin{cases} \beta + w & \text{if } \ I_w \cap J(\mathbf{f}) = \emptyset; \\ \beta + w - 1 & \text{if } \ |I_w \cap J(\mathbf{f})| = 1,\ \mathbf{h}_{j_\beta} = \mathbf{h}_{i_w},\ c_\beta + \mu m_w \neq 0; \\ \beta + w - 2 & \text{if } \ |I_w \cap J(\mathbf{f})| = 1,\ \mathbf{h}_{j_\beta} = \mathbf{h}_{i_w},\ c_\beta + \mu m_w = 0; \\ \beta + w - 2 & \text{if } \ |I_w \cap J(\mathbf{f})| = 2,\ \mathbf{h}_{j_\beta} = \mathbf{h}_{i_w},\ c_\beta + \mu m_w \neq 0, \\ & \qquad \mathbf{h}_{j_{\beta-1}} = \mathbf{h}_{i_{w-1}},\ c_{\beta-1} + \mu m_{w-1} \neq 0. \end{cases} \tag{5.4}$$

To use (5.3), (5.4), submatrices of $\widehat{\mathbf{H}}_\rho$ can be treated as parity check matrices of codes; we call them *component codes* and write in Table 1, where $u = 1, \dots, \rho$, "MDS" notes a minimum distance separable code, "AMDS" says on an Almost MDS code.

Remark 2. The following is useful to estimate ℓ in the code $\widehat{V}_\rho$.

(i) In an $[n, n-r, d]_q$ MDS code, any d columns of a parity check matrix correspond to a weight d codeword [32].

(ii) In an $[n, n-r, d]_q$ MDS code with $n \leq q$, there are codewords of *all weights* $w \in \{d, d+1, \dots, n\}$ [22, Th. 6].

Table 1: Components codes corresponding to submatrices of $\widehat{\mathbf{H}}_\rho$ based on (3.1)–(3.5)

rows of $\widehat{\mathbf{H}}_\rho$	columns of $\widehat{\mathbf{H}}_\rho$	geometrical object	code parameters	q	code name	code type
1,2	$\mathbf{h}_1 \dots \mathbf{h}_q$	$\{A_0^0\} \cup L_0^*$	$[q, q-2, 3]_q 2$	all	$\mathbb{L}_0$	MDS
$2u, 2u+1, 2u+2$	$\mathbf{h}_{qu+1} \dots \mathbf{h}_{qu+q-1}$	C_u^*	$[q-1, q-4, 4]_q 3$	all	$\mathbb{C}_u$	MDS
$2u, 2u+1, 2u+2$	$\mathbf{h}_{qu+1} \dots \mathbf{h}_{qu+q}$	$C_u^* \cup \{T_u\}$	$[q, q-3, 4]_q 3$	even	$\mathbb{C}_u^T$	MDS
$2u, 2u+1, 2u+2$	$\mathbf{h}_{qu+1} \dots \mathbf{h}_{qu+q}$	$C_u^* \cup \{T_u\}$	$[q, q-3, 3]_q 3$	odd	$\mathbb{C}_u^T$	AMDS
$2\rho, 2\rho+1, 2\rho+2$	$\mathbf{h}_{q\rho+1} \dots \mathbf{h}_{q\rho+q-1}$, $\mathbf{h}_{q\rho+q+1}$	$C_\rho^* \cup \{A_\rho^\infty\}$	$[q, q-3, 4]_q 3$	all	$\mathbb{C}_\rho^\infty$	MDS
$2\rho, 2\rho+1, 2\rho+2$	$\mathbf{h}_{q\rho+1} \dots \mathbf{h}_{q\rho+q+1}$	$C_\rho^* \cup \{A_\rho^\infty, T_\rho\}$	$[q+1, q-2, 4]_q 3$	even	$\mathbb{C}_\rho^{\infty T}$	MDS
$2\rho, 2\rho+1, 2\rho+2$	$\mathbf{h}_{q\rho+1} \dots \mathbf{h}_{q\rho+q+1}$	$C_\rho^* \cup \{A_\rho^\infty, T_\rho\}$	$[q+1, q-2, 3]_q 3$	odd	$\mathbb{C}_\rho^{\infty T}$	AMDS

(iii) If q is odd, for AMDS component codes $\mathbb{C}_u^T$ and $\mathbb{C}_\rho^{\infty T}$, we note that T_u lies on two tangents to C_u (in A_u^0 and A_u^∞) and on $\frac{q-1}{2}$ bisecants of C_u^*. Every of these bisecants gives rise to a weight 3 codeword. The $(q-1)$-set of points of C_u^* is partitioned to $\frac{q-1}{2}$ point pairs; every pair together with T_u forms a weight 3 codeword.

