Tables, bounds and graphics of short linear codes with covering radius 3 and codimension 4 and 5 *

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Abstract. The ***length function*** $\ell_q(r, R)$ is the smallest length of a q-ary linear code of codimension (redundancy) r and covering radius R. The ***d-length function*** $\ell_q(r, R, d)$ is the smallest length of a q-ary linear code with codimension (redundancy) r, covering radius R, and minimum distance d.

By computer search in wide regions of q, we obtained following short codes of covering radius $R = 3$: $[n, n-4, 5]_q3$ quasi-perfect MDS codes, $[n, n-5, 5]_q3$ quasi-perfect Almost MDS codes, and $[n, n-5, 3]_q3$ codes. In computer search, we use the step-by-step leximatrix and inverse leximatrix algorithms to obtain parity check matrices of codes. These algorithms are versions of the recursive g-parity check matrix algorithm for greedy codes.

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The new codes imply the following new upper bounds (called ***lexi-bounds***) on the length function and the d-length function:

$\ell_q(4,3) \le \ell_q(4,3,5) < 2.8\sqrt[3]{\ln q} \cdot q^{(4-3)/3} = 2.8\sqrt[3]{\ln q} \cdot \sqrt[3]{q} = 2.8\sqrt[3]{q \ln q}$ for $11 \le q \le 6607$;

$\ell_q(5,3) \le \ell_q(5,3,5) < 3\sqrt[3]{\ln q} \cdot q^{(5-3)/3} = 3\sqrt[3]{\ln q} \cdot \sqrt[3]{q^2} = 3\sqrt[3]{q^2 \ln q}$ for $37 \le q \le 839$.

Moreover, we improve the lexi-bounds, applying randomized greedy algorithms, and show that

$$\ell_q(4,3) \le \ell_q(4,3,5) < 2.61\sqrt[3]{q \ln q} \quad \text{if} \quad 13 \le q \le 4373;$$
$$\ell_q(4,3) \le \ell_q(4,3,5) < 2.65\sqrt[3]{q \ln q} \quad \text{if} \quad 4373 < q \le 6607;$$
$$\ell_q(5,3) < 2.785\sqrt[3]{q^2 \ln q} \quad \text{if} \quad 11 \le q \le 401;$$
$$\ell_q(5,3) \le \ell_q(5,3,5) < 2.884\sqrt[3]{q^2 \ln q} \quad \text{if} \quad 401 < q \le 839.$$

The general form of the new bounds is

$$\ell_q(r,3) < c_\ell \sqrt[3]{\ln q} \cdot q^{(r-3)/3}, \quad c_\ell \text{ is a constant independent of } q, \quad r = 4,5 \ne 3t.$$

The codes, obtained in this paper by leximatrix and inverse leximatrix algorithms, provide the following new upper bounds (called ***density lexi-bounds***) on the ***smallest covering density*** $\mu_q(r,R)$ of a q-ary linear code of codimension r and covering radius R:

$$\mu_q(4,3) < 3.3 \cdot \ln q \quad \text{for} \quad 11 \le q \le 6607;$$
$$\mu_q(5,3) < 4.2 \cdot \ln q \quad \text{for} \quad 37 \le q \le 839.$$

In the general form, we have

$$\mu_q(r,3) < c_\mu \cdot \ln q, \quad c_\mu \text{ is a constant independent of } q, \quad r = 4,5.$$

The new bounds on the length function, the d-length function and covering density hold for the field basis q of an arbitrary structure, including $q \ne (q')^3$ where q' is a prime power.

Keywords: Covering codes, saturating sets, the length function, the d-length function, covering density, upper bounds, projective spaces.

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1 Introduction

1.1 Covering codes. The length function. The d-length function

Let $\mathbb{F}_q$ be the Galois field with q elements. Let F_q^n be the n-dimensional vector space over $\mathbb{F}_q$. Denote by $[n, n-r]_q$ a q-ary linear code of length n and codimension (redundancy) r, that is, a subspace of F_q^n of dimension $n-r$. The sphere of radius R with center c in F_q^n is the set $\{v : v \in F_q^n, d(v,c) \le R\}$ where $d(v,c)$ is the Hamming distance between the vectors v and c.

Definition 1.1. A linear $[n, n-r]_q$ code has *covering radius* R and is denoted as an $[n, n-r]_q R$ code if any of the following equivalent properties holds:

(i) The value R is the least integer such that the space F_q^n is covered by the spheres of radius R centered at the codewords.

(ii) Every column of F_q^r is equal to a linear combination of at most R columns of a parity check matrix of the code, and R is the smallest integer with this property.

Let an $[n, n-r, d]_q R$ code be an $[n, n-r]_q R$ code of minimum distance d. For an introduction to coverings of vector Hamming spaces over finite fields, see [8–10, 15, 23].

The covering density μ of an $[n, n-r]_q R$-code is defined as the ratio of the total volume of all q^{n-r} spheres of radius R centered at the codewords to the volume q^n of the space F_q^n. By Definition 1.1(i), we have $\mu \geq 1$. In the other words,

$$\mu = \left(q^{n-r} \sum_{i=0}^{R} (q-1)^i \binom{n}{i} \right) \frac{1}{q^n} = \frac{1}{q^r} \sum_{i=0}^{R} (q-1)^i \binom{n}{i} \geq 1. \qquad (1.1)$$

The covering quality of a code is better if its covering density is smaller. For fixed q, r, and R, the covering density of an $[n, n-r]_q R$ code decreases with decreasing n.

Codes investigated from the point view of the covering quality are usually called *covering codes* [9]; see an online bibliography [30], works [4,8,10,11,13–15,17–23,27–29,34], and the references therein.

This paper is devoted to non-binary covering codes with radius $R = 3$.

Note that for relatively small $q > 2$ many results are given in [11, 15, 19, 20] and the references therein.

Definition 1.2. (i) [8, 9] ***The length function*** $\ell_q(r, R)$ is the smallest length of a q-ary linear code of codimension (redundancy) r and covering radius R.

(ii) The ***d-length function*** $\ell_q(r, R, d)$ is the smallest length of a q-ary linear code with codimension (redundancy) r, covering radius R, and minimum distance d.

Obviously,

$$\ell_q(r, R) \leq \ell_q(r, R, d).$$

Let $\mathcal{A}_{R,q}$ denote a family of covering codes, in which the covering radius R and the size q of the underlying Galois field are fixed, while the code length tends to infinity. The construction of families with small asymptotic covering densities is a classical problem in the area of covering codes.

In [15], infinite sets of families $\mathcal{A}_{R,q}$, where R is fixed but q ranges over an infinite set of prime powers are considered. It is shown that for the upper limit $\mu_q^*(R, \mathcal{A}_{R,q})$ of the covering density of $\mathcal{A}_{R,q}$, the best possibility is

$$\mu_q^*(R, \mathcal{A}_{R,q}) = O(q). \qquad (1.2)$$

Moreover, in [15], for any covering radius $R \geq 2$, it is proposed the construction of *optimal* infinite sets of families $\mathcal{A}_{R,q}$ such that (1.2) holds. For this, in [15], the following results are obtained: In first, it is shown that for a given R, to obtain optimal infinite sets of families it is enough to construct R infinite families $\mathcal{A}_{R,q}^{(0)}, \mathcal{A}_{R,q}^{(1)}, \ldots, \mathcal{A}_{R,q}^{(R-1)}$ such that, for all $u \geq u_0$, the family $\mathcal{A}_{R,q}^{(\gamma)}$ contains codes of codimension $r_u = Ru + \gamma$ and length $n_q^{(\gamma)}(r_u)$ where

$$n_q^{(\gamma)}(r) = O\left(q^{(r-R)/R}\right)$$

and u_0 is a constant. Then, the needed families $\mathcal{A}_{R,q}^{(\gamma)}$ are constructed for any covering radius $R \geq 2$, with q ranging over the (infinite) set of R-th powers. A result of independent interest is that in each of these families $\mathcal{A}_{R,q}^{(\gamma)}$, the *lower limit of the covering density is bounded from above by a constant independent of* q.

So, infinite families of $[n, n-r]_q R$ codes of length

$$n = O\left(q^{(r-R)/R}\right) \tag{1.3}$$

play an important role in covering code theory.

Infinite families of covering $[n, n-r]_q R$ codes of length (1.3) are known for the following cases (see [8, 9, 11, 13–15, 17] and the references therein):

$r = tR,$ the field basis q has an arbitrary structure including $q \neq (q')^R$	[11, 13–15, 17] [19, 20];
$r \neq tR, \quad q = (q')^R$	[13–15];
$R = sR^*, \quad r = Rt + s, \quad q = (q')^{R^*}$	[13, 15, 17].

Here t and s are integers, q' is a prime power.

In the general case, *for arbitrary* r, R, q*, the problem to construct infinite families of* $[n, n-r]_q R$ *codes of length* (1.3) *is open.*

In the last decades, upper bounds on $\ell_q(r, R)$ have been intensively investigated, see [4, 8–15, 17–23, 27–30, 34] and the references therein.

The *goal of this paper* is to obtain new *upper bounds on the length functions* $\ell_q(4, 3)$*,* $\ell_q(5, 3)$ *and the d-length functions* $\ell_q(4, 3, 5)$*,* $\ell_q(5, 3, 5)$ *where codimension* $r \neq tR$ *and the field basis* q *has an arbitrary structure, including* $q \neq (q')^3$ *with* q' *is a prime power.* It is an open problem.

1.2 Saturating sets in projective spaces. Complete arcs

Let $\mathrm{PG}(N, q)$ be the N-dimensional projective space over the field $\mathbb{F}_q$; see [24–26] for an introduction to the projective spaces over finite fields, see also [11, 15, 21, 25, 28, 29] for connections between coding theory and Galois geometries.

Effective methods to obtain upper bounds on $\ell_q(r, R)$ are connected with saturating sets in $\mathrm{PG}(N, q)$.

Definition 1.3. A point set $\mathcal{S} \subseteq \mathrm{PG}(N,q)$ is *ρ-saturating* if any of the following equivalent properties holds:

(i) For any point A of $\mathrm{PG}(N,q) \setminus \mathcal{S}$ there exist $\rho+1$ points in $\mathcal{S}$ generating a subspace of $\mathrm{PG}(N,q)$ containing A, and ρ is the smallest value with this property.

(ii) Every point $A \in \mathrm{PG}(N,q)$ (in homogeneous coordinates) can be written as a linear combination of at most $\rho+1$ points of $\mathcal{S}$, and ρ is the smallest value with this property (cf. Definition 1.1(ii)).

Saturating sets are considered, for instance, in [1–4, 8, 11–19, 21, 22, 27–29, 35]. In the literature, saturating sets are also called “saturated sets”, “spanning sets”, “dense sets”.

Let $s_q(N,\rho)$ be *the smallest size of a ρ-saturating set* in $\mathrm{PG}(N,q)$.

If q-ary positions of a column of an $r \times n$ parity check matrix of an $[n, n-r]_q R$ code are treated as homogeneous coordinates of a point in $\mathrm{PG}(r-1,q)$ then this parity check matrix defines an $(R-1)$-saturating set of size n in $\mathrm{PG}(r-1,q)$ and vice versa [4, 11, 13–15, 17, 18, 21, 22, 27–29].

So, there is a *one-to-one correspondence between $[n, n-r]_q R$ codes and $(R-1)$-saturating sets in* $\mathrm{PG}(r-1,q)$. Therefore,

$$\ell_q(r,R) = s_q(r-1, R-1),$$

in particular, $\ell_q(4,3) = s_q(3,2)$, $\ell_q(5,3) = s_q(4,2)$.

Complete arcs in $\mathrm{PG}(N,q)$ are an important class of saturating sets. An n-arc in $\mathrm{PG}(N,q)$ with $n > N+1$ is a set of n points such that no $N+1$ points belong to the same hyperplane of $\mathrm{PG}(N,q)$. An n-arc of $\mathrm{PG}(N,q)$ is complete if it is not contained in an $(n+1)$-arc of $\mathrm{PG}(N,q)$. A complete arc in $\mathrm{PG}(N,q)$ is an $(N-1)$-saturating set. Points (in homogeneous coordinates) of a complete n-arc in $\mathrm{PG}(N,q)$, treated as columns, form a parity check matrix of an $[n, n-(N+1), N+2]_q N$ maximum distance separable (MDS) code [4, 5, 15, 18, 21, 22, 24–26, 28, 29]. If $N = 2, 3$ these codes are quasi-perfect.

Let $s_q^{\mathrm{arc}}(N)$ be *the smallest size of a complete arc* in $\mathrm{PG}(N,q)$. By above,

$$\ell_q(R+1,R) = s_q(R,R-1) \le \ell_q(R+1,R,R+2) = s_q^{\mathrm{arc}}(R).$$

The known results about upper bounds on $\ell_q(R+1,R,R+2)$ and $s_q^{\mathrm{arc}}(R)$, $R \ge 2$, can be found in [1–5], see also the references therein.

1.3 Covering codes with radius 3

For the *field basis q of an arbitrary structure*, infinite families of covering $[n, n-r]_q 3$ codes of length (1.3) are known only for $r = tR = 3t$ [15, 19]. In particular, the following parameters n, r are obtained by algebraic constructions [15, Sect. 5, eq. (5.2)], [19, Th. 12]:

$$n = 3q^{(r-3)/3} + q^{(r-6)/3},\ r = 3t \ge 6,\ r \ne 9,\ q \ge 5,\ \text{and}\ r = 9,\ q = 16,\ q \ge 23;$$

$$n = 3q^{(r-3)/3} + 2q^{(r-6)/3} + 1,\ r = 9,\ q = 7, 8, 11, 13, 17, 19;$$
$$n = 3q^{(r-3)/3} + 2q^{(r-6)/3} + 2,\ r = 9,\ q = 5, 9.$$

If $r = 3t+1$ or $r = 3t+2$, infinite families of covering codes of length (1.3) are known only when $q = (q')^3$, where q' is a prime power [13–15, 22]. In particular, $[n.n - r, 3]_q3$ codes with the following parameters n and r are obtained by algebraic constructions, see [13, 14, 22], [15, Sect. 5, eqs. (5.3),(5.4)]:

$$n = \left(4 + \frac{4}{\sqrt[3]{q}}\right) q^{(r-3)/3},\ r = 3t+1 \geq 4,\ q = (q')^3 \geq 64; \tag{1.4}$$
$$n = \left(9 - \frac{8}{\sqrt[3]{q}} + \frac{4}{\sqrt[3]{q^2}}\right) q^{(r-3)/3},\ r = 3t+2 \geq 5,\ q = (q')^3 \geq 27.$$

For the field basis q of an arbitrary structure, including $q \neq (q')^3$, in the literature, computer results are given for $[n, n-4]_q3$ codes with $q \leq 563$ [20, Tab. 1] and $q \leq 6229$ [4], and also for $[n, n-5]_q3$ codes with $q \leq 43$ [14, Tab. 1], [20, Tab. 2] and $q \leq 761$ [4].

The results of this paper are used in [18] and presented in XVI International Symposium "Problems of Redundancy in Information and Control Systems" (Redundancy 2019), Moscow, Russia, 21–25 October 2019.

The paper is organized as follows. In Section 2, we give the main results of this paper. In Section 3, a leximatrix algorithm to obtain parity check matrices of covering codes is described. In Sections 4 and 5, upper bounds on the length functions $\ell_q(4,3)$, $\ell_q(5,3)$ and the d-length functions $\ell_q(4,3,5)$, $\ell_q(5,3,5)$, based on leximatrix codes, are given. In Section 6, an inverse leximatrix algorithm to obtain parity check matrices of covering codes is considered and invleximatrix codes are obtained with the help of this algorithm. In Section 7 randomized greedy algorithms to obtain parity check matrices of covering codes are presented; new upper bounds improving the bounds of the previous sections are obtained. In Conclusion, the results of this paper are briefly analyzed; some tasks for investigation of the leximatrix algorithm are formulated. In Appendix, tables with sizes of codes obtained in this paper are given.

2 The main results

In this paper, by computer search, we obtain new results for $[n, n-4, 5]_q3$ quasi-perfect MDS codes with $q \leq 6607$ and $[n, n-5, 5]_q3$ quasi-perfect Almost MDS codes with $q \leq 839$. Also, we obtain $[n, n-5, 3]_q3$ codes for $q \leq 401$. This gives upper bounds on $\ell_q(4,3)$, $\ell_q(4,3,5)$, $\ell_q(5,3)$, and $\ell_q(5,3,5)$ for a set of values q greater than in [4, 14, 20]. New bounds are better than known ones.

The following Theorem 2.1 is based on the results of Sections 3–7, see Propositions 4.3, 5.1, 6.1, 7.1, and 7.2.