(iv) From the proofs of Sect. 3 it can be seen that for the representation of a column $\mathbf{f} \in \mathbb{F}_q^{2R}$ it is sufficient to use (for every u) at most 3 points (columns) of C_u^*. Similarly, one can use 2 points of $\{A_0^0\} \cup L_0^*$. Therefore, we have in $\{A_0^0\} \cup L_0^*$ and in every C_u^* at least $q-4$ "free" points (columns) that are not used to represent $\mathbf{f}$; these columns can be used to form sets I_w useful to increase β^{new} for $\mathbf{f}$ by (5.3), (5.4).

(v) If $\beta < R$ in (5.1), then at least $R - \beta$ component codes are not used to represent $\mathbf{f}$; the columns corresponding to these codes are "free" and can be used to form sets I_w.

(vi) If $q \geq 7$, always there exists μ providing conditions "$= 0$", "$\neq 0$" in (5.4).

Lemma 3. *Let $q \geq 7$. Let $R \geq 4$. Let $\widehat{V}_\rho$ be the $[Rq+1, Rq+1-2R, 3]_q R, \ell$ locally optimal code such that the columns of its parity check matrix $\widehat{\mathbf{H}}_\rho$ correspond to points (in homogeneous coordinates) of the minimal ρ-saturating set of Construction S* (3.1)–(3.5) *with $\rho = R-1$. Then $\ell = R$ if q is odd and $\ell = R-1$ if q is even.*

Proof. We should show that every column $\mathbf{f}$ of $\mathbb{F}_q^{2R}$ (including the zero column) is equal to a linear combination with nonzero coefficients of $R-1$ or R columns of $\widehat{\mathbf{H}}_\rho$ for even q and R columns of $\widehat{\mathbf{H}}_\rho$ for odd q.

Let $I_w = \{\mathbf{h}_{i_1}, \dots, \mathbf{h}_{i_w}\}$ be a set of distinct columns of $\widehat{\mathbf{H}}_\rho$ corresponding to a weight w codeword of an MDS component code. Then there is a linear combination $L_w = \sum_{k=1}^{w} m_k \mathbf{h}_{i_k} = \mathbf{0}$, $m_k \in \mathbb{F}_q^*$, cf. (5.2). Let $w_1 + w_2 + \ldots + w_b = T$. We denote $\Upsilon_T = L_{w_1} + L_{w_2} + \ldots + L_{w_b} = \mathbf{0}$ the sum of the linear combinations.

Let a column $\mathbf{f} \in \mathbb{F}_q^{2R}$ have the representation (5.1) of the form $\mathbf{f} = \sum_{k=1}^{\beta} c_k \mathbf{h}_{j_k}$ where $\mathbf{h}_{j_k} \in J(\mathbf{f})$ and $\beta \leq R$. If $\beta = R$, the assertions of the lemma hold.

Let $0 \le \beta \le R-3$ where $\beta = 0$ corresponds to the zero column. We represent the column as $\mathbf{f} = \sum_{k=1}^{\beta} c_k \mathbf{h}_{j_k} + \Upsilon_{R-\beta}$ where the linear combinations L_{w_j} of $\Upsilon_{R-\beta}$ consist of "free" columns that are not used in the set $J(\mathbf{f})$. We have several "free" columns, see Remark 2(iv),(v). The component code $\mathbb{L}_0$ has $d = 3$. Therefore, taking into account also Remark 2(i),(ii), the sum $\Upsilon_{R-\beta}$ with $3 \le R-\beta \le R$ always can be found.

Let $\beta \in \{R-2, R-1\}$. The increase of β by $w-1$, $w-2$ is possible if some columns of $J(\mathbf{f})$ and I_w correspond to the same component code and $|I_w \cap J(\mathbf{f})| \in \{1,2\}$, see (5.3), (5.4). Let d be minimum distance of a component code. Due to Remark 2(i),(iii),(iv), one always can take in (5.2) a set I_w with $w = d \in \{3,4\}$ so that $|I_w \cap J(\mathbf{f})| \in \{1,2\}$. This provides the cases with $w = d = 3$, $w-1 = 2$, $\beta^{\text{new}} = \beta + 2$, and $w = d = 4$, $w-2 = 2$, $\beta^{\text{new}} = \beta + 2$.