Theorem 2.1. *For the length function $\ell_q(r,3)$, the d-length function $\ell_q(r,3,5)$, the smallest size $s_q(r-1,2)$ of a 2-saturating set in the projective space $\mathrm{PG}(r-1,q)$, and the smallest size $s_q^{arc}(3)$ of a complete arc in $\mathrm{PG}(3,q)$, the following upper bounds hold:*

(1) *Upper bounds provided by $[n,n-r,5]_q3$ leximatrix and invleximatrix quasi-perfect codes (**lexi-bounds**).*

(i) $$\ell_q(4,3)=s_q(3,2)\le \ell_q(4,3,5)=s_q^{\mathrm{arc}}(3)<2.8\sqrt[3]{\ln q}\cdot q^{(4-3)/3}=2.8\sqrt[3]{\ln q}\cdot\sqrt[3]{q}$$
$$=2.8\sqrt[3]{q\ln q}\quad \textit{for } r=4,\quad 11\le q\le 6607;$$

(ii) $$\ell_q(5,3)=s_q(4,2)\le \ell_q(5,3,5)<3\sqrt[3]{\ln q}\cdot q^{(5-3)/3}=3\sqrt[3]{\ln q}\cdot\sqrt[3]{q^2}=3\sqrt[3]{q^2\ln q}$$
$$\textit{for } r=5,\quad 37\le q\le 839.$$

(2) *Upper bounds provided by $[n,n-4,5]_q3$ quasi-perfect MDS codes obtained with the help of the leximatrix, invleximatrix and d-Rand-Greedy algorithms.*

$$\ell_q(4,3)=s_q(3,2)\le \ell_q(4,3,5)=s_q^{\mathrm{arc}}(3)<\begin{cases}2.61\sqrt[3]{q\ln q} & \textit{if}\quad 13\le q\le 4373\\ 2.65\sqrt[3]{q\ln q} & \textit{if}\quad 4373<q\le 6607\end{cases}.$$

(3) *Upper bounds provided by $[n,n-5]_q3$ codes obtained with the help of the leximatrix and Rand-Greedy algorithms.*

$$\ell_q(5,3)=s_q(4,2)<2.785\sqrt[3]{q^2\ln q}\qquad \textit{if } 11\le q\le 401;$$
$$\ell_q(5,3)=s_q(4,2)\le\ell_q(5,3,5)<2.884\sqrt[3]{q^2\ln q}\qquad \textit{if } 401<q\le 839.$$

Note that, for $r\ne 3t$ and the field basis q of an arbitrary structure, including $q\ne(q')^3$ where q' is a prime power, the new bounds of Theorem 2.1 have the form

$$\ell_q(r,3)<c_\ell\sqrt[3]{\ln q}\cdot q^{(r-3)/3},\quad c_\ell \text{ is a constant independent of } q,\quad r=4,5.$$

The constants c_ℓ in the new bounds are smaller than in the paper [4].

Our results, in particular, figures and observations in Sections 4 and 5, comparison of leximatrix and invleximatrix codes in Table 3, improvements of the lexi-bounds in Section 7, allow us to conjecture the following.

Conjecture 2.2. *For the length function $\ell_q(r,3)$, the d-length function $\ell_q(r,3,5)$, the smallest size $s_q(r-1,2)$ of a 2-saturating set in the projective space $\mathrm{PG}(r-1,q)$, and the smallest size $s_q^{arc}(3)$ of a complete arc in $\mathrm{PG}(3,q)$, the following upper bounds (**lexi-bounds**) hold:*

(i) $$\ell_q(4,3)=s_q(3,2)\le \ell_q(4,3,5)=s_q^{\mathrm{arc}}(3)<2.8\sqrt[3]{\ln q}\cdot q^{(4-3)/3}=2.8\sqrt[3]{\ln q}\cdot\sqrt[3]{q}$$
$$=2.8\sqrt[3]{q\ln q}\quad \textit{for } r=4 \textit{ and } \textbf{\textit{all}}\ q\ge 11;$$

(ii) $\ell_q(5,3) = s_q(4,2) \leq \ell_q(5,3,5) < 3\sqrt[3]{\ln q} \cdot q^{(5-3)/3} = 3\sqrt[3]{\ln q} \cdot \sqrt[3]{q^2} = 3\sqrt[3]{q^2 \ln q}$
for $r = 5$ *and* ***all*** $q \geq 37$.

Let $\mu_q(r,R)$ be the ***smallest covering density of a*** q***-ary linear code of codimension (redundancy)*** r ***and covering radius*** R.

The following Theorem 2.3 is based on the results of Sections 3–7, see Propositions 4.6 and 5.3.

Theorem 2.3. *The* $[n, n-r, 5]_q 3$ *leximatrix and invleximatrix quasi-perfect codes, providing lexi-bounds of Theorem* 2.1(1)*, give also the following upper bounds on* $\mu_q(r,3)$ *(***density lexi-bounds***):*

$$\mu_q(4,3) < 3.3 \cdot \ln q \quad \text{for} \quad 11 \leq q \leq 6607;$$
$$\mu_q(5,3) < 4.2 \cdot \ln q \quad \text{for} \quad 37 \leq q \leq 839.$$

Note that, for $r \neq 3t$ and the field basis q of an arbitrary structure, including $q \neq (q')^3$ where q' is a prime power, the new bounds of Theorem 2.3 have the form

$$\mu_q(r,3) < c_\mu \cdot \ln q, \quad c_\mu \text{ is a constant independent of } q, \quad r = 4, 5.$$

3 A leximatrix algorithm to obtain parity check matrices of covering codes

The following is a *version of the recursive g-parity check matrix algorithm for greedy codes*, see e.g. [7, p. 25], [31], [32, Section 7].

Let $\mathbb{F}_q = \{0, 1, \ldots, q-1\}$ be the Galois field with q elements.

If q is prime, the elements of $\mathbb{F}_q$ are treated as integers modulo q.

If $q = p^m$ with p prime and $m \geq 2$, the elements of $\mathbb{F}_{p^m}$ are represented by integers as follows: $\mathbb{F}_{p^m} = \mathbb{F}_q = \{0, 1 = \alpha^0, 2 = \alpha^1, \ldots, u = \alpha^{u-1}, \ldots, q-1 = \alpha^{q-2}\}$, where α is a root of a primitive polynomial of $\mathbb{F}_{p^m}$.

For a q-ary code of codimension r, covering radius R, and minimum distance $d = R+2$, we construct a parity check matrix from nonzero columns h_i of the form

$$h_i = (x_1^{(i)}, x_2^{(i)}, \ldots, x_r^{(i)})^{tr}, \; x_u^{(i)} \in \mathbb{F}_q, \tag{3.1}$$

where the first (leftmost) non-zero element is 1; tr is the sign of transposition. The number of distinct columns is $(q^r-1)/(q-1)$. We order the columns in the list as

$$h_1, h_2, \ldots, h_{(q^r-1)/(q-1)}. \tag{3.2}$$

For h_i we put

$$i = \sum_{u=1}^{r} x_u^{(i)} q^{r-u}. \tag{3.3}$$

The columns of the list are candidates to be included in the parity check matrix.

By the above arguments connected with the formula for i and the order of the columns, a column h_i is treated as its number i in our list written in the q-ary scale of notation. The considered ***order of the columns*** is ***lexicographical***.

The first column of the list should be included into the matrix. Then step-by-step, one takes the next column from the list which cannot be represented as a linear combination of at most R columns already chosen. The process ends when no new column may be included into the matrix. The obtained matrix H_n is a parity check matrix of an $[n, n-r, R+2]_qR$ code.

The obtained parity check matrix is called the ***parity check leximatrix*** or the ***leximatrix*** for short. We call a ***leximatrix code*** the corresponding code.

For prime q, the following holds: **length** n **of a leximatrix code and the form of the leximatrix** H_n **depend on** q**,** r**, and** R **only**. No other factors affect code length and structure. Actually, assume that after some step a current matrix is obtained. At the next step we should remove from our current list all columns that are linear combination of R or less columns of the current matrix. For prime q and the given r and R, the result of removing is unequivocal; hence, the next column is taken uniquely.

For non-prime q, the length n of a leximatrix code depends on q and on the primitive polynomial of the field. In this paper, we use primitive polynomials that are created by the program system MAGMA [6] by default, see Table A. In any case, the choice of the polynomial changes the leximatrix code length unessentially.

By the leximatrix algorithm, if $R = 1$, we obtain the q-ary Hamming code. If $R = 2$, we obtain a quasi-perfect $[n, n-r, 4]_q2$ code; for $r = 3$, such code is an MDS code and corresponds to a complete arc in $\mathrm{PG}(2, q)$. If $R = 3$, we obtain a quasi-perfect $[n, n-r, 5]_q3$ code; for $r = 4$, such code is an MDS code and corresponds to a complete arc in $\mathrm{PG}(3, q)$; for $r = 5$, it is an Almost MDS code.

Let $n_q^{\mathrm{L}}(r, R)$ be **length of the** q**-ary leximatrix code of codimension** r **and covering radius** R.

It is assumed that for a non-prime field $\mathbb{F}_q$, one uses the primitive polynomial created by the program system MAGMA [6] by default; in particular, for non-prime $q \le 6889$, the polynomial from Table A should be taken.

We represent length $n_q^{\mathrm{L}}(r, R)$ of an $[n_q^{\mathrm{L}}(r, R), n_q^{\mathrm{L}}(r, R) - r, R+2]_qR$ leximatrix code in the form

$$n_q^{\mathrm{L}}(r, R) = c_q^{\mathrm{L}}(r, R) \sqrt[R]{\ln q} \cdot q^{(r-R)/R}, \tag{3.4}$$

Table A. Primitive polynomials used in this paper for leximatrix $[n, n-r, 5]_q3$ quasi-perfect codes with non-prime q

$q=p^m$	primitive polynomial	$q=p^m$	primitive polynomial	$q=p^m$	primitive polynomial
$4=2^2$	x^2+x+1	$8=2^3$	x^3+x+1	$9=3^2$	x^2+2x+2
$16=2^4$	x^4+x^3+1	$25=5^2$	x^2+x+2	$27=3^3$	x^3+2x^2+x+1
$32=2^5$	x^5+x^3+1	$49=7^2$	x^2+x+3	$64=2^6$	$x^6+x^4+x^3+1$
$81=3^4$	x^4+x+2	$121=11^2$	x^2+4x+2	$125=5^3$	x^3+3x+2
$128=2^7$	x^7+x+1	$169=13^2$	x^2+x+2	$243=3^5$	x^5+2x+1
$256=2^8$	$x^8+x^4+x^3+$ x^2+1	$289=17^2$	x^2+x+3	$343=7^3$	x^3+3x+2
$361=19^2$	x^2+x+2	$512=2^9$	x^9+x^4+1	$529=23^2$	x^2+2x+5
$625=5^4$	x^4+x^2+2x+2	$729=3^6$	x^6+x+2	$841=29^2$	$x^2+24x+2$
$961=31^2$	$x^2+29x+3$	$1024=2^{10}$	$x^{10}+x^6+x^5+$ x^3+x^2+x+1	$1331=11^3$	x^3+2x+9
$1369=37^2$	$x^2+33x+2$	$1681=41^2$	$x^2+38x+6$	$1849=43^2$	x^2+x+3
$2048=2^{11}$	$x^{11}+x^2+1$	$2187=3^7$	x^7+x^2+2x+1	$2197=13^3$	x^3+x^2+7
$2209=47^2$	x^2+x+13	$2401=7^4$	x^4+5x^2+4x+3	$2809=53^2$	$x^2+49x+2$
$3125=5^5$	x^5+4x+2	$3481=59^2$	$x^2+58x+2$	$3721=61^2$	$x^2+60x+2$
$4096=2^{12}$	$x^{12}+x^8+x^2+$ $x+1$	$4489=67^2$	$x^2+63x+2$	$4913=17^3$	x^3+x+14
$5041=71^2$	$x^2+69x+7$	$5329=73^2$	$x^2+70x+5$	$6241=79^2$	$x^2+78x+3$
$6561=3^8$	$x^8+2x^5+x^4+$ $2x^2+2x+2$	$6859=19^3$	$x^3+4x+17$	$6889=83^2$	$x^2+82x+2$

where $c_q^{\mathrm{L}}(r,R)$ is a coefficient. The coefficient $c_q^{\mathrm{L}}(r,R)$ and length $n_q^{\mathrm{L}}(r,R)$ are entirely given by r, R, q (if q is prime) or by r, R, q, and the primitive polynomial of $\mathbb{F}_q$ (if q is non-prime).

Remark 3.1. In the literature on the projective geometry, the columns are considered as points in homogeneous coordinates; the algorithm, described above, is called an "algorithm with fixed order of points" (FOP) [2,3,18].

Let $\mu_q^{\mathrm{L}}(r,R)$ be **covering density of the q-ary leximatrix code of codimension r and covering radius R.**

By (1.1). we have

$$\mu_q^{\mathrm{L}}(r,R) = \frac{1}{q^r}\sum_{i=0}^{R}(q-1)^i\binom{n_q^{\mathrm{L}}(r,R)}{i} \geq 1. \tag{3.5}$$

We represent covering density $\mu_q^{\mathrm{L}}(r,R)$ of an $[n_q^{\mathrm{L}}(r,R), n_q^{\mathrm{L}}(r,R)-r, R+2]_q R$ leximatrix code in the form

$$\mu_q^{\mathrm{L}}(r,R) = m_q^{\mathrm{L}}(r,R) \cdot \ln q, \tag{3.6}$$

where $m_q^{\mathrm{L}}(r,R)$ is a coefficient. The coefficient $m_q^{\mathrm{L}}(r,R)$ and density $\mu_q^{\mathrm{L}}(r,R)$ are entirely given by r,R,q (if q is prime) or by r,R,q, and the primitive polynomial of $\mathbb{F}_q$ (if q is non-prime).

4 Upper bounds on the length function $\ell_q(4,3)$ and d-length function $\ell_q(4,3,5)$ based on leximatrix codes

The following properties of the leximatrix algorithm are useful for implementation.

Proposition 4.1. *Let q be a prime. Then the v-th column of the parity check leximatrix of an $[n,n-4,5]_q 3$ code is the same for all $q \geq q_0(v)$ where $q_0(v)$ is large enough.*

Proof. Let $H_j = [h^{(1)}, h^{(2)}, \ldots, h^{(j)}]$ be the matrix obtained in the j-th step of the leximatrix algorithm. Here $h^{(v)}$ is a column of the matrix. A column from the list, not included in H_j, is covered by H_j if it can be represented as a linear combination of at most 3 columns of H_j. Suppose that $h^{(j)} = h_s$, where h_s is the s-th column in the lexicographical list of candidates. A column $Q = h_u \notin H_j$ is the next chosen column, if and only if all the columns h_m with $m \in [s+1, u-1]$ are covered by H_j. This means that, for any $m \in [s+1, u-1]$, at least one of the determinants $\det(h^{(v_1)}, h^{(v_2)}, h^{(v_3)}, h_m)$, with $h^{(v_1)}, h^{(v_2)}, h^{(v_3)} \in H_j$, is equal to zero modulo q. This can happen only in two cases:

- $\det(h^{(v_1)}, h^{(v_2)}, h^{(v_3)}, h_m) = 0$, we say that h_m is "absolutely" covered by H_j;
- $\det(h^{(v_1)}, h^{(v_2)}, h^{(v_3)}, h_m) = B \neq 0$, but $B \equiv 0 \bmod q$.

For q large enough, q does not divide any of the possible values of B and then, at least for j relatively small, the columns covered are just the absolutely covered columns. Therefore, when q is large enough the leximatrices share a certain number of columns. □

The values of $q_0(v)$ can be found with the help of calculations based on the proof of Proposition 4.1. Also, we can directly consider leximatrices for a convenient region of q.

Example 4.2. Values of $q_0(v)$, $v \leq 20$, together with columns $(x_1^{(v)}, x_2^{(v)}, x_3^{(v)}, x_4^{(v)})^{tr}$, are given in Table B. So, for all prime $q \geq 233$ (resp. $q \geq 1321$) the first 14 (resp. 20) columns of a parity check leximatrix of an $[n,n-4,5]_q 3$ quasi-perfect MDS leximatrix code are as in Table B.

Table B. The first 20 columns of the parity check leximatrices of $[n, n-4, 5]_q 3$ quasi-perfect MDS leximatrix codes, q prime

v	$x_1^{(v)}$	$x_2^{(v)}$	$x_3^{(v)}$	$x_4^{(v)}$	$q_0(v)$	v	$x_1^{(v)}$	$x_2^{(v)}$	$x_3^{(v)}$	$x_4^{(v)}$	$q_0(v)$
1	0	0	0	1	2	11	1	7	11	8	67
2	0	0	1	0	2	12	1	8	6	13	109
3	0	1	0	0	2	13	1	9	13	16	199
4	1	0	0	0	2	14	1	10	12	22	233
5	1	1	1	1	2	15	1	11	7	29	269
6	1	2	3	4	5	16	1	12	22	15	769
7	1	3	2	5	11	17	1	13	16	20	769
8	1	4	5	3	29	18	1	14	17	7	1283
9	1	5	4	2	41	19	1	15	21	10	1283
10	1	6	8	9	41	20	1	16	9	38	1321

Proposition 4.3. (i) *For $q = 9$, there exists a $[7, 7-4, 4]_9 3$ code of length $n = 7 < 2.8\sqrt[3]{9 \ln 9}$.*

(ii) *There exist $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3) - 4, 5]_q 3$ quasi-perfect MDS leximatrix codes of length $n_q^{\mathrm{L}}(4,3) < 2.8\sqrt[3]{q \ln q}$ for $q = 8$ and $11 \le q \le 6607$.*

Proof. **(i)** The existence of the code is noted in [20, Tab. 1], see also the references therein.

(ii) The needed codes are obtained by computer search, using the leximatrix algorithm, Proposition 4.1, and Example 4.2.

□

Proposition 4.3 implies the assertions of Theorem 2.1(1i) on the upper ***lexi-bound*** on the length function $\ell_q(4,3)$ and the d-length function $\ell_q(4,3,5)$.

Lengths $n_q^{\mathrm{L}}(4,3)$ of the $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3) - 4, 5]_q 3$ leximatrix quasi-perfect MDS codes are collected in Table 1 (see Appendix) and presented in Figure 1 by the bottom solid black curve. The bound

$$n_q^{\mathrm{L}}(4,3) < 2.8\sqrt[3]{q \ln q},$$

called the ***lexi-bound***, is shown in Figure 1 by the top dashed red curve.

We denote by $\delta_q(4,3)$ the difference between the lexi-bound $2.8\sqrt[3]{q \ln q}$ and length $n_q^{\mathrm{L}}(4,3)$ of the leximatrix code. Let $\delta_q^{\%}(4,3)$ be the corresponding percent difference. Thus,

$$\delta_q(4,3) = 2.8\sqrt[3]{q \ln q} - n_q^{\mathrm{L}}(4,3);$$

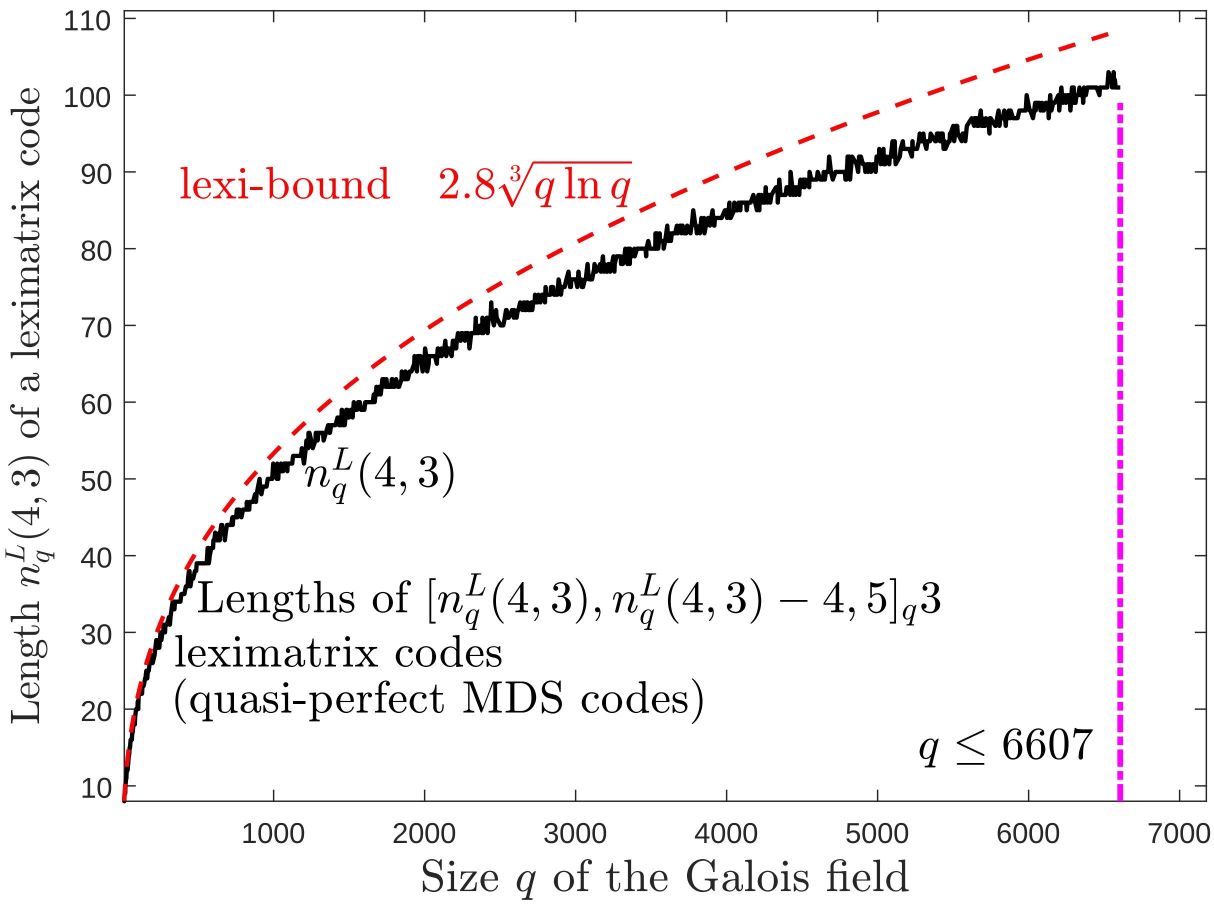

Figure 1: Lengths $n_q^{\mathrm{L}}(4,3)$ of the $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3)-4,5]_q3$ leximatrix quasi-perfect MDS codes (*bottom solid black curve*) vs the lexi-bound $2.8\sqrt[3]{q\ln q}$ (*top dashed red curve*); $11 \leq q \leq 6607$. *Vertical magenta line* marks region $q \leq 6607$

$$\delta_q^{\%}(4,3) = \frac{2.8\sqrt[3]{q\ln q} - n_q^{\mathrm{L}}(4,3)}{2.8\sqrt[3]{q\ln q}}100\%.$$

The difference $\delta_q(4,3)$ and the percent difference $\delta_q^{\%}(4,3)$ are presented in Figures 2 and 3.

By (3.4), we represent length of an $[n_q^L(4,3), n_q^L(4,3)-4,5]_q3$ leximatrix code in the form

$$n_q^{\mathrm{L}}(4,3) = c_q^{\mathrm{L}}(4,3)\sqrt[3]{q\ln q}, \tag{4.1}$$

where $c_q^{\mathrm{L}}(4,3)$ is a coefficient entirely given by q (if q is prime) or by q and the primitive polynomial of the field $\mathbb{F}_q$ (if q is non-prime). The coefficients $c_q^{\mathrm{L}}(4,3) = n_q^{\mathrm{L}}(4,3)/\sqrt[3]{q\ln q}$ are shown in Figure 4.

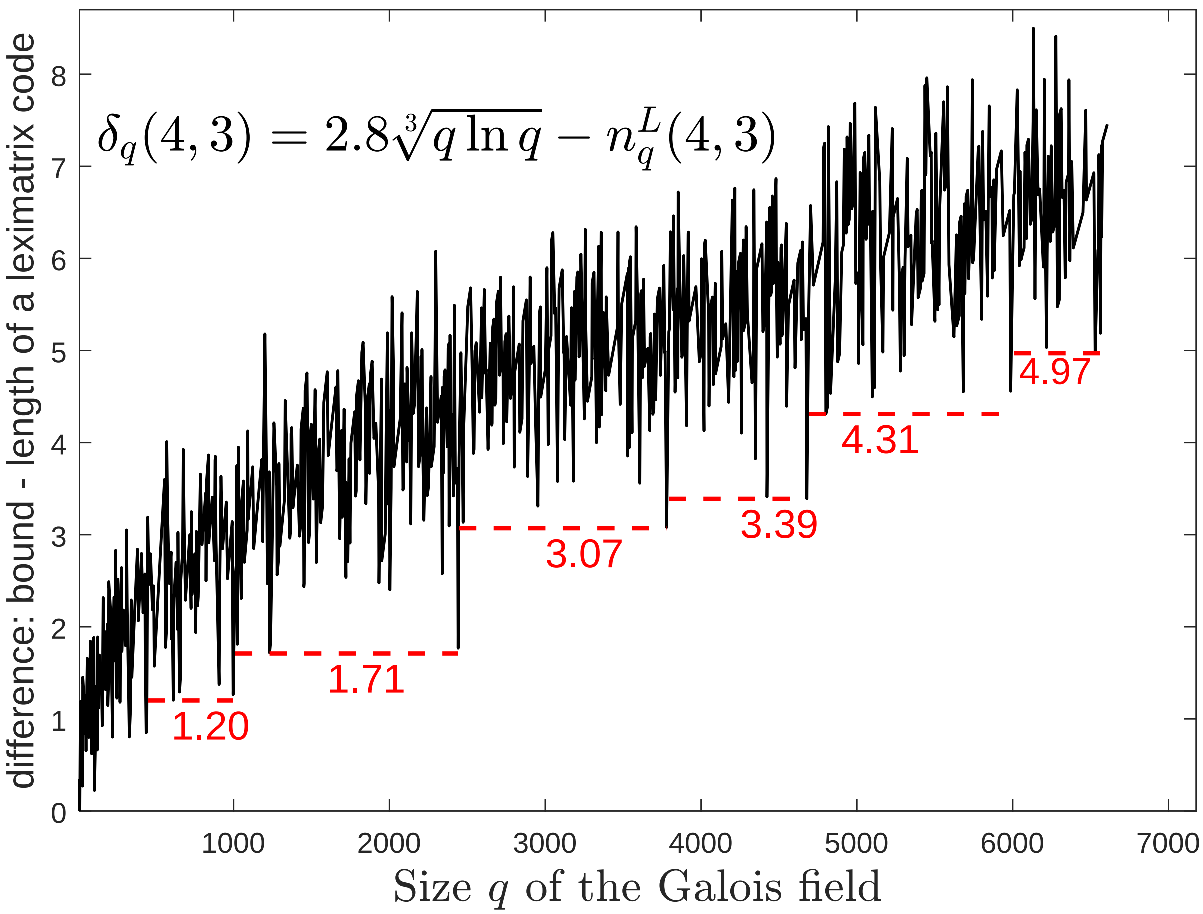

Figure 2: Difference $\delta_q(4,3)$ between the lexi-bound $2.8\sqrt[3]{q \ln q}$ and length $n_q^{\mathrm{L}}(4,3)$ of an $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3) - 4, 5]_q 3$ leximatrix code; $11 \le q \le 6607$

Observation 4.4. **(i)** *The difference* $\delta_q(4,3)$ *tends to increase when* q *grows, see Figures* 1 *and* 2.

(ii) *The percent difference* $\delta_q^{\%}(4,3)$ *oscillates around the horizontal line* $y = 6\%$. *When* q *increases, the oscillation amplitude tends to decrease, see Figure* 3.

(iii) *Coefficients* $c_q^L(4,3)$ *oscillate around the horizontal line* $y = 2.64$ *with a small amplitude.* ***When*** q ***increases, the oscillation amplitude tends to decrease****, see Figure* 4.

Observation 4.4 gives rise to Conjecture 2.2(i) on the length function $\ell_q(4,3)$ and the d-length function $\ell_q(4,3,5)$.

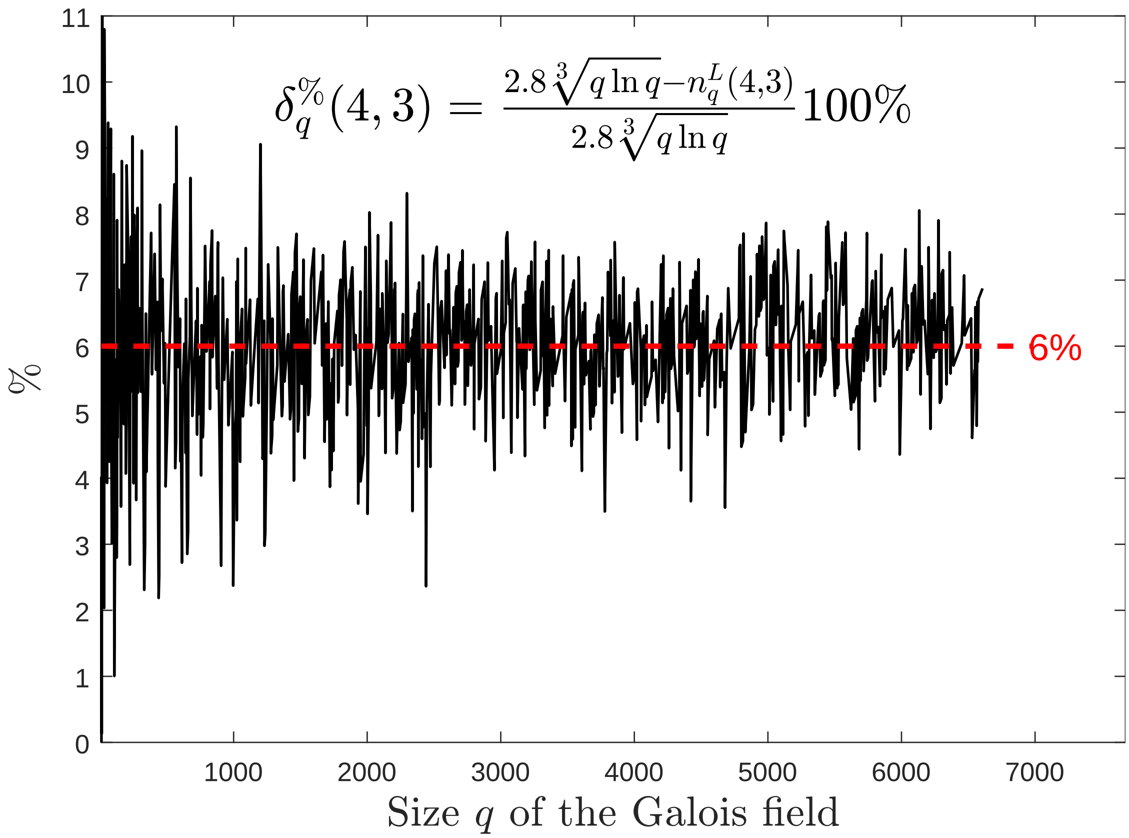

Figure 3: Percent difference $\delta_q^{\%}(4,3) = \frac{2.8\sqrt[3]{q\ln q} - n_q^{\mathrm{L}}(4,3)}{2.8\sqrt[3]{q\ln q}} 100\%$ between the lexi-bound $2.8\sqrt[3]{q\ln q}$ and length $n_q^{\mathrm{L}}(4,3)$ of an $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3)-4,5]_q 3$ leximatrix code; $11 \le q \le 6607$

Note that Observations 4.4(ii) and 4.4(iii) are connected with each other. Actually,

$$\delta_q^{\%}(4,3) = \frac{2.8\sqrt[3]{q\ln q} - n_q^{\mathrm{L}}(4,3)}{2.8\sqrt[3]{q\ln q}} 100\% = \left(1 - \frac{c_q^{\mathrm{L}}(4,3)}{2.8}\right) 100\%.$$

Remark 4.5. It is interesting that the oscillation of the coefficients $c_q^{\mathrm{L}}(4,3)$ around a horizontal line, in principle, is similar to the oscillation of the values $h^{\mathrm{L}}(q)$ around a horizontal line in [2, Fig. 6, Observation 3.5], [3, Fig. 5, Observation 3.7].

In the papers [2,3], small complete $t_2^L(2,q)$-arcs in the projective plane $\mathrm{PG}(2,q)$ are constructed by computer search using algorithm with fixed order of points (FOP). These arcs correspond to $[t_2^L(2,q), t_2^L(2,q)-3,4]_q 2$ quasi-perfect MDS codes while the algorithm FOP is analogous to the leximatrix algorithm of Section 3. Moreover, the value $h^{\mathrm{L}}(q)$ is defined in [2,3] as $h^{\mathrm{L}}(q) = t_2^L(2,q)/\sqrt{3q\ln q}$. So, see (4.1), the coefficients $c_q^{\mathrm{L}}(4,3)$ and the

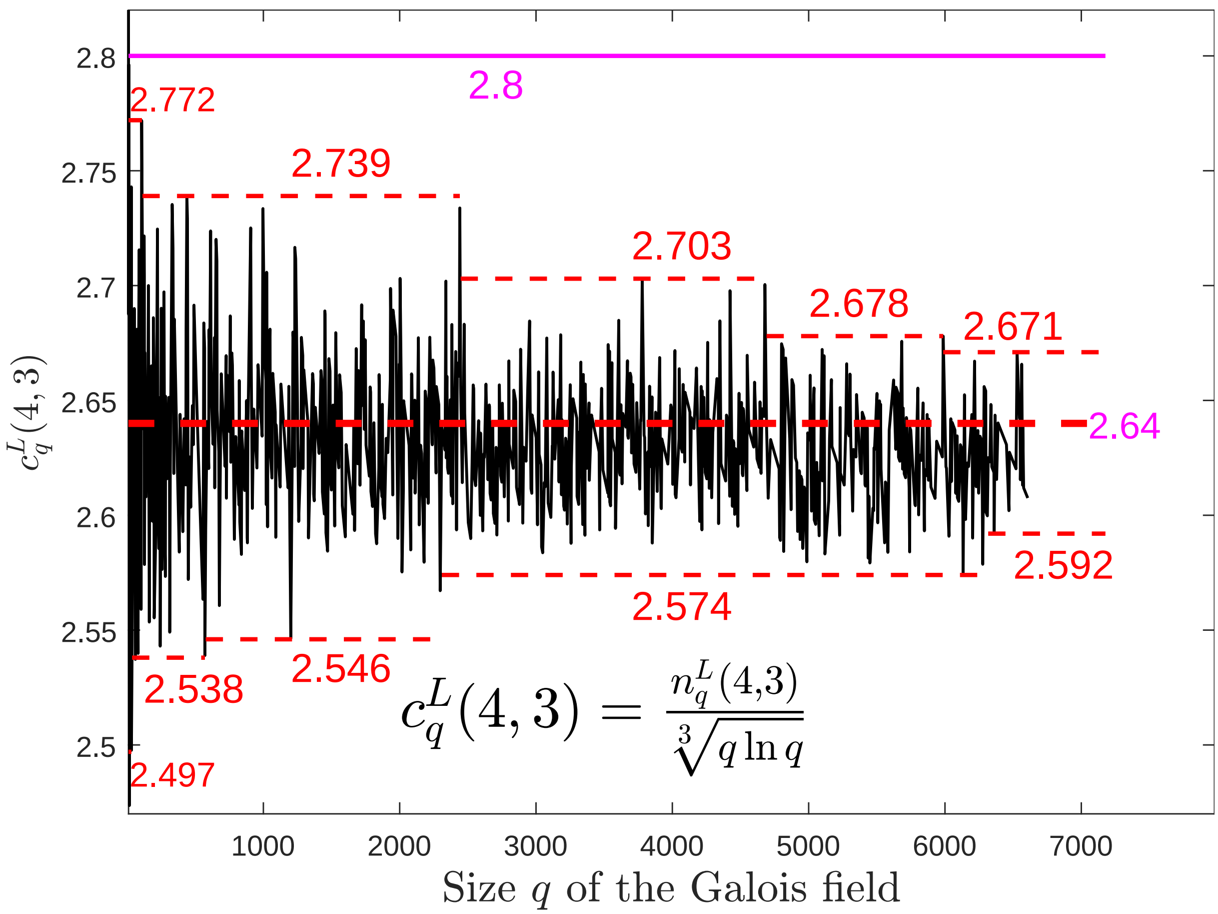

Figure 4: Coefficients $c_q^{\mathrm{L}}(4,3) = n_q^{\mathrm{L}}(4,3)/\sqrt[3]{q \ln q}$ for the $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3) - 4, 5]_q 3$ leximatrix quasi-perfect MDS codes; $11 \le q \le 6607$

values $h^{\mathrm{L}}(q)$ have the similar nature. It is possible that the oscillations mentioned also have similar reasons.