So, for even and odd q, if $\beta = R-2$, we can obtain $\beta^{\text{new}} = R$.

Let $\beta = R-1$. The case with $w = 3$, $w-2 = 1$, $\beta^{\text{new}} = \beta + 1$, can be provided if some column or a column pair of $J(\mathbf{f})$ and I_w correspond to the same code $\mathbb{L}_0$ (for all q) or to the same code $\mathbb{C}_u^T$, $\mathbb{C}_\rho^{\infty T}$ (for q odd) since these codes have $d = 3$. There exist columns $\mathbf{f} \in \mathbb{F}_q^{2R}$ such that $\mathbb{L}_0$ is not used for their representation. Therefore we should consider only codes $\mathbb{C}_u^T$, $\mathbb{C}_\rho^{\infty T}$. For q odd we always can obtain $\beta^{\text{new}} = R$ using $\mathbb{C}_u^T$, $\mathbb{C}_\rho^{\infty T}$ with $d = 3$, see Remark 2(iii). But in general, for even q (where MDS codes $\mathbb{C}_u^T$, $\mathbb{C}_\rho^{\infty T}$ have $d = 4$) we are not able to do $\beta^{\text{new}} = R$ when $\beta = R-1$, see (5.3), (5.4). □

In Theorems 7 and 8 we consider $R \ge 4$ since for $R = 2,3$, several short covering codes with $r = tR$ are given in detail in [11–16, 18–20].

Theorem 7. *Let $q \ge 7$ be odd. Let t be an integer. Then for all $R \ge 4$ there is an infinite family of $[n, n-r, 3]_q R, R$ locally optimal surface-covering codes with the parameters*

$$n = Rq^{(r-R)/R} + q^{(r-2R)/R},\ r = tR,\ t = 2 \text{ and } t \ge \lceil \log_q R \rceil + 3.$$

Proof. We take the $[Rq+1, Rq+1-2R, 3]_q R, R$ code $\widehat{V}_\rho$, see Lemma 3, as the starting code V_0 of Construction QM$_1$. By Theorem 5, we obtain an $[n, n-r, 3]_q, R, R$ code with $n = (qR+1)q^m$, $r = 2R + mR$. Obviously, $m+1 = \frac{r-R}{R}$. The condition $q^m \ge n_0 - 1$ implies $q^m \ge qR$ whence $m \ge \lceil \log_q R \rceil + 1$. Finally, we put $t = m+2$. □

Theorem 8. *Let $q \ge 8$ be even. Let t be an integer. Let $m_1 = \lceil \log_q(R+1) \rceil + 1$. Then for all $R \ge 4$ there are infinite families of $[n, n-r, 3]_q R, R$ locally optimal surface-covering codes with the parameters*

$$\textbf{(i)}\ n = Rq^{(r-R)/R} + 2q^{(r-2R)/R} + \sum_{j=3}^{t} q^{(r-jR)/R},\ r = tR,\ m_1 + 2 < t < 3m_1 + 2;$$

$$\textbf{(ii)}\ n = Rq^{(r-R)/R} + 2q^{(r-2R)/R} + \sum_{j=3}^{m_1+2} q^{(r-jR)/R},\ r = tR,\ t = m_1 + 2 \text{ and } t \ge 3m_1 + 2.$$

Proof. **(i)** We take the $[qR+1,qR+1-2R,3]_qR,\ell$ code $\widehat{V}_\rho$ with $\ell = R-1$, see Lemma 3, as the starting code V_0 of Construction QM$_2$. By Theorem 6, we obtain an $[n,n-r,3]_q,R,R$ code with $n=(qR+1)q^m+\frac{q^{m+1}-1}{q-1}$, $r=2R+mR$. Obviously, $m-(j-2)=\frac{r-jR}{R}$. The condition $q^m \geq n_0$ implies $q^m \geq qR+1$ whence $m \geq \lceil \log_q(qR+1)\rceil = \lceil \log_q(R+1)\rceil+1$. The restriction $m<3m_1$ is introduced as for $m \geq 3m_1$ we have codes of (ii) that are better than ones in (i). For $m=m_1$, codes of (i) and (ii) are the same. Finally, we put $t=m+2$.

(ii) In the relation (i), we put $t=m_1+2$ and obtain an $[n_1,n_1-r_1,3]_qR,R$ code with $n_1=(qR+1)q^{m_1}+\frac{q^{m_1+1}-1}{q-1}$, $r_1=2R+m_1R$. We take this code as the starting code V_0 of Construction QM$_1$. By Theorem 5, we obtain an $[n,n-r,3]_q,R,R$ code with $r=2R+m_1R+m_2R$, $q^{m_2} \geq n_1$, $n=n_1q^{m_2}=(qR+1)q^{m_1+m_2}+\sum_{i=0}^{m_1}q^{m_1+m_2-i}$. Obviously, $m_1+m_2-i=\frac{r-(i+2)R}{R}$. Since $(R+1)q^{m_1+1}>n_1$, the condition $q^{m_2} \geq n_1$ is satisfied when $q^{m_2} \geq (R+1)q^{m_1+1}$ whence $m_2 \geq \lceil \log_q(R+1)\rceil+m_1+1=2m_1$. Then we denote $2+m_1+m_2$ by t. □

6 New infinite code families with fixed even radius $R \geq 2$ and increasing codimension $tR+\frac{R}{2}$

In the projective plane PG$(2,q)$, a *blocking* (resp. *double blocking*) set S is a set of points such that every line of PG$(2,q)$ contains at least one (resp. two) points of S.

There is an useful connection between double blocking sets and 1-saturating sets.

Proposition 7. [16, Cor. 3.3], [28] *Let q be a square. Any double blocking set in the subplane* PG$(2,\sqrt{q}) \subset$ PG$(2,q)$ *is a* 1*-saturating set in the plane* PG$(2,q)$.

In the following we shall use these results:

Proposition 8. [1,3,4,16] *Let p be prime. Let $\phi(q)$ be as in* (2.1)*. The following bounds on the smallest size $\tau_2(2,q)$ of a double blocking set in* PG$(2,q)$ *hold:*

$$\tau_2(2,q) \leq 2(q+q^{2/3}+q^{1/3}+1), \qquad q=p^3,\ p\leq 73 \qquad [16, \text{Th. } 3.5];$$
$$\tau_2(2,q) \leq 2(q+q^{2/3}+q^{1/3}+1), \qquad q=p^{3h},\ p^h\equiv 2 \bmod 7 \qquad [4, \text{Th. } 5.5];$$
$$\tau_2(2,q) \leq 2\left(q+\frac{q-1}{\phi(q)-1}\right), \qquad q=p^h,\ h\geq 2,\ p\geq 3 \qquad [1, \text{Cor. } 1.9];$$
$$\tau_2(2,q) \leq 2\left(q+\frac{q}{p}+1\right), \qquad q=p^h,\ h\geq 2,\ p\geq 7 \qquad [3, \text{Th. } 1.8, \text{Cor. } 4.10].$$

Now we give a list of 1-saturating sets in the projective plane of square order. The sets (iv)–(vi) are new, they directly follow from Propositions 7 and 8.

Proposition 9. *Let q be a square. Let p be prime. Let $\phi(\sqrt{q})$ be as in* (2.1)*. Then in* PG$(2,q)$ *there are* 1*-saturating sets of the following sizes:*

(i) $3\sqrt{q}-1$, $\qquad q=p^{2h}\geq 4,\ h\geq 1$ [12, Th. 5.2];

(ii) $2\sqrt{q}+2\sqrt[4]{q}+2,\qquad q=p^{4h}\geq 16, h\geq 1$ [15, Th. 3.3], [16, Th. 3.4], [28];

(iii) $2\sqrt{q}+2\sqrt[3]{q}+2\sqrt[6]{q}+2,\qquad q=p^6,\ p\leq 73$ [15, Th. 3.4], [16, Cor. 3.6];

(iv) $2\sqrt{q}+2\sqrt[3]{q}+2\sqrt[6]{q}+2,\qquad q=p^{6h},\ p^h\equiv 2 \bmod 7$;

(v) $2\sqrt{q}+2\dfrac{\sqrt{q}-1}{\phi(\sqrt{q})-1},\qquad q=p^{2h},\ h\geq 2,\ p\geq 3$;

(vi) $2\sqrt{q}+2\dfrac{\sqrt{q}}{p}+2,\qquad q=p^{2h},\ h\geq 2,\ p\geq 7$.