However, in the present time the **enigma of the oscillations** is incomprehensible,

Proposition 4.6. *There exist* $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3) - 4, 5]_q 3$ *quasi-perfect MDS leximatrix codes of covering density* $\mu_q^L(4,3) < 3.3 \cdot \ln q$ *for* $11 \le q \le 6607$.

Proof. The needed codes are the codes of Proposition 4.3. □

Proposition 4.6 implies the assertion of Theorem 2.3 on the upper ***density lexi-bound*** on the covering density $\mu_q(4,3)$.

Covering densities $\mu_q^{\mathrm{L}}(4,3)$ of the $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3) - 4, 5]_q 3$ leximatrix quasi-perfect MDS codes are presented in Figure 5 by the bottom solid black curve. The values $\mu_q^{\mathrm{L}}(4,3)$

are obtained by (3.5) where lengths $n_q^{\mathrm{L}}(4,3)$ are taken from Table 1 (see Appendix). The bound

$$\mu_q^{\mathrm{L}}(4,3) < 3.3 \cdot \ln q,$$

called the ***density lexi-bound***, is shown in Figure 5 by the top dashed red curve.

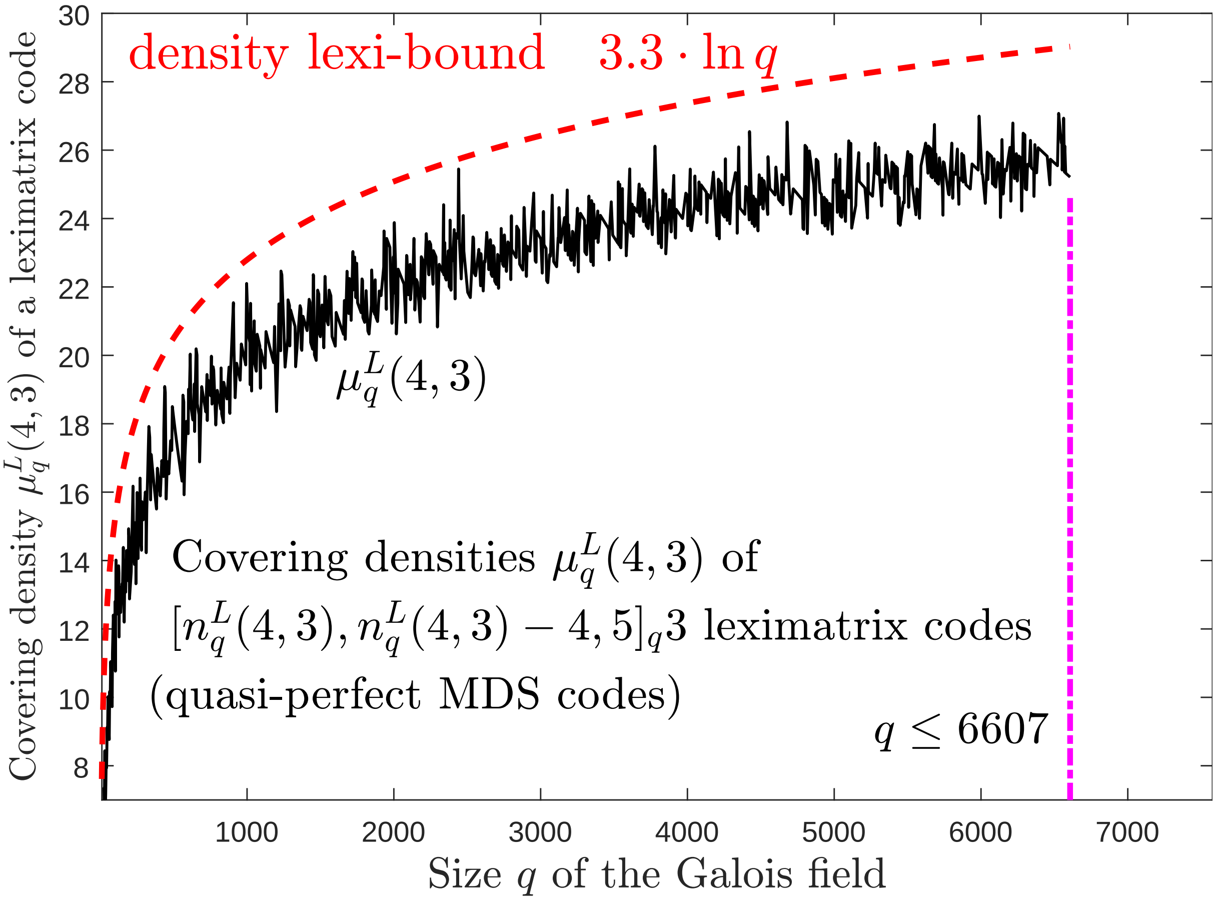

Figure 5: Covering densities $\mu_q^{\mathrm{L}}(4,3)$ of the $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3)-4,5]_q3$ leximatrix quasi-perfect MDS codes (*bottom solid black curve*) vs the density lexi-bound $3.3 \cdot \ln q$ (*top dashed red curve*); $11 \le q \le 6607$. *Vertical magenta line* marks region $q \le 6607$

By (3.6), we represent covering density of an $[n_q^L(4,3), n_q^L(4,3)-4,5]_q3$ leximatrix code in the form

$$\mu_q^{\mathrm{L}}(4,3) = m_q^{\mathrm{L}}(4,3) \cdot \ln q,$$

where $m^{\mathrm{L}}_q(4,3)$ is a coefficient entirely given by q (if q is prime) or by q and the primitive polynomial of the field $\mathbb{F}_q$ (if q is non-prime). The coefficients $m^{\mathrm{L}}_q(4,3) = \frac{\mu^{\mathrm{L}}_q(4,3)}{\ln q}$ are shown in Figure 6.

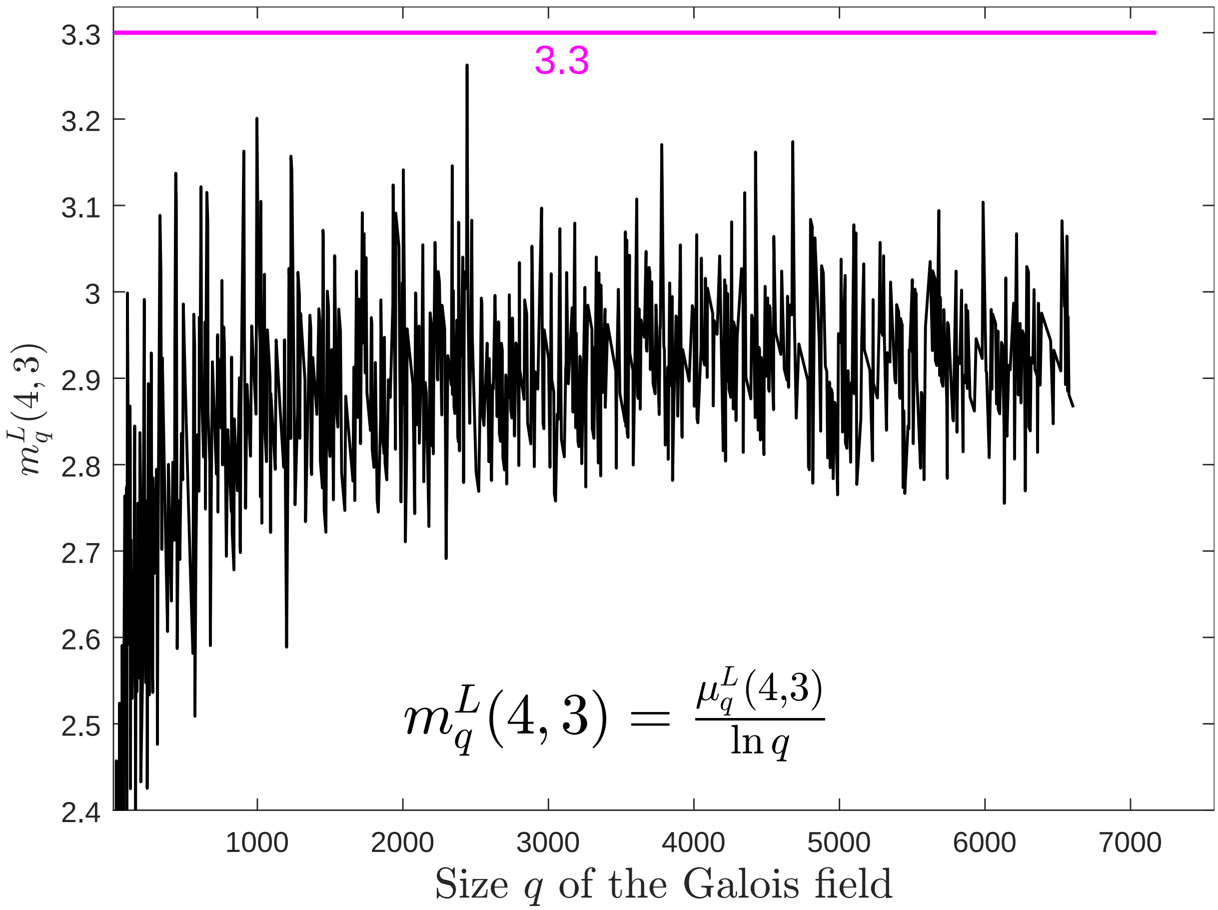

Figure 6: Coefficients $m^{\mathrm{L}}_q(4,3) = \mu^{\mathrm{L}}_q(4,3)/\ln q$ for covering density of the $[n^{\mathrm{L}}_q(4,3), n^{\mathrm{L}}_q(4,3)-4,5]_q3$ leximatrix quasi-perfect MDS codes; $11 \le q \le 6607$

5 Upper bounds on the length function $\ell_q(5,3)$ and d-length function $\ell_q(5,3,5)$ based on leximatrix codes

Proposition 5.1. **(i)** *There exist $[n, n-5, 4]_q3$ codes with $n < 3\sqrt[3]{q^2 \ln q}$ for $5 \le q < 37$.*

(ii) *There exist $[n^L_q(5,3), n^L_q(5,3)-5, 5]_q3$ quasi-perfect Almost MDS leximatrix codes with $n^L_q(5,3) < 3\sqrt[3]{q^2 \ln q}$ for $37 \le q \le 839$.*

Proof. **(i)** The existence of the codes is noted in [14, Tab. 1], [20, Tab. 2], see also the references therein.

(ii) The needed codes are obtained by computer search, using the leximatrix algorithm. □

Proposition 5.1 implies the assertions of Theorem 2.1(1ii) on the upper ***lexi-bound*** on the length function $\ell_q(5,3)$ and the d-length function $\ell_q(5,3,5)$.

Lengths $n_q^L(5,3)$ of the $[n_q^L(5,3), n_q^L(5,3)-5,5]_q3$ leximatrix Almost MDS codes are collected in Table 2 (see Appendix) and presented in Figure 7 by the bottom solid black curve. The bound $3\sqrt[3]{q^2\ln q}$, called the ***lexi-bound***, is shown in Figure 7 by the top dashed red curve.

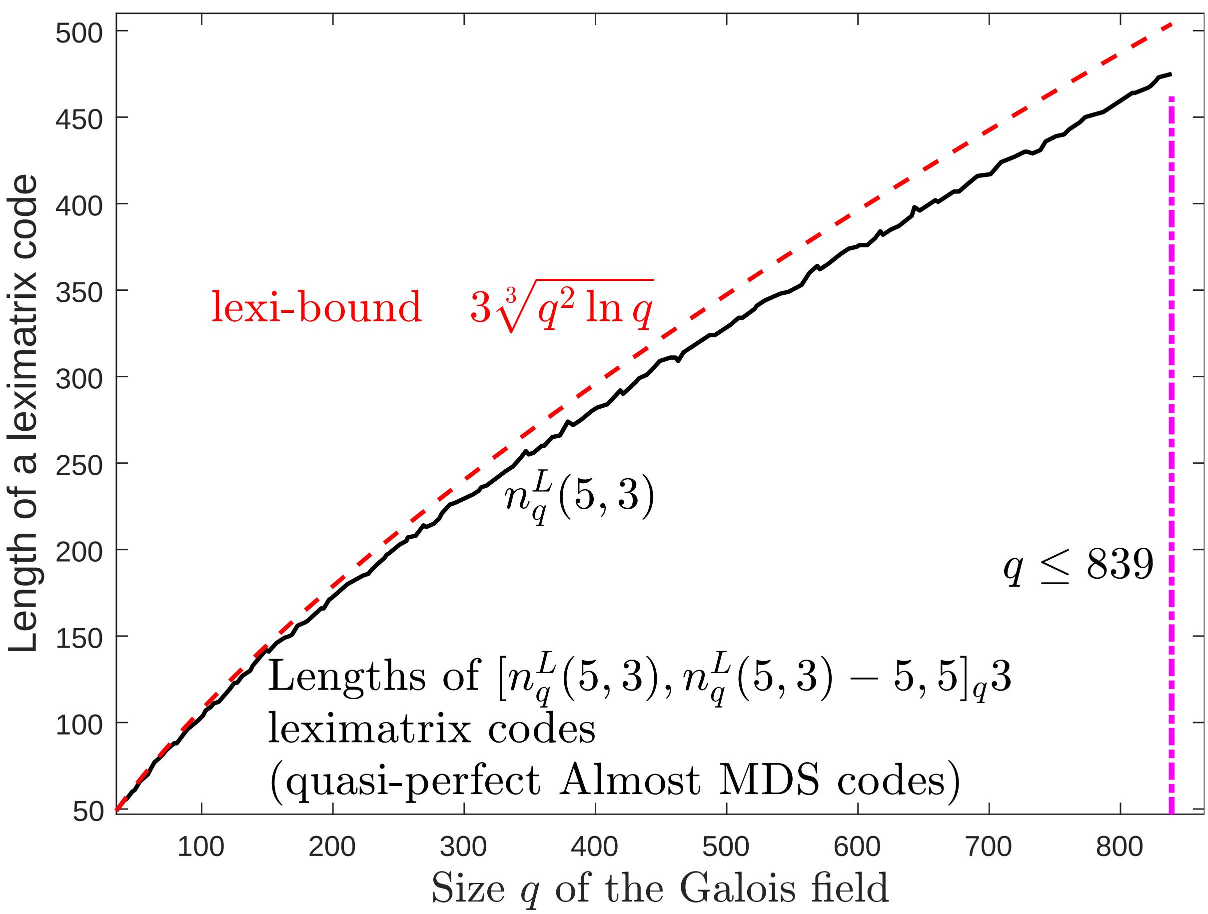

Figure 7: Lengths $n_q^{\mathrm{L}}(5,3)$ of the $[n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{L}}(5,3)-5,5]_q3$ leximatrix quasi-perfect Almost MDS codes (*bottom solid black curve*) vs the lexi-bound $3\sqrt[3]{q^2\ln q}$ (*top dashed red curve*); $37 \le q \le 839$. *Vertical magenta line* marks region $q \le 839$

We denote by $\delta_q(5,3)$ the difference between the lexi-bound $3\sqrt[3]{q^2\ln q}$ and length $n_q^{\mathrm{L}}(5,3)$ of the leximatrix code. Let $\delta_q^{\%}(5,3)$ be the corresponding percent difference. Thus,

$$\delta_q(5,3) = 3\sqrt[3]{q^2\ln q} - n_q^{\mathrm{L}}(5,3);$$
$$\delta_q^{\%}(5,3) = \frac{3\sqrt[3]{q^2\ln q} - n_q^{\mathrm{L}}(5,3)}{3\sqrt[3]{q^2\ln q}}100\%.$$

The difference $\delta_q(5,3)$ and the percent difference $\delta_q^{\%}(5,3)$ are presented in Figures 8 and 9.