Remark 3. In Proposition 9, if $\sqrt{q}=p^\eta$ with $\eta\geq 3$ odd, then *the new* 1-*saturating sets* of (iv)–(vi) *have smaller sizes than the known ones* of (i)–(iii). For example, if $q=p^6$, $\eta=3$, then the new size of (vi) is $2\sqrt{q}+2\sqrt[3]{q}+2$, cf. (iii). If $\eta\geq 5$ odd, the known sets have size $3\sqrt{q}-1$ whereas new sizes are $2\sqrt{q}+o(\sqrt{q})$. For example, if $q=p^{30}$, $\eta=15$, then the new size of (iv), (v) is $2\sqrt{q}+2\sqrt[3]{q}+2\sqrt[6]{q}+2$, cf. (i). In general, if $\eta\geq 3$ is prime, then the case (vi) gives smaller sizes than other variants. If η is odd non-prime, then the variant (v) is the best.

The case (iv) gives the same size as (v), if $3|\eta$. Therefore, in future we consider new codes and bounds resulting from Proposition 9(v),(vi).

Note also that if $q=p^2$, i.e. $\eta=1$, then the size (i) is the smallest in Proposition 9. It is why we pay attention to this case, see Remarks 4–6 and Problem 5 below.

Remark 4. Let a point of $\mathrm{PG}(2,q)$ have the form (x_0,x_1,x_2) where $x_i\in\mathbb{F}_q$, the leftmost nonzero coordinate is equal to 1. Let β be a primitive element of $\mathbb{F}_q$.

In [12, Th. 5.2, eq. (30)], the following construction of a 1-saturating $(3\sqrt{q}-1)$-set S in $\mathrm{PG}(2,q)$, q square, is proposed:

$$S=\{(1,0,x_2)|x_2\in\mathbb{F}_{\sqrt{q}}\}\cup\{(1,0,c\beta)|c\in\mathbb{F}^*_{\sqrt{q}}\}\cup\{(0,1,x_2)|x_2\in\mathbb{F}_{\sqrt{q}}\}. \tag{6.1}$$

We describe this construction in more detail than in [12] using, for the description, the Baer sublines similarly to [5, Prop. 3.2]. In [12], see (6.1), specific Baer sublines are noted. Here we explain the structure and role of these specific sublines. Two Baer subplanes $\mathscr{B}_1$ and $\mathscr{B}_2$ are considered. In the points of $\mathscr{B}_1$, all coordinates $x_i\in\mathbb{F}_{\sqrt{q}}$. Also, $\mathscr{B}_2=\mathscr{B}_1\Phi$ where Φ is the collineation such that $(x_0,x_1,x_2)\Phi=(x_0,x_1\beta,x_2\beta)$. Let $L_i\subset\mathrm{PG}(2,q)$ be the "long" line of equation $x_i=0$. Let $L_{i,j}=L_i\cap\mathscr{B}_j$ be the Baer subline of L_i in the Baer subplane $\mathscr{B}_j$. We denote points $A_1=(0,0,1)$, $A_2=(1,0,0)$. Obviously, $\{A_1,A_2\}\subset\mathscr{B}_1\cap\mathscr{B}_2$.

We have $L_{0,1}=L_{0,2}$, $\mathscr{B}_1\cap\mathscr{B}_2=L_{0,1}\cup\{A_2\}$. Thus, the Baer subplanes $\mathscr{B}_1$ and $\mathscr{B}_2$ have the common Baer subline $L_{0,1}$ and also the common point A_2 not on $L_{0,1}$. Also, $L_{0,1}\cap L_{1,1}\cap L_{1,2}=\{A_1\}$. So, we consider three Baer sublines through A_1; one of them $L_{0,1}$ is common for $\mathscr{B}_1$ and $\mathscr{B}_2$; the other two ($L_{1,1}$ and $L_{1,2}$) belong to the same long line L_1 that passes through $A_2\notin L_{0,1}$ and $A_1\in L_{0,1}$. The needed set consists of these three Baer sublines without their intersection point, i.e. $S=(L_{0,1}\cup L_{1,1}\cup L_{1,2})\setminus\{A_1\}$. Since $L_{1,1}\cap L_{1,2}=\{A_1,A_2\}$ it holds that $|S|=3\sqrt{q}-1$. Note that if A_1 is not removed from S then we have no bisecants of S through A_1.

All points on L_0 and L_1 are 1-covered by S. Consider a point $A=(1,a,b)\notin(L_0\cup L_1)$ with $a=a_1\beta+a_0\in\mathbb{F}_q^*$, $b=b_1\beta+b_0\in\mathbb{F}_q$. (If $a=0$ then $A\in L_1$.) Let $a_0\neq 0$. Then $A=(1,0,(b_1-a_1a_0^{-1}b_0)\beta)+a(0,1,a_0^{-1}b_0)$. Let $a_0=0$. Then $a_1\neq 0$ and $A=(1,0,b_0)+a(0,1,a_1^{-1}b_1)$. Thus, A is 1-covered by S. Also, from the above consideration it follows that all points of S are 1-essential and S is a *minimal* 1-saturating set.

Remark 5. In [33, Ex. B] and [5, Prop. 3.2], constructions of a 1-saturating $3\sqrt{q}$-set in $\mathrm{PG}(2,q)$, q square, are proposed. In [33], the set is minimal; it consists of three non-concurrent Baer sublines in a Baer subplane. In [5], the set is non-minimal; it is similar to one of the construction [12, Th. 5.2], see its description in Remark 4. However, in [5], the intersection point of the three Baer sublines is not removed from the 1-saturating set.

Remark 6. Let p be prime. To construct a 1-saturating $(3p-1)$-set in $\mathrm{PG}(2,p^2)$ one can apply Proposition 7 to a double blocking set in $\mathrm{PG}(2,p)$. However, double blocking $(3p-1)$-sets in $\mathrm{PG}(2,p)$ are known only for $q=13,19,31,37,43$, see [9]. Moreover, in $\mathrm{PG}(2,p)$, no double blocking sets of size less than $3p-1$ are known.

In $\mathrm{PG}(2,p^2)$, p prime, by [16, Tab. 2], we have the following sporadic examples of 1-saturating k-sets with $k<3p-1$: $p^2=9,k=6$; $p^2=25,k=12$; $p^2=49,k=18$.

Problem 5. *Develop a general construction of a* 1*-saturating k-set in* $\mathrm{PG}(2,p^2)$*, p prime, such that* $k<3p-1$.

In [13, 16], a lift-construction is given. It provides the following result.

Proposition 10. [13, Ex. 6], [16, Th. 4.4] *Let an* $[n_q,n_q-3]_q2$ *code exist. Let* $n_q<q$ *and* $q+1\geq 2n_q$*. Let* $f_q(r,2)$ *be as in* (2.2)*. Then there is an infinite family of* $[n,n-r]_q2$ *codes with odd codimension* $r=2t+1\geq 3$, $t\geq 1$*, and length* $n=n_qq^{(r-3)/2}+2q^{(r-5)/2}+f_q(r,2)$.

Theorem 9. *Assume that p is prime,* $q=p^{2h}$, $h\geq 2$*, and covering radius* $R=2$*. Let* $\phi(\sqrt{q})$ *and* $f_q(r,2)$ *be as in* (2.1), (2.2)*. Then there exist infinite families of* $[n,n-r]_q2$ *codes with odd codimension* $r=2t+1\geq 3$, $t\geq 1$*, and length*

$$n=\left(2+2\frac{\sqrt{q}-1}{\sqrt{q}(\phi(\sqrt{q})-1)}\right)q^{(r-2)/2}+2\lfloor q^{(r-5)/2}\rfloor+f_q(r,2),\ p\geq 3;$$

$$n=\left(2+\frac{2}{p}+\frac{2}{\sqrt{q}}\right)q^{(r-2)/2}+2\lfloor q^{(r-5)/2}\rfloor+f_q(r,2),\ p\geq 7.$$

Proof. Let n_q be the size of the 1-saturating sets of Proposition 9(iii),(iv). We treat every point (in homogeneous coordinates) of the set as a column of an $3\times n_q$ parity check matrix of an $[n_q,n_q-3]_q2$ code. For these codes it can be shown that $n_q<q$ and $q+1\geq 2n_q$. Then we use Proposition 10. □

The direct sum construction [16, Sect. 4.2] gives the following lemma.