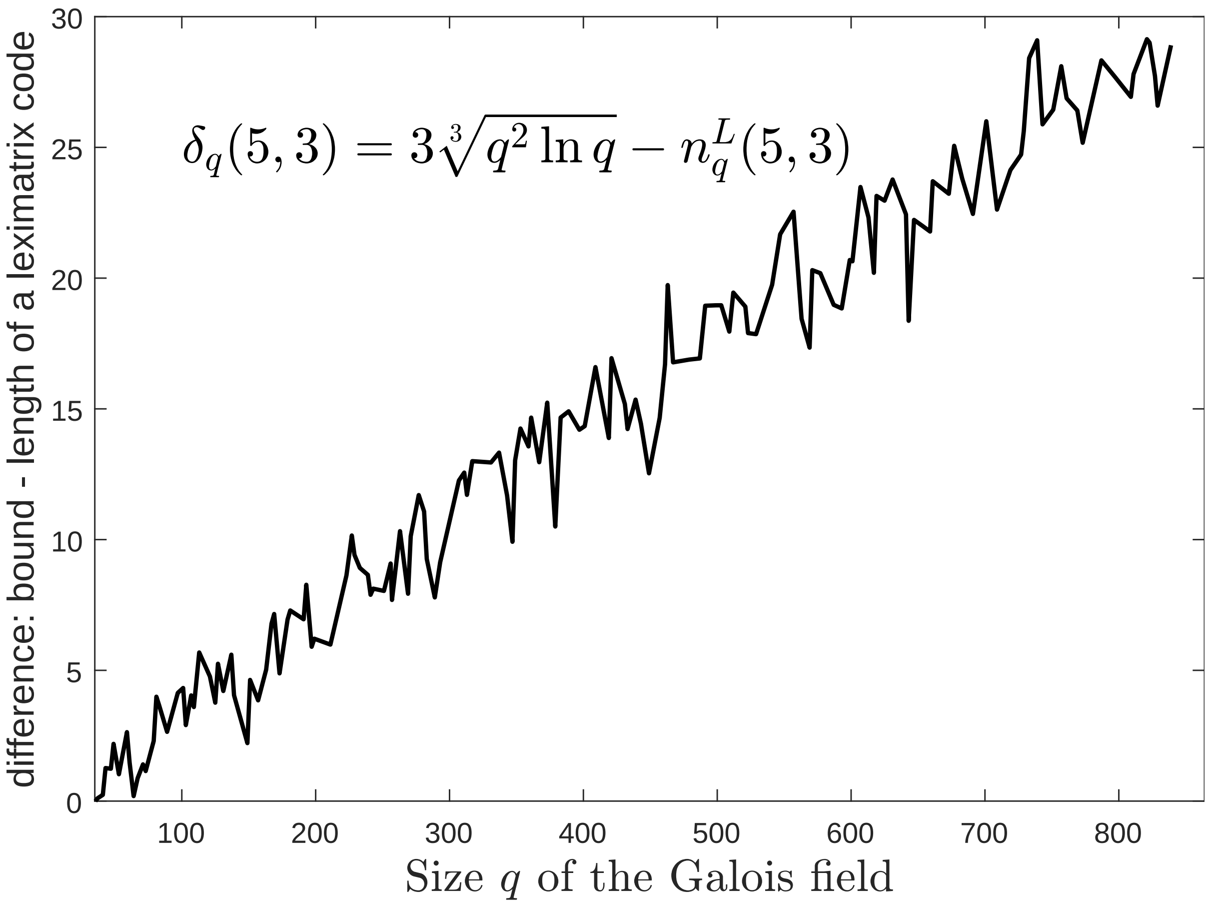

Figure 8: Difference $\delta_q(5,3)$ between the lexi-bound $3\sqrt[3]{q^2\ln q}$ and length $n_q^{\mathrm{L}}(5,3)$ of an $[n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{L}}(5,3)-5,5]_q3$ leximatrix code; $37 \le q \le 839$

By (3.4), we represent length of an $[n_q^L(5,3), n_q^L(5,3)-5,5]_q3$ leximatrix code in the

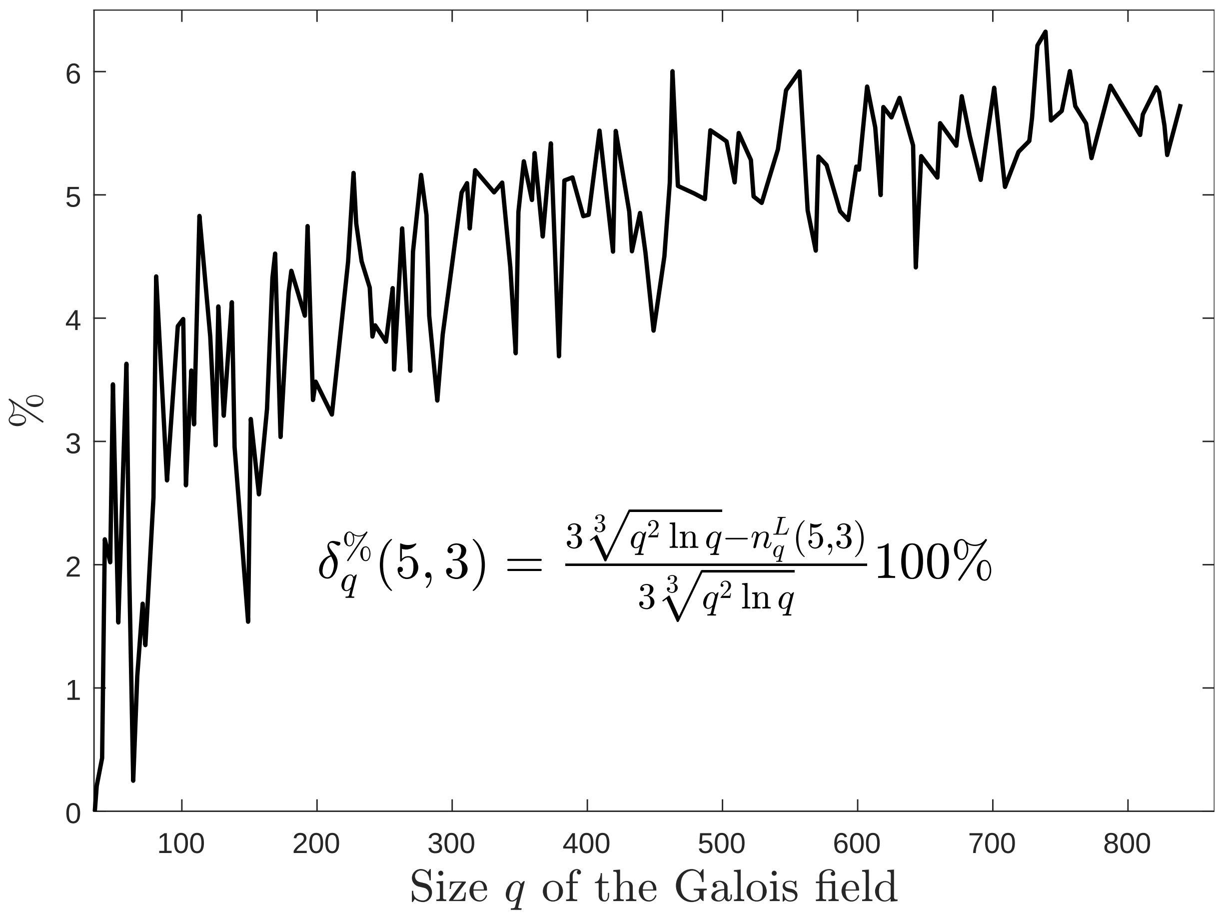

Figure 9: Percent difference $\delta_q^{\%}(5,3) = \frac{3\sqrt[3]{q^2\ln q} - n_q^{\mathrm{L}}(5,3)}{3\sqrt[3]{q^2\ln q}} 100\%$ between the lexi-bound $3\sqrt[3]{q^2\ln q}$ and length $n_q^{\mathrm{L}}(5,3)$ of an $[n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{L}}(5,3)-5,5]_q3$ leximatrix code; $37 \le q \le 839$

form

$$n_q^{\mathrm{L}}(5,3) = c_q^{\mathrm{L}}(5,3)\sqrt[3]{q^2\ln q},$$

where $c_q^{\mathrm{L}}(5,3)$ is a coefficient entirely given by q (if q is prime) or by q and the primitive polynomial of the field $\mathbb{F}_q$ (if q is non-prime). The coefficients $c_q^{\mathrm{L}}(5,3) = n_q^{\mathrm{L}}(5,3)/\sqrt[3]{q^2\ln q}$ are shown in Figure 10.

Observation 5.2. **(i)** *The difference* $\delta_q(5,3)$ *tends to increase when* q *grows, see Figures* 7 *and* 8.

(ii) *The percent difference* $\delta_q^{\%}(5,3)$ *tends to increase when* q *grows, see Figure* 9.

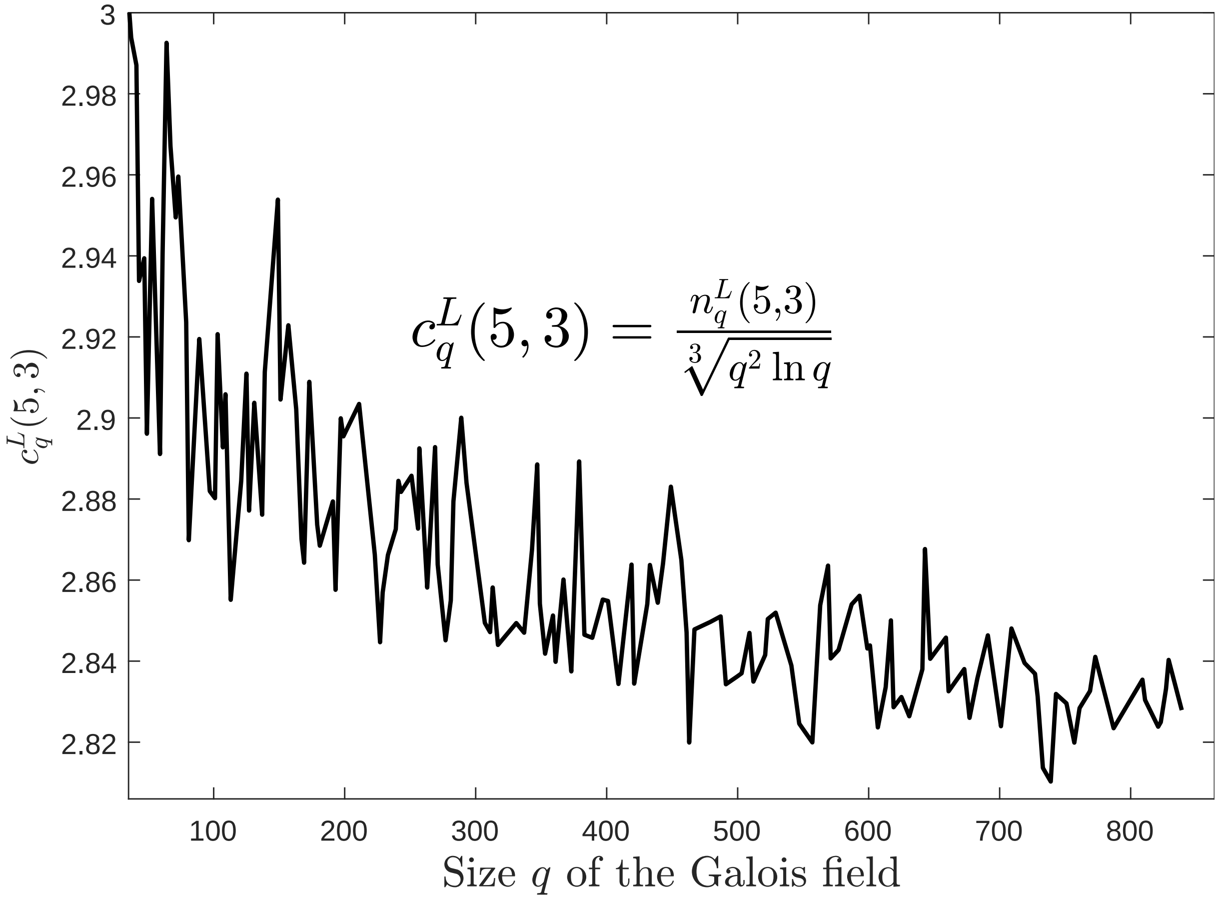

Figure 10: Coefficients $c_q^{\mathrm{L}}(5,3) = n_q^{\mathrm{L}}(5,3)/\sqrt[3]{q^2 \ln q}$ for the $[n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{L}}(5,3) - 5, 5]_q 3$ leximatrix quasi-perfect Almost MDS codes; $37 \leq q \leq 839$

(iii) *Coefficients* $c_q^L(5,3)$ *tend to decrease when* q *grows, see Figure* 10.

Observation 5.2 gives rise to Conjecture 2.2(ii) on the length function $\ell_q(5,3)$ and the d-length function $\ell_q(5,3,5)$.

Note that Observations 5.2(ii) and 5.2(iii) directly follow each from other. Actually,

$$\delta_q^{\%}(5,3) = \frac{3\sqrt[3]{q^2 \ln q} - n_q^{\mathrm{L}}(5,3)}{3\sqrt[3]{q^2 \ln q}} 100\% = \left(1 - \frac{c_q^{\mathrm{L}}(5,3)}{3}\right) 100\%.$$

Proposition 5.3. *There exist* $[n_q^L(5,3), n_q^L(5,3) - 5, 5]_q 3$ *quasi-perfect Almost MDS leximatrix codes of covering density* $\mu_q^L(5,3) < 4.2 \cdot \ln q$ *for* $37 \leq q \leq 839$.

Proof. The needed codes are the codes of Proposition 5.1. □

Proposition 5.3 implies the assertion of Theorem 2.3 on the upper ***density lexi-bound*** on covering density $\mu_q(5,3)$.

Covering densities $\mu_q^{\mathrm{L}}(5,3)$ of the $[n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{L}}(5,3)-5,5]_q3$ leximatrix quasi-perfect Almost MDS codes are presented in Figure 11 by the bottom solid black curve. The values $\mu_q^{\mathrm{L}}(5,3)$ are obtained by (3.5) where lengths $n_q^{\mathrm{L}}(5,3)$ are taken from Table 2 (see Appendix). The bound

$$\mu_q^{\mathrm{L}}(5,3) < 4.2 \cdot \ln q,$$

called the ***density lexi-bound***, is shown in Figure 11 by the top dashed red curve.

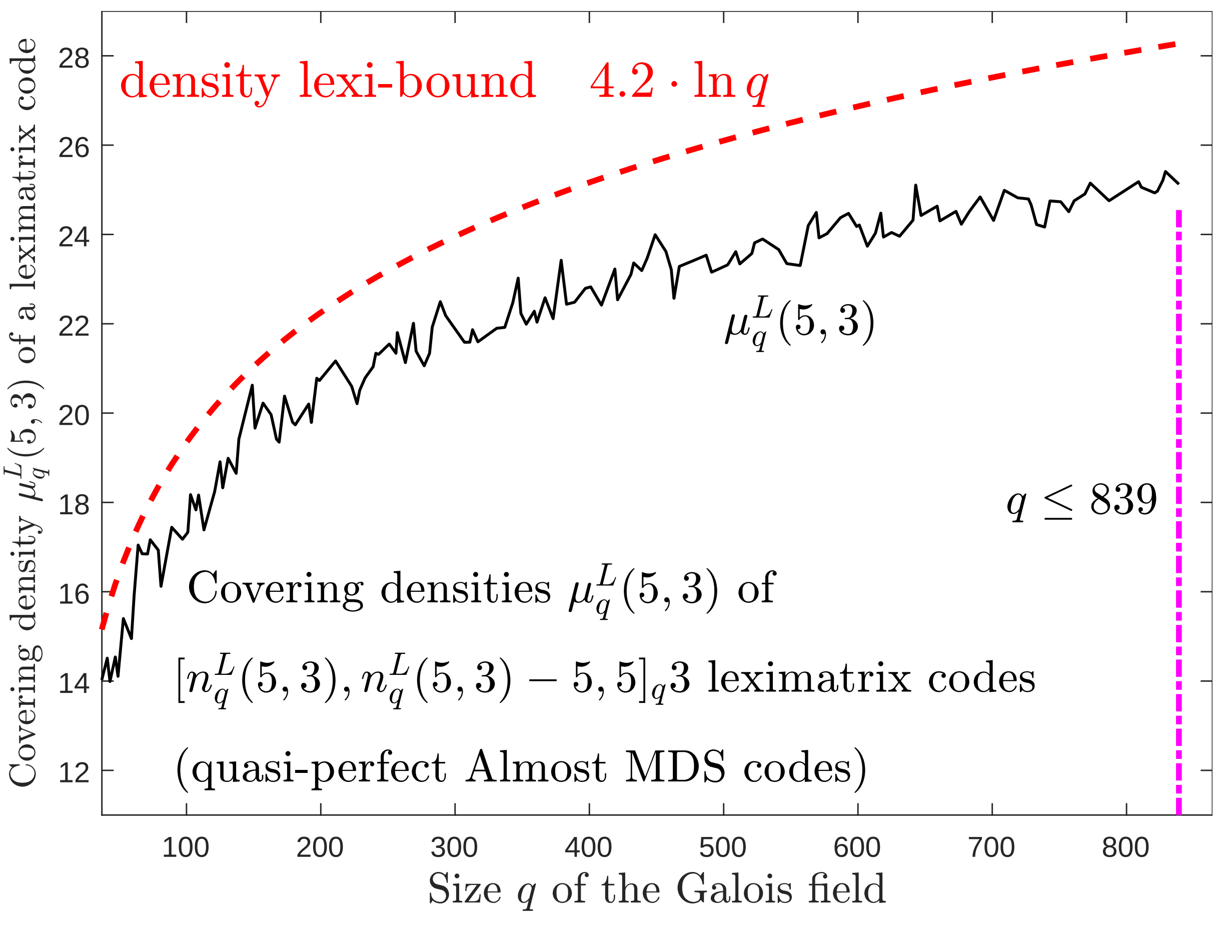

Figure 11: Covering densities $\mu_q^{\mathrm{L}}(5,3)$ of the $[n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{L}}(5,3)-5,5]_q3$ leximatrix quasi-perfect Almost MDS codes (*bottom solid black curve*) vs the density lexi-bound $4.2 \cdot \ln q$ (*top dashed red curve*); $37 \leq q \leq 839$. *Vertical magenta line* marks region $q \leq 839$

By (3.6), we represent covering density of an $[n_q^L(5,3), n_q^L(5,3)-5,5]_q3$ leximatrix code in the form

$$\mu_q^{\mathrm{L}}(5,3) = m_q^{\mathrm{L}}(5,3) \cdot \ln q,$$

where $m_q^{\mathrm{L}}(5,3)$ is a coefficient entirely given by q (if q is prime) or by q and the primitive polynomial of the field $\mathbb{F}_q$ (if q is non-prime). The coefficients $m_q^{\mathrm{L}}(5,3) = \frac{\mu_q^{\mathrm{L}}(5,3)}{\ln q}$ are shown in Figure 12.