Lemma 4. *Let covering radius $R \geq 2$ be even. Let an $[n'', n'' - r'']_q 2$ code exist. Then there is an $[\frac{R}{2}n'', \frac{R}{2}n'' - \frac{R}{2}r'']_q R$ code.*

Theorem 10. *Assume that p is prime, $q = p^{2h}$, $h \geq 2$, $R \geq 2$ even, and code codimension is $r = tR + \frac{R}{2}$ with integer $t \geq 1$. Let $\phi(\sqrt{q})$ and $f_q(r,R)$ be as in* (2.1), (2.2). *Then for all even $R \geq 2$ there are infinite families of $[n, n-r]_q R$ codes with fixed covering radius R, codimension $r = tR + \frac{R}{2}$, $t \geq 1$, and length*

$$n = R\left(1 + \frac{\sqrt{q} - 1}{\sqrt{q}(\phi(\sqrt{q}) - 1)}\right) q^{(r-R)/R} + R\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor + \frac{R}{2} f_q(r,R), \; p \geq 3;$$

$$n = R\left(1 + \frac{1}{p} + \frac{1}{\sqrt{q}}\right) q^{(r-R)/R} + R\left\lfloor q^{(r-2R)/R-0.5}\right\rfloor + \frac{R}{2} f_q(r,R), \; p \geq 7.$$

Proof. We take codes of Theorem 9 as the codes $[n'', n'' - r'']_q 2$ of Lemma 4. □

References

[1] Bacsó, G., Héger, T., Szőnyi, T.: The 2-blocking number and the upper chromatic number of PG$(2,q)$. J. Combin. Designs **21**(12), 585-602 (2013)

[2] Bartoli, D., Davydov, A.A., Giulietti, M., Marcugini, S., Pambianco, F.: New bounds for linear codes of covering radii 2 and 3, Cryptography and Communications, to appear, `https://link.springer.com/article/10.1007/s12095-018-0335-0`

[3] De Beule, J., Héger, T., Szőnyi, T., Van de Voorde, G.: Blocking and Double Blocking Sets in Finite Planes. Electron. J. Combin. **23**(2) Paper #P2.5 (2016)

[4] Blokhuis, A., Lovász, L., Storme, L., Szőnyi, T.: On multiple blocking sets in Galois planes. Adv. Geom. **7**(1), 39–53 (2007)

[5] Boros, E., Szőnyi, T., Tichler, K.: On Defining Sets for Projective Planes. Discrete Math. **303**(1–3), 17–31 (2005)

[6] Brualdi, R.A., Litsyn, S., Pless, V.S.: Covering radius. In: Pless, V.S., Huffman, W.C., Brualdi, R.A. (eds.) Handbook of coding theory, vol. 1, pp. 755–826. Elsevier, Amsterdam, The Netherlands (1998)

[7] Brualdi, R.A., Pless, V.S, Wilson, R.M.: Short codes with a given covering radius. IEEE Trans. Inform. Theory **35**(1), 99–109 (1989)

[8] Cohen, G., Honkala, I., Litsyn, S., Lobstein, A.: Covering codes. North-Holland Mathematical Library, vol. 54. Elsevier, Amsterdam, The Netherlands (1997)

[9] Csajbók, B., Héger,T.: Double blocking sets of size $3q-1$ in $\mathrm{PG}(2,q)$. Europ. J. Combin. **78**, 73-89 (2019)

[10] Davydov, A.A.: Construction of linear covering codes. Probl. Inform. Transmis. **26**(4), 317–331 (1990)

[11] Davydov, A.A.: Construction of codes with covering radius 2. In: Cohen, G., Litsyn, S., Lobstein, A., Zemor G. (eds.) Algebraic Coding. Lect. Notes Comput. Science, vol. 573, pp. 23–31. Springer–Verlag, New–York (1992)

[12] Davydov, A.A.: Constructions and families of covering codes and saturated sets of points in projective geometry. IEEE Trans. Inform. Theory **41**(6), 2071–2080 (1995)

[13] Davydov, A.A.: Constructions and families of nonbinary linear codes with covering radius 2. IEEE Trans. Inform. Theory **45**(5), 1679–1686 (1999)

[14] Davydov, A.A., Faina, G., Marcugini, S., Pambianco, F.: Locally optimal (nonshortening) linear covering codes and minimal saturating sets in projective spaces. IEEE Trans. Inform. Theory **51**(12), 4378–4387 (2005)

[15] Davydov, A.A., Giulietti, M., Marcugini, S., Pambianco, F.: Linear covering codes over nonbinary finite fields. In: Proc. XI Int. Workshop on Algebraic and Combinatorial Coding Theory, ACCT2008. pp. 70–75. Pamporovo, Bulgaria (2008) `http://www.moi.math.bas.bg/acct2008/b12.pdf`

[16] Davydov, A.A., Giulietti, M., Marcugini, S., Pambianco, F.: Linear nonbinary covering codes and saturating sets in projective spaces. Adv. Math. Commun. **5**(1), 119–147 (2011)

[17] Davydov, A.A., A.A., Marcugini, S., Pambianco, F.: On saturating sets in projective spaces. J. Combin. Theory Ser. A **103**(1), 1–15 (2003)

[18] Davydov, A.A., Östergård, P.R.J.: On saturating sets in small projective geometries. Europ. J. Combin. **21**(5), 563–570 (2000)

[19] Davydov, A.A., Östergård, P.R.J.: Linear codes with covering radius $R=2,3$ and codimension tR. IEEE Trans. Inform. Theory **47**(1), 416–421 (2001)

[20] Davydov, A.A., Östergård, P.R.J.: Linear codes with covering radius 3. Des. Codes Crypt. **54**(3), 253–271 (2010)

[21] Etzion, T., Storme, L.: Galois geometries and coding theory. Des. Codes Crypt. **78**(1), 311–350 (2016)

[22] Ezerman, M.F., Grassl, M., Sole, P.: The Weights in MDS Codes. IEEE Trans. Inform. Theory **57**(1), 392–396 (2011)

[23] Giulietti, M.: The geometry of covering codes: small complete caps and saturating sets in Galois spaces. In: Blackburn, S.R., Holloway, R., Wildon, M. (eds.) Surveys in Combinatorics 2013, London Math. Soc. Lect. Note Series, vol. 409, pp. 51–90. Cambridge Univ Press, Cambridge (2013)

[24] Hirschfeld, J.W.P.: Projective Geometries Over Finite Fields. Oxford mathematical monographs, Clarendon Press, Oxford, 2nd edn. (1998)

[25] Hirschfeld, J.W.P., Storme, L.: The packing problem in statistics, coding theory and finite projective spaces. J. Statist. Planning Infer. **72**(1), 355–380 (1998)

[26] Hirschfeld, J.W.P., Storme, L.: The packing problem in statistics, coding theory and finite geometry: update 2001. In: Blokhuis, A., Hirschfeld, J.W.P. et al. (eds.) Finite Geometries, Developments of Mathematics, vol. 3, Proc. of the Fourth Isle of Thorns Conf., Chelwood Gate, 2000, pp. 201–246. Kluwer Academic Publisher, Boston (2001)

[27] Janwa, H.: Some optimal codes from algebraic geometry and their covering radii. Europ. J. Combin. **11**(3), 249–266 (1990)

[28] Kiss, G., Kóvacs, I., Kutnar, K., Ruff, J., Šparl, P.: A note on a geometric construction of large Cayley graphs of given degree and diameter. Studia Univ. Babes-Bolyai Math. **54**(3), 77–84 (2009)

[29] Klein, A., Storme, L.: Applications of Finite Geometry in Coding Theory and Cryptography. In: D. Crnković, V. Tonchev (eds.) NATO Science for Peace and Security, Ser. - D: Information and Communication Security, vol. 29, Information Security, Coding Theory and Related Combinatorics, pp. 38–58 (2011)

[30] Landjev, I., Storme, L.: Galois geometry and coding theory. In: De. Beule, J., Storme, L. (eds.) Current Research Topics in Galois geometry, Chapter 8, pp. 187–214, NOVA Academic Publisher, New York (2012)

[31] Lobstein, A.: Covering radius, an online bibliography. `https://www.lri.fr/~lobstein/bib-a-jour.pdf`

[32] MacWilliams, F.J., Sloane, N.J.A.: The Theory of Error-Correcting Codes. North- Holland, Amsterdam, The Netherlands, 3-rd edition (1981)

[33] Ughi, E.: Saturated configurations of points in projective Galois spaces. Europ. J. Combin. **8**(3), 325–334 (1987)