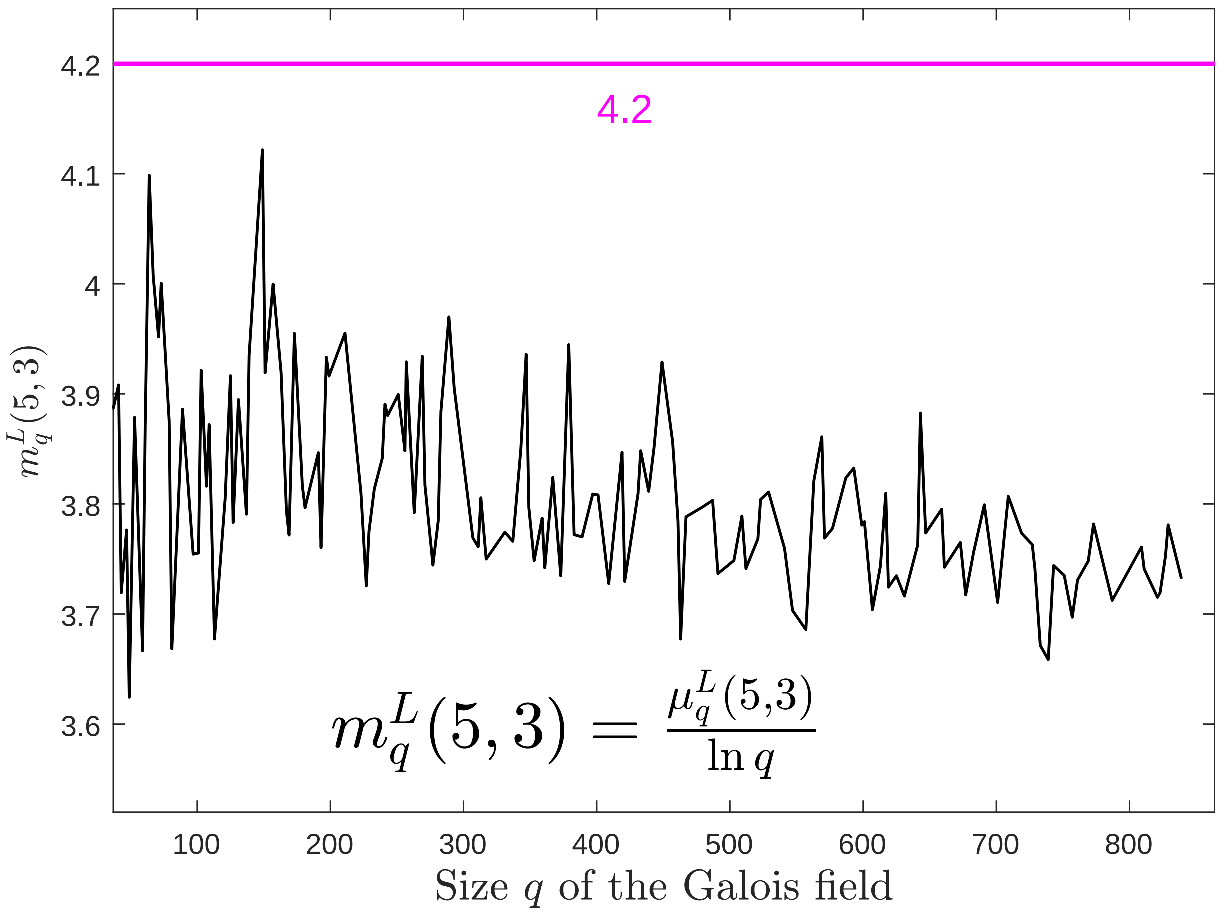

Figure 12: Coefficients $m_q^{\mathrm{L}}(5,3) = \mu_q^{\mathrm{L}}(5,3)/\ln q$ for covering density of the $[n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{L}}(5,3) - 5, 5]_q 3$ leximatrix quasi-perfect Almost MDS codes; $37 \le q \le 839$

Observation 5.4. **(i)** *The difference* $4.2 \cdot \ln q - \mu_q^L(5,3)$ *tends to increase when* q *grows, see Figure* 11.

(ii) *Coefficients* $m_q^L(5,3)$ *tend to decrease when* q *grows, see Figure* 12.

6 An inverse leximatrix algorithm to obtain parity check matrices of covering codes

An inverse leximatrix algorithm is a modification of the leximatrix algorihm of Section 3.

Let $\mathbb{F}_q = \{0, 1, \ldots, q-1\}$ be the Galois field with q elements.

If q is prime, the elements of $\mathbb{F}_q$ are treated as integers modulo q.

If $q = p^m$ with p prime and $m \geq 2$, the elements of $\mathbb{F}_{p^m}$ are represented by integers as follows: $\mathbb{F}_{p^m} = \mathbb{F}_q = \{0, 1 = \alpha^0, 2 = \alpha^1, \ldots, u = \alpha^{u-1}, \ldots, q-1 = \alpha^{q-2}\}$, where α is a root of a primitive polynomial of $\mathbb{F}_{p^m}$.

For a q-ary code of codimension r, covering radius R, and minimum distance $d = R+2$, we construct a parity check matrix from nonzero columns h_i of the form

$$h_i = (x_1^{(i)}, x_2^{(i)}, \ldots, x_r^{(i)})^{tr},\ x_u^{(i)} \in \mathbb{F}_q, \tag{6.1}$$

where the first (leftmost) non-zero element is 1; tr is the sign of transposition. The number of distinct columns is $(q^r - 1)/(q-1)$. We order the columns in the list as

$$h_1, h_2, \ldots, h_{(q^r-1)/(q-1)}. \tag{6.2}$$

One sees that the forms of the columns of a parity check matrix in (3.1) and (6.1) coincide with each other. Also, external view of the list of the columns in (3.2) and (6.2) is the same. However, contrary to (3.3), we represent a number i of a column h_i as follows:

$$i = \frac{q^r - 1}{q - 1} - \sum_{u=1}^{r} x_u^{(i)} q^{r-u}. \tag{6.3}$$

We call the ***order of the columns*** corresponding to (6.3) ***the inverse lexicographical order***.

Apart the fixed order of columns (cf. (3.3) and (6.3)), the inverse leximatrix algorithm is similar to the leximatrix algorithm.

The first column of the list should be included into the matrix. Then step-by-step, one takes the next column from the list which cannot be represented as a linear combination of at most R columns already chosen. The process ends when no new column may be included into the matrix. The obtained matrix H_n is a parity check matrix of an $[n, n-r, R+2]_q R$ code.

The obtained parity check matrix is called the ***parity check invleximatrix*** or the ***invleximatrix*** for short. We call an ***invleximatrix code*** the corresponding code.

For prime q, the following holds: **length** n **of an invleximatrix code and the form of the invleximatrix** H_n **depend on** q**,** r**, and** R **only**. No other factors affect code length and structure. Actually, assume that after some step a current matrix is obtained. At the next step we should remove from our current list all columns that are linear combination of R or less columns of the current matrix. For prime q and the given r and R, the result of removing is unequivocal; hence, the next column is taken uniquely.

For non-prime q, the length n of an invleximatrix code depends on q and on the primitive polynomial of the field. In this paper, we use primitive polynomials that are created by the program system MAGMA [6] by default, see Table A. In any case, the choice of the polynomial changes the invleximatrix code length unessentially.

By the invleximatrix algorithm, if $R = 1$, we obtain the q-ary Hamming code. If $R = 2$, we obtain a quasi-perfect $[n, n-r, 4]_q 2$ code; for $r = 3$ such code is an MDS code and corresponds to a complete arc in $\mathrm{PG}(2, q)$. If $R = 3$, we obtain a quasi-perfect $[n, n-r, 5]_q 3$ code; for $r = 4$ such code is an MDS code and corresponds to a complete arc in $\mathrm{PG}(3, q)$; for $r = 5$ it is an Almost MDS code.

Let $n_q^{\mathrm{IL}}(r, R)$ be **length of the q-ary invleximatrix code of codimension r and covering radius R**. It is assumed that for a non-prime field $\mathbb{F}_q$, one uses the primitive polynomial created by the program system MAGMA [6] by default; in particular, for non-prime $q \le 6889$, the polynomial from Table A should be taken.

We represent length of an $[n_q^{\mathrm{IL}}(r, R), n_q^{\mathrm{IL}}(r, R) - r, R + 2]_q R$ invleximatrix code in the form

$$n_q^{\mathrm{IL}}(r, R) = c_q^{\mathrm{IL}}(r, R) \sqrt[R]{\ln q} \cdot q^{(r-R)/r}, \tag{6.4}$$

where $c_q^{\mathrm{IL}}(r, R)$ is a coefficient entirely given by r, R, q (if q is prime) or by r, R, q, and the primitive polynomial of $\mathbb{F}_q$ (if q is non-prime).

Proposition 6.1. *There exist $[n_q^{\mathrm{IL}}(4, 3), n_q^{\mathrm{IL}}(4, 3) - 4, 5]_q 3$ quasi-perfect MDS invleximatrix codes of length $n_q^{\mathrm{IL}}(4, 3) < 2.8\sqrt[3]{q \ln q}$ for $127 \le q \le 6101$ and $q = 6143, 6217, 6287, 6299, 6529, 6563$.*

Proof. The needed codes are obtained by computer search, using the inverse leximatrix algorithm. □

Proposition 6.1 as well as Proposition 4.3 implies the assertions of Theorem 2.1(1i) on the upper ***lexi-bound*** on the length function $\ell_q(4, 3)$ and the d-length function $\ell_q(4, 3, 5)$.

Lengths of the $[n_q^{\mathrm{IL}}(4, 3), n_q^{\mathrm{IL}}(4, 3) - 4, 5]_q 3$ invleximatrix quasi-perfect MDS codes are collected in Table 3 (see Appendix). The cases

$$n_q^{\mathrm{IL}}(4, 3) < n_q^{\mathrm{L}}(4, 3)$$

are noted in Table 3 in ***bold italic*** font.

We have relatively ***many the cases*** $n_q^{\mathbf{IL}}(4, 3) < n_q^{\mathbf{L}}(4, 3)$***; this strengthens our assurance in truth of Conjecture* 2.2(i)**.

7 Randomized greedy algorithms to obtain parity check matrices of covering codes

7.1 Randomized greedy algorithms

Randomized greedy algorithms are described (in geometrical language) in [1–3], see also the references therein.

In every step a randomized greedy algorithm maximizes an objective function f but some steps are executed in a random manner. The number of these steps, their ordinal numbers, and some other parameters of the algorithm have been taken intuitively. Also, if the same maximum of f can be obtained in distinct ways, one way is chosen randomly.

We begin to construct a parity check matrix of an $[n, n-r]_q R$ code by using a starting matrix H_0. In the i-th step one column is added to the current matrix H_{i-1} and we obtain a matrix H_i. We say that an r-dimensional ***column is R-covered*** if it can be represented as linear combination at most R columns of the current parity check matrix. As the value of the objective function f we consider ***the number of R-covered columns***.

On every "random" i-th step we take $d_{q,i}$ *randomly chosen columns* of F_q^r *not covered by* H_{i-1} and compute the objective function f adding each of these $d_{q,i}$ columns to H_{i-1}. The column providing the maximum of f is included into H_i. On every "non-random" j-th step we consider *all columns not covered by* H_{j-1} and add to H_{j-1} the column providing the maximum of f.

As H_0 we can use a matrix obtained in previous stages of the search.

A generator of random numbers is used for a random choice. To get codes with distinct lengths, the starting conditions of the generator are changed for the same matrix H_0. In this way the algorithm works in a convenient limited region of the search space to obtain examples decreasing the size of the matrix from which the fixed starting submatrix have been taken.

To obtain codes with new lengths, sufficiently many attempts should be made with randomized greedy algorithms. "Predicted" lengths could be useful for understanding if a good result has been obtained. If the result is not close to the predicted size, the attempts are continued.

We consider the following two versions of the randomized greedy algorithms:

• ***Rand-Greedy algorithm.*** In this version, one does not take into account if a new column is R-covered. Therefore, the constructed code has minimum distance $d = 3$.

• ***d-Rand-Greedy algorithm.*** In this version, we chose a *new column from columns that are not R-covered*. Therefore, minimum distance of the obtained code is $d = R + 2$.

The randomized greedy algorithms give better results than the leximatrix and inverse leximatrix algorithms but the randomized greedy algorithms take essentially greater computer time.

Lengths of codes obtained by randomized greedy algorithms depend of many factors connected with parameters of the algorithms.

Let $n_q^{\mathrm{G}}(r,R)$ be **length of a q-ary code of codimension r and covering radius R obtained by the Rand-Greedy algorithm**.

We represent length of an $[n_q^{\mathrm{G}}(r,R), n_q^{\mathrm{G}}(r,R)-r,3]_qR$ code obtained by the Rand-Greedy algorithm in the form

$$n_q^{\mathrm{G}}(r,R) = c_q^{\mathrm{G}}(r,R)\sqrt[R]{\ln q}\cdot q^{(r-R)/r},$$

where $c_q^{\mathrm{G}}(r,R)$ is a coefficient dependent on parameters of the Rand-Greedy algorithm.

Let $n_q^{d\mathrm{G}}(r,R)$ be **length of a q-ary code of codimension r and covering radius R obtained by the d-Rand-Greedy algorithm**.

We represent length of an $[n_q^{d\mathrm{G}}(r,R), n_q^{d\mathrm{G}}(r,R)-r,R+2]_qR$ code obtained by the d-Rand-Greedy algorithm in the form

$$n_q^{d\mathrm{G}}(r,R) = c_q^{d\mathrm{G}}(r,R)\sqrt[R]{\ln q}\cdot q^{(r-R)/r},$$

where $c_q^{d\mathrm{G}}(r,R)$ is a coefficient dependent on parameters of the d-Rand-Greedy algorithm.

Let $\overline{n}_q(r,R)$ **be length of the shortest *known* q-ary code of codimension r and covering radius R.**

Let $\overline{n}_q(r,R,d)$ **be length of the shortest *known* q-ary code of codimension r, covering radius R, and minimum distance d.**

Clearly,

$$\overline{n}_q(r,R) \le \overline{n}_q(r,R,d).$$

We represent length $\overline{n}_q(r,R,d)$ in the form

$$\overline{n}_q(r,R,d) = \overline{c}_q(r,R,d)\sqrt[R]{\ln q}\cdot q^{(r-R)/R},$$

where $\overline{c}_q(r,R,d)$ is a coefficient.

7.2 The shortest known $[\overline{n}_q(4,3,5), \overline{n}_q(4,3,5)-4,5]_q3$ and $[\overline{n}_q(4,3), \overline{n}_q(4,3)-4]_q3$ codes

For $2 \le q \le 6607$, lengths $\overline{n}_q(4,3,5)$ of the ***shortest known*** $[\overline{n}_q(4,3,5), \overline{n}_q(4,3,5)-4,5]_q3$ quasi-perfect MDS codes, obtained by the leximatrix, inverse leximatrix, and d-Rand-Greedy algorithms, are as follows

$$\overline{n}_q(4,3,5) = \min\{n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{IL}}(4,3), n_q^{d\mathrm{G}}(4,3)\}. \quad (7.1)$$

Proposition 7.1. *There exist* $[\overline{n}_q(4,3,5), \overline{n}_q(4,3,5)-4,5]_q3$ *quasi-perfect MDS codes of length*

$$\overline{n}_q(4,3,5) < \begin{cases} 2.61\sqrt[3]{q\ln q} & \text{if} \quad 13 \le q \le 4373 \\ 2.65\sqrt[3]{q\ln q} & \text{if} \quad 4373 < q \le 6607 \end{cases}.$$

Proof. The needed codes are obtained by computer search, using the approach of (7.1). To obtain codes with $q \le 4451$ we used the d-Rand-Greedy algorithm. For $4451 < q \le 6607$ we used, in preference, the leximatrix and inverse leximatrix algorithms, see Sections 5 and 6, but for $q = 4489, 4679, 4877, 4889, 4913, 5801$ we applied the d-Rand-Greedy algorithm. For $q = 841$, the complete 42-arc of [33] is used. □

Proposition 7.1 implies the assertions of Theorem 2.1(2) on upper bounds on the length function $\ell_q(4,3)$ and the d-length function $\ell_q(4,3,5)$.

Proposition 7.1 ***improves the lexi-bound of Theorem 2.1(1i); this strengthens our assurance in truth of Conjecture 2.2(i)***.

The lengths $\overline{n}_q(4,3)$ of the shortest known $[\overline{n}_q(4,3), \overline{n}_q(4,3)-4]_q3$ codes we obtain using results of computer search for $\overline{n}_q(4,3,5)$, data from [20, Tab. 1], the Rand-Greedy algorithm, and formula (1.4) for $q = (q')^3$ where (q') is a prime power.

The ***smallest known lengths*** $\overline{n}_q(4,3)$ are given in Table 4 (see Appendix) where the cases

$$\overline{n}_q(4,3,5) = \overline{n}_q(4,3) + j$$

are noted by the superscript "$+j$". For the rest of q we have $\overline{n}_q(4,3,5) = \overline{n}_q(4,3)$. So, in fact, *Table 4 gives also* ***smallest known lengths*** $\overline{n}_q(4,3,5)$.

Note, that in Table 4, the improvements of code distance up to $d = 5$ in comparison with [20, Tab. 1] are noted in bold italic font. Also, in Table 4, the cases $\ell_q(4,3) = \overline{n}_q(4,3)$ are noted by the subscript "•", see [20, Tab. 1].

Coefficients $\overline{c}_q(4,3,5)$ corresponding to the codes of Table 4 (taking into account the superscripts "$+j$") are shown in Figure 13.

7.3 The shortest known $[\overline{n}_q(5,3), \overline{n}_q(5,3)-5]_q3$ codes

For $3 \le q \le 839$, lengths $\overline{n}_q(5,3)$ of the ***shortest known*** $[\overline{n}_q(5,3), \overline{n}_q(5,3)-5,3]_q3$ codes, obtained by the leximatrix and Rand-Greedy algorithms, are as follows

$$\begin{aligned} &\overline{n}_q(5,3) = \min\{n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{G}}(5,3)\} \text{ if } 11 \le q \le 401; \\ &\overline{n}_q(5,3) = \overline{n}_q(5,3,5) = n_q^{\mathrm{L}}(5,3) \text{ if } 401 < q \le 839. \end{aligned} \tag{7.2}$$

Proposition 7.2. *There exist* $[\overline{n}_q(5,3), \overline{n}_q(5,3)-5]_q3$ *codes of length*

$$\begin{aligned} &\overline{n}_q(5,3) < 2.785\sqrt[3]{q^2\ln q} \text{ if } 11 \le q \le 401; \\ &\overline{n}_q(5,3) = \overline{n}_q(5,3,5) < 2.884\sqrt[3]{q^2\ln q} \text{ if } 401 < q \le 839. \end{aligned}$$

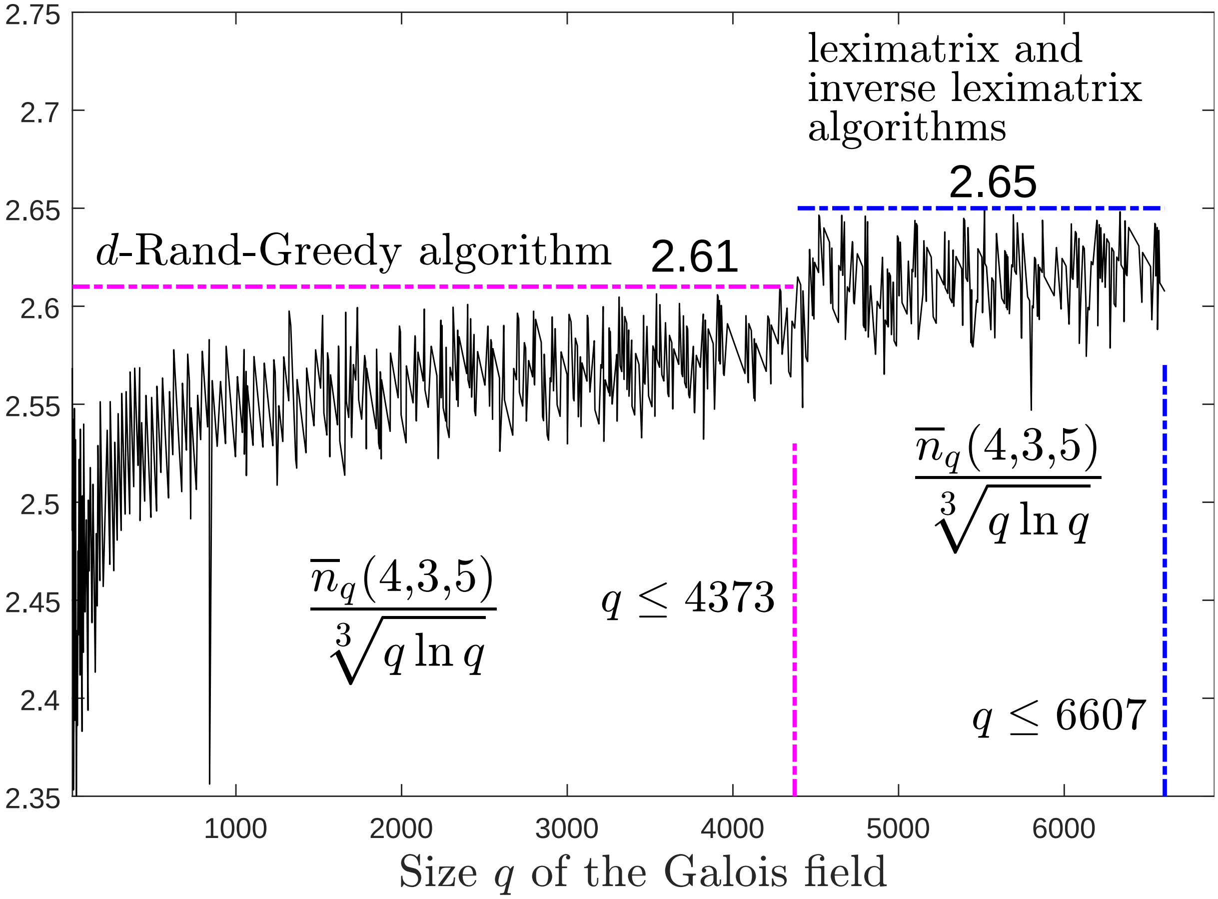

Figure 13: Coefficients $\overline{c}_q(4,3,5) = \overline{n}_q(4,3,5)/\sqrt[3]{q \ln q}$ for $[\overline{n}_q(4,3,5), \overline{n}_q(4,3,5) - 4, 5]_q3$ quasi-perfect MDS codes; $13 \leq q \leq 6607$

Proof. The needed codes are obtained by computer search, using the approach of (7.2). To obtain codes with $q \leq 401$ we used the Rand-Greedy algorithm; it gives $[n, n-5, 3]_q3$ codes with minimum distance $d = 3$. For $401 < q \leq 839$ we used the leximatrix algorithm, see Section 5. □

Proposition 7.2 implies the assertions of Theorem 2.1(3) on upper bounds on the length function $\ell_q(5,3)$ and the d-length function $\ell_q(5,3,5)$.

Proposition 7.2 ***improves the lexi-bound of Theorem 2.1(1ii); this strengthens our assurance in truth of Conjecture 2.2(ii)***.

Lengths $\overline{n}_q(5,3)$ of the ***shortest known*** $[\overline{n}_q(5,3), \overline{n}_q(5,3) - 5, 3]_q3$ codes are collected in Table 5, where the improvements of code length in comparison with [20, Tab. 2] are noted in bold italic font. Also,in Table 5, the cases $\ell_q(5,3) = \overline{n}_q(5,3)$ are noted by the subscript "$\bullet$", see [20, Tab. 2].

For $q \le 839$, coefficients $\overline{c}_q(5,3,3)$ and $\overline{c}_q(5,3,5)$ corresponding to the codes of Table 5 are shown in Figure 14.

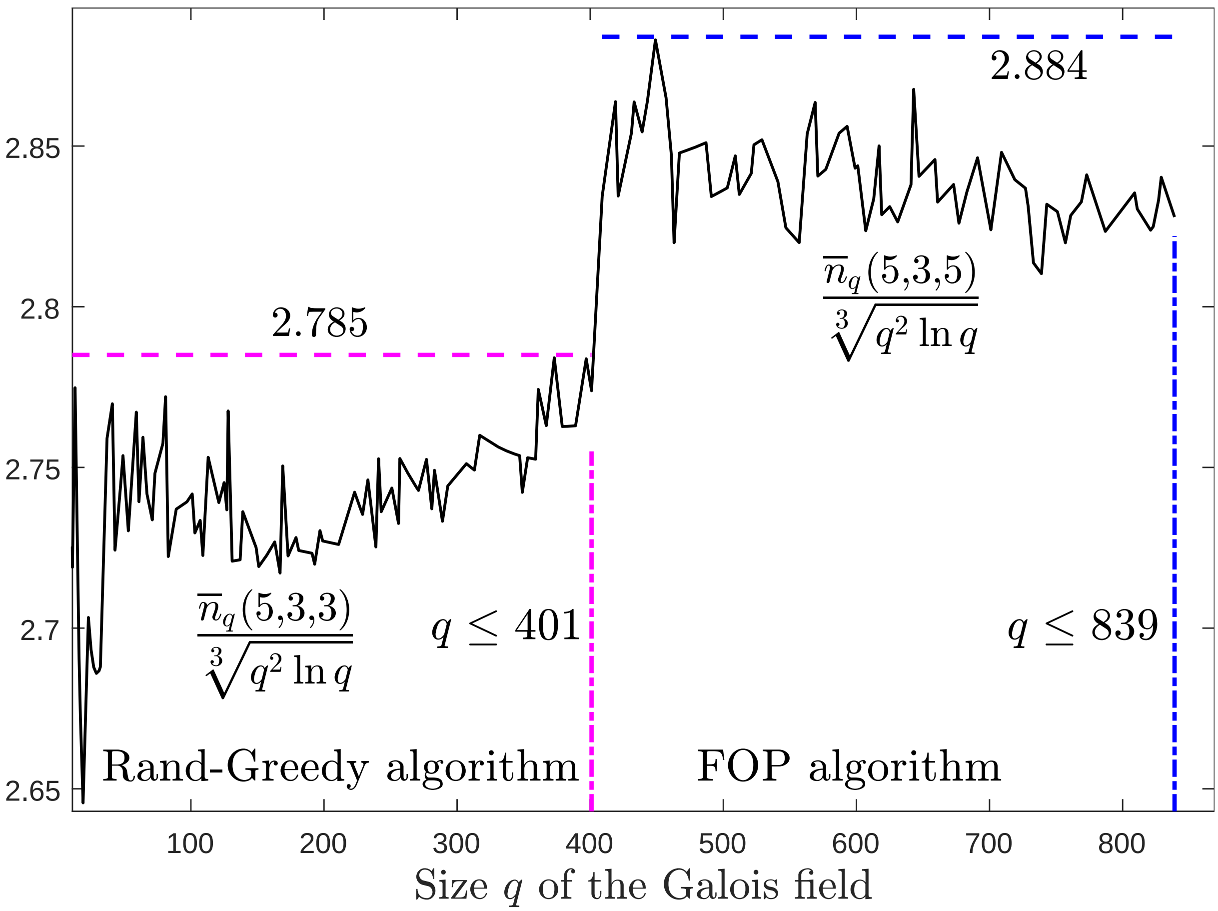

Figure 14: Coefficients $\overline{c}_q(5,3,3) = \overline{n}_q(5,3,3)/\sqrt[3]{q^2 \ln q}$ for $[\overline{n}_q(5,3,3), \overline{n}_q(5,3,3) - 5, 3]_q3$ codes, $11 \le q \le 401$; and $\overline{c}_q(5,3,5) = \overline{n}_q(5,3,5)/\sqrt[3]{q^2 \ln q}$ for $[\overline{n}_q(5,3,5), \overline{n}_q(5,3,5) - 5, 5]_q3$ Almost MDS codes, $401 < q \le 839$

8 Conclusion

The length function $\ell_q(r,R)$ is the smallest length of a q-ary linear code of covering radius R and codimension r. ***The d-length function*** $\ell_q(r,R,d)$ is the smallest length of a q-ary linear code with codimension r, covering radius R, and minimum distance d. In this paper, we consider upper bounds on the length functions $\ell_q(4,3)$, $\ell_q(5,3)$ and the d-length functions $\ell_q(4,3,5)$, $\ell_q(5,3,5)$. For $r \ne 3t$ and $q \ne (q')^3$, where q' is a prime power, upper bounds on $\ell_q(r,3)$ and $\ell_q(r,3,5)$ are open problems.

By computer search in wide regions of q, we obtained following short codes of covering radius $R = 3$: $[n, n-4, 5]_q 3$ quasi-perfect MDS codes, $[n, n-5, 5]_q 3$ quasi-perfect Almost MDS codes, and $[n, n-5, 3]_q 3$ codes.

For $r \neq 3t$ and the field basis q of an arbitrary structure, including $q \neq (q')^3$, the new codes imply upper bounds (called the ***lexi-bounds***) of the form

$$\ell_q(r, 3) < c_\ell \sqrt[3]{\ln q} \cdot q^{(r-3)/3}, \quad c_\ell \text{ is a constant independent of } q, \quad r = 4, 5 \neq 3t. \tag{8.1}$$

Also, the new codes imply the following upper bounds (called the ***density lexi-bounds***) on the smallest covering density $\mu_q(r, 3)$ of a q-ary linear code of covering radius 3 and codimension r:

$$\mu_q(r, 3) < c_\mu \cdot \ln q, \quad c_\mu \text{ is a constant independent of } q, \quad r = 4, 5 \neq 3t. \tag{8.2}$$

In comparison with upper bounds of [13–15] for the special field basis $q = (q')^3$*, bounds* (8.1) *and* (8.2) *have "extra" multipliers* $\sqrt[3]{\ln q}$ *and* $\ln q$*, respectively.* ***This is a "price" of an arbitrary structure of the field basis*** q***.***

In computer search, we use the step-by-step leximatrix and inverse leximatrix algorithms to obtain parity check matrices of codes. The algorithms are versions of the recursive g-parity check matrix algorithm for greedy codes. Also, we apply the randomized greedy algorithms.

In future, it would be useful to investigate and understand properties of the leximatrix and inverse leximatrix algorithms and structure of leximatrices and invleximatrices.

In particular, the following is of great interest:

• Initial part of the parity check leximatrices and invleximatrices, see Proposition 4.1 and Example 4.2.

• The working mechanism and its quantitative estimates for the leximatrix and inverse leximatrix algorithms; see, for instance, the papers [1, 12] where the working mechanisms of greedy algorithms for complete arcs in the projective plane $\mathrm{PG}(2, q)$ and for complete caps in the projective spaces $\mathrm{PG}(N, q)$ are studied.

• The oscillation of the coefficients $c_q^{\mathrm{L}}(4, 3)$ around a horizontal line and its likenesses with the oscillation of the values $h^{\mathrm{L}}(q)$ around a horizontal line in [2, Fig. 6, Observation 3.5], [3, Fig. 5, Observation 3.7], see Figure 4 and Remark 4.5.

It is important to emphasize that although the ***lexi-bounds*** of Theorem 2.1(1) are obtained by computer search, several factors give us insurance that these bounds ***are truth for all*** q (see Conjecture 2.2). In particular we note figures and observations in Sections 4 and 5, comparison of leximatrix and invleximatrix codes in Table 3, improvements of the lexi-bounds in Section 7.

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Appendix

Table 1. Lengths $n_q^{\mathrm{L}}(4,3)$ of the $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3)-4, 5]_q3$ leximatrix quasi-perfect MDS codes, $2 \le q \le 6607$

q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$
2	5	3	5	4	5	5	6	7	8	8	7
9	9	11	8	13	9	16	9	17	9	19	10
23	11	25	11	27	12	29	12	31	13	32	12
37	13	41	14	43	14	47	15	49	15	53	16
59	16	61	16	64	17	67	17	71	18	73	18
79	18	81	18	83	19	89	20	97	20	101	21
103	20	107	22	109	22	113	22	121	22	125	23
127	23	128	22	131	23	137	23	139	23	149	24
151	24	157	25	163	24	167	25	169	25	173	25
179	26	181	26	191	26	193	27	197	27	199	26
211	27	223	29	227	28	229	28	233	28	239	29
241	29	243	28	251	30	256	29	257	29	263	30
269	30	271	31	277	30	281	30	283	31	289	31
293	31	307	32	311	32	313	31	317	32	331	34
337	34	343	33	347	34	349	34	353	34	359	34
361	34	367	34	373	34	379	34	383	34	389	35
397	35	401	35	409	35	419	36	421	36	431	36
433	37	439	38	443	38	449	36	457	37	461	37
463	37	467	37	479	38	487	38	491	39	499	39
503	39	509	39	512	39	521	39	523	39	529	39
541	39	547	39	557	39	563	41	569	41	571	39
577	40	587	41	593	41	599	41	601	42	607	42
613	43	617	42	619	42	625	42	631	42	641	43
643	42	647	43	653	44	659	44	661	43	673	43
677	42	683	43	691	44	701	44	709	44	719	44
727	45	729	44	733	45	739	45	743	45	751	45
757	46	761	45	769	46	773	46	787	45	797	46
809	46	811	46	821	46	823	47	827	46	829	46
839	46	841	47	853	47	857	47	859	47	863	47
877	48	881	47	883	47	887	48	907	50	911	49
919	48	929	49	937	49	941	49	947	49	953	49
961	50	967	50	971	50	977	50	983	50	991	50
997	52	1009	51	1013	51	1019	51	1021	50	1024	52
1031	50	1033	51	1039	51	1049	52	1051	51	1061	51

Table 1. Continue 1

q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$
1063	51	1069	52	1087	52	1091	51	1093	52	1097	52
1103	52	1109	52	1117	52	1123	52	1129	53	1151	53
1153	53	1163	53	1171	53	1181	53	1187	54	1193	53
1201	52	1213	54	1217	55	1223	55	1229	54	1231	56
1237	56	1249	55	1259	54	1277	55	1279	56	1283	56
1289	56	1291	55	1297	56	1301	56	1303	56	1307	56
1319	56	1321	56	1327	56	1331	55	1361	57	1367	57
1369	56	1373	56	1381	57	1399	57	1409	57	1423	58
1427	58	1429	58	1433	57	1439	57	1447	57	1451	59
1453	59	1459	57	1471	57	1481	59	1483	59	1487	59
1489	59	1493	58	1499	58	1511	59	1523	58	1531	60
1543	59	1549	59	1553	59	1559	60	1567	60	1571	60
1579	59	1583	59	1597	59	1601	59	1607	60	1609	60
1613	60	1619	60	1621	60	1627	60	1637	60	1657	60
1663	61	1667	61	1669	60	1681	62	1693	61	1697	62
1699	62	1709	61	1721	63	1723	62	1733	63	1741	62
1747	63	1753	62	1759	62	1777	62	1783	63	1787	63
1789	62	1801	62	1811	63	1823	62	1831	62	1847	63
1849	64	1861	63	1867	63	1871	63	1873	64	1877	63
1879	63	1889	63	1901	64	1907	64	1913	64	1931	65
1933	66	1949	64	1951	66	1973	66	1979	65	1987	64
1993	65	1997	66	1999	65	2003	67	2011	66	2017	64
2027	65	2029	66	2039	66	2048	66	2053	66	2063	66
2069	66	2081	65	2083	66	2087	67	2089	67	2099	66
2111	67	2113	66	2129	67	2131	67	2137	68	2141	67
2143	66	2153	67	2161	67	2179	66	2187	68	2197	68
2203	67	2207	68	2209	67	2213	68	2221	69	2237	68
2239	68	2243	69	2251	69	2267	68	2269	69	2273	69
2281	69	2287	69	2293	68	2297	67	2309	69	2311	69
2333	69	2339	71	2341	69	2347	70	2351	69	2357	69
2371	70	2377	69	2381	69	2383	71	2389	69	2393	70
2399	70	2401	70	2411	71	2417	69	2423	71	2437	71
2441	73	2447	71	2459	70	2467	71	2473	72	2477	71
2503	70	2521	70	2531	71	2539	72	2543	72	2549	71
2551	71	2557	71	2579	72	2591	71	2593	72	2609	71
2617	72	2621	72	2633	73	2647	72	2657	73	2659	73
2663	72	2671	72	2677	73	2683	73	2687	72	2689	72
2693	72	2699	72	2707	73	2711	73	2713	72	2719	73

Table 1. Continue 2

q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$
2729	73	2731	74	2741	73	2749	73	2753	74	2767	73
2777	74	2789	74	2791	74	2797	73	2801	75	2803	74
2809	74	2819	74	2833	74	2837	75	2843	75	2851	75
2857	74	2861	74	2879	74	2887	76	2897	75	2903	74
2909	75	2917	75	2927	75	2939	76	2953	77	2957	76
2963	75	2969	75	2971	76	2999	76	3001	76	3011	75
3019	77	3023	76	3037	76	3041	75	3049	75	3061	76
3067	76	3079	78	3083	77	3089	76	3109	76	3119	77
3121	77	3125	78	3137	77	3163	78	3167	77	3169	77
3181	79	3187	77	3191	78	3203	77	3209	77	3217	78
3221	78	3229	77	3251	79	3253	78	3257	77	3259	78
3271	79	3299	79	3301	78	3307	78	3313	78	3319	79
3323	79	3329	80	3331	79	3343	78	3347	80	3359	78
3361	80	3371	79	3373	79	3389	80	3391	79	3407	80
3413	80	3433	80	3449	80	3457	80	3461	80	3463	80
3467	79	3469	80	3481	81	3491	80	3499	80	3511	80
3517	80	3527	80	3529	82	3533	80	3539	82	3541	80
3547	80	3557	82	3559	81	3571	81	3581	81	3583	80
3593	81	3607	83	3613	81	3617	81	3623	82	3631	81
3637	82	3643	82	3659	82	3671	83	3673	82	3677	82
3691	83	3697	83	3701	82	3709	83	3719	82	3721	82
3727	82	3733	82	3739	83	3761	82	3767	83	3769	83
3779	85	3793	83	3797	83	3803	82	3821	83	3823	82
3833	84	3847	83	3851	84	3853	82	3863	83	3877	84
3881	84	3889	83	3907	85	3911	84	3917	83	3919	83
3923	84	3929	84	3931	84	3943	84	3947	84	3967	84
3989	85	4001	85	4003	84	4007	85	4013	85	4019	86
4021	84	4027	84	4049	85	4051	86	4057	85	4073	85
4079	86	4091	85	4093	86	4096	86	4099	86	4111	86
4127	86	4129	86	4133	85	4139	86	4153	86	4157	86
4159	86	4177	87	4201	85	4211	87	4217	85	4219	87
4229	86	4231	87	4241	86	4243	86	4253	86	4259	88
4261	87	4271	86	4273	87	4283	87	4289	86	4297	87
4327	88	4337	88	4339	86	4349	89	4357	87	4363	87
4373	87	4391	87	4397	88	4409	88	4421	87	4423	90
4441	87	4447	88	4451	88	4457	87	4463	88	4481	87
4483	88	4489	89	4493	88	4507	89	4513	88	4517	88
4519	89	4523	89	4547	88	4549	90	4561	89	4567	89

Table 1. Continue 3

q	$n^{\mathrm{L}}_q(4,3)$	q	$n^{\mathrm{L}}_q(4,3)$	q	$n^{\mathrm{L}}_q(4,3)$	q	$n^{\mathrm{L}}_q(4,3)$	q	$n^{\mathrm{L}}_q(4,3)$	q	$n^{\mathrm{L}}_q(4,3)$
4583	89	4591	89	4597	89	4603	90	4621	89	4637	89
4639	89	4643	90	4649	89	4651	89	4657	90	4663	90
4673	90	4679	92	4691	90	4703	89	4721	90	4723	90
4729	90	4733	90	4751	90	4759	90	4783	90	4787	89
4789	89	4793	89	4799	91	4801	92	4813	92	4817	89
4831	92	4861	91	4871	90	4877	92	4889	92	4903	91
4909	91	4913	91	4919	90	4931	91	4933	91	4937	90
4943	91	4951	91	4957	90	4967	91	4969	91	4973	91
4987	90	4993	92	4999	92	5003	92	5009	92	5011	93
5021	91	5023	92	5039	93	5041	91	5051	91	5059	92
5077	91	5081	92	5087	92	5099	94	5101	92	5107	93
5113	94	5119	91	5147	92	5153	93	5167	94	5171	93
5179	93	5189	93	5197	93	5209	93	5227	92	5231	94
5233	93	5237	93	5261	93	5273	94	5279	95	5281	94
5297	94	5303	95	5309	94	5323	93	5329	94	5333	94
5347	94	5351	95	5381	94	5387	94	5393	95	5399	95
5407	95	5413	94	5417	94	5419	95	5431	95	5437	93
5441	94	5443	94	5449	93	5471	94	5477	94	5479	95
5483	95	5501	96	5503	95	5507	94	5519	96	5521	95
5527	96	5531	95	5557	94	5563	95	5569	95	5573	95
5581	94	5591	96	5623	97	5639	96	5641	97	5647	97
5651	97	5653	97	5657	97	5659	96	5669	96	5683	98
5689	96	5693	97	5701	96	5711	96	5717	97	5737	96
5741	95	5743	97	5749	97	5779	96	5783	96	5791	97
5801	98	5807	96	5813	97	5821	97	5827	97	5839	98
5843	97	5849	96	5851	97	5857	97	5861	97	5867	97
5869	98	5879	97	5881	98	5897	97	5903	97	5923	97
5927	97	5939	98	5953	98	5981	98	5987	100	6007	98
6011	98	6029	97	6037	98	6043	99	6047	98	6053	99
6067	99	6073	99	6079	98	6089	99	6091	98	6101	98
6113	99	6121	99	6131	98	6133	97	6143	100	6151	98
6163	99	6173	99	6197	100	6199	100	6203	98	6211	100
6217	101	6221	100	6229	99	6241	100	6247	99	6257	100
6263	100	6269	100	6271	100	6277	98	6287	101	6299	101
6301	99	6311	99	6317	100	6323	100	6329	100	6337	101
6343	100	6353	100	6359	100	6361	99	6367	101	6373	100
6379	100	6389	101	6397	101	6421	101	6427	101	6449	101
6451	101	6469	100	6473	101	6481	101	6491	101	6521	101
6529	103	6547	102	6551	102	6553	101	6561	102	6563	103
6569	101	6571	102	6577	101	6581	101	6599	101	6607	101

Table 2. Lengths $n_q^{\mathrm{L}}(5,3)$ of the $[n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{L}}(5,3)-5,5]_q3$ leximatrix quasi-perfect Almost MDS codes, $3 \le q \le 839$

q	$n_q^{\mathrm{L}}(5,3)$	q	$n_q^{\mathrm{L}}(5,3)$	q	$n_q^{\mathrm{L}}(5,3)$	q	$n_q^{\mathrm{L}}(5,3)$	q	$n_q^{\mathrm{L}}(5,3)$	q	$n_q^{\mathrm{L}}(5,3)$	q	$n_q^{\mathrm{L}}(5,3)$
3	11	4	10	5	11	7	16	8	17	9	19	11	22
13	24	16	28	17	28	19	31	23	36	25	37	27	40
29	43	31	46	32	46	37	51	41	55	43	56	47	60
49	61	53	66	59	70	61	73	64	77	67	79	71	82
73	84	79	88	81	88	83	90	89	96	97	101	101	104
103	107	107	109	109	111	113	112	121	119	125	123	127	123
128	124	131	127	137	130	139	133	149	142	151	141	157	146
163	149	167	150	169	151	173	156	179	158	181	159	191	166
193	166	197	171	199	172	211	180	223	185	227	186	229	188
233	191	239	195	241	197	243	198	251	203	256	205	257	207
263	208	269	214	271	213	277	215	281	218	283	221	289	226
293	227	307	232	311	234	313	236	317	237	331	245	337	248
343	253	347	257	349	255	353	256	359	260	361	260	367	265
373	266	379	274	383	272	389	275	397	280	401	282	409	284
419	292	421	290	431	297	433	299	439	301	443	304	449	309
457	311	461	311	463	309	467	314	479	320	487	324	491	324
499	328	503	330	509	334	512	334	521	339	523	341	529	344
541	348	547	349	557	353	563	360	569	364	571	362	577	365
587	371	593	374	599	375	601	376	607	376	613	380	617	384
619	382	625	385	631	387	641	393	643	398	647	396	653	399
659	402	661	401	673	407	677	407	683	411	691	416	701	417
709	424	719	427	727	430	729	430	733	429	739	431	743	436
751	439	757	440	761	443	769	447	773	450	787	453	797	458
809	464	811	464	821	467	823	468	827	471	829	473	839	475

Table 3. Lengths $n_q^{\mathrm{IL}}(4,3)$ of the $[n_q^{\mathrm{IL}}(4,3), n_q^{\mathrm{IL}}(4,3)-4,5]_q3$ invleximatrix quasi-perfect MDS codes, $7 \le q \le 6101$ and $q = 6143, 6217, 6287, 6299, 6529, 6563$. The cases $n_q^{\mathrm{IL}}(4,3) < n_q^{\mathrm{L}}(4,3)$ are noted in bold italic font

q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$
7	8	8	7	***9***	***8***	11	9	13	10	16	10
17	11	19	10	23	12	25	11	27	12	29	13
31	13	32	12	37	14	41	14	43	14	47	15
49	15	53	16	59	16	61	17	64	17	67	17
71	***17***	73	18	79	19	81	19	83	19	***89***	***19***
97	21	***101***	***20***	103	21	***107***	***21***	***109***	***21***	113	23
121	22	***125***	***22***	127	23	128	23	***131***	***22***	137	23
139	23	149	24	151	24	***157***	***24***	163	25	167	25
169	***24***	173	26	179	26	***181***	***25***	191	27	***193***	***26***
197	28	199	26	211	28	***223***	***28***	227	29	229	29
233	28	239	29	241	29	243	29	251	30	256	29
257	30	263	31	269	30	***271***	***30***	277	31	281	30
283	31	289	31	293	32	***307***	***31***	311	32	313	32
317	32	***331***	***32***	***337***	***32***	343	34	***347***	***33***	349	34
353	34	359	34	361	34	367	34	373	35	379	35
383	35	389	35	397	35	401	35	409	36	419	36
421	36	431	36	433	37	***439***	***37***	***443***	***37***	449	37
457	37	***461***	***36***	463	38	467	38	***479***	***37***	487	38
491	***38***	***499***	***38***	***503***	***37***	509	39	512	39	521	40
523	39	529	40	541	40	547	40	557	40	***563***	***40***
569	41	571	40	577	41	***587***	***40***	593	41	599	41
601	42	607	43	613	43	***617***	***41***	619	42	***625***	***41***
631	42	641	43	643	42	***647***	***42***	***653***	***43***	***659***	***43***
661	43	***673***	***42***	677	44	683	43	691	45	***701***	***43***
709	***43***	719	44	***727***	***44***	729	44	733	45	***739***	***44***
743	46	***751***	***44***	***757***	***45***	761	45	***769***	***45***	***773***	***44***
787	45	797	46	809	47	811	47	821	46	823	48
827	47	829	47	839	47	***841***	***46***	853	48	857	48
859	48	863	48	877	48	881	49	883	48	887	48
907	***49***	***911***	***48***	919	48	929	49	937	49	941	49
947	50	953	49	961	50	***967***	***49***	***971***	***49***	977	50
983	50	991	50	***997***	***50***	1009	51	***1013***	***50***	***1019***	***50***
1021	51	***1024***	***50***	1031	51	1033	51	1039	52	***1049***	***51***
1051	51	1061	51	1063	52	***1069***	***51***	1087	52	1091	52
1093	52	1097	52	1103	52	1109	52	1117	53	1123	52
1129	***52***	***1151***	***52***	1153	54	1163	53	1171	53	1181	53
1187	***53***	1193	54	1201	54	1213	54	***1217***	***54***	***1223***	***54***

Table 3. Continue 1

q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$
1229	54	***1231***	***53***	***1237***	***54***	***1249***	***54***	1259	55	***1277***	***54***
1279	***55***	***1283***	***55***	***1289***	***54***	1291	56	1297	57	***1301***	***55***
1303	***55***	1307	56	***1319***	***55***	1321	57	1327	56	1331	56
1361	***56***	***1367***	***56***	1369	56	1373	58	1381	57	***1399***	***56***
1409	***56***	***1423***	***57***	1427	59	1429	58	1433	57	1439	57
1447	58	***1451***	***58***	***1453***	***58***	1459	58	1471	59	1481	59
1483	59	1487	59	1489	59	1493	58	1499	59	1511	60
1523	59	***1531***	***59***	1543	60	***1549***	***58***	1553	59	1559	60
1567	60	1571	60	1579	60	1583	59	1597	60	1601	60
1607	60	***1609***	***59***	1613	60	1619	61	1621	60	1627	61
1637	60	1657	61	***1663***	***60***	***1667***	***60***	1669	61	***1681***	***61***
1693	***60***	***1697***	***61***	***1699***	***61***	1709	62	1721	63	1723	62
1733	***62***	1741	62	1747	63	1753	62	1759	63	1777	62
1783	***62***	***1787***	***62***	1789	62	1801	63	1811	64	1823	63
1831	63	1847	63	***1849***	***63***	1861	63	1867	64	1871	63
1873	***63***	1877	64	1879	63	1889	63	1901	64	1907	64
1913	64	***1931***	***64***	***1933***	***65***	1949	64	***1951***	***64***	***1973***	***64***
1979	***64***	1987	66	1993	66	***1997***	***65***	1999	66	***2003***	***66***
2011	***65***	2017	66	2027	67	2029	66	2039	66	2048	67
2053	66	2063	67	***2069***	***65***	2081	66	2083	66	***2087***	***66***
2089	***66***	2099	67	***2111***	***66***	2113	66	2129	67	2131	68
2137	***66***	2141	68	2143	67	2153	67	***2161***	***66***	2179	68
2187	68	***2197***	***66***	2203	68	***2207***	***67***	2209	69	***2213***	***67***
2221	69	2237	68	2239	69	***2243***	***68***	***2251***	***68***	***2267***	***67***
2269	***68***	***2273***	***68***	2281	70	2287	69	2293	68	2297	69
2309	70	2311	69	2333	69	***2339***	***68***	2341	70	2347	70
2351	***68***	2357	70	***2371***	***69***	2377	70	2381	69	***2383***	***70***
2389	70	2393	70	***2399***	***69***	2401	70	2411	71	2417	71
2423	***70***	***2437***	***70***	***2441***	***70***	2447	71	2459	71	2467	71
2473	***70***	***2477***	***70***	2503	70	2521	72	2531	72	2539	72
2543	72	2549	71	2551	71	2557	72	2579	73	2591	72
2593	***71***	2609	72	***2617***	***71***	***2621***	***71***	***2633***	***72***	2647	74
2657	73	2659	74	2663	72	2671	73	2677	73	2683	73
2687	72	2689	73	2693	73	2699	72	2707	73	2711	74
2713	73	2719	73	2729	74	***2731***	***73***	2741	73	2749	74
2753	***73***	2767	74	***2777***	***73***	2789	74	***2791***	***72***	2797	74
2801	***74***	***2803***	***73***	2809	74	***2819***	***73***	2833	75	2837	76

Table 3. Continue 2

q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$
2843	76	***2851***	***74***	2857	74	***2861***	***73***	2879	75	***2887***	***75***
2897	75	2903	75	2909	76	2917	76	***2927***	***74***	***2939***	***75***
2953	***76***	***2957***	***75***	2963	76	2969	75	2971	76	2999	76
3001	76	3011	76	3019	78	3023	77	3037	76	3041	77
3049	76	3061	76	3067	77	***3079***	***77***	***3083***	***76***	3089	77
3109	77	***3119***	***76***	3121	77	***3125***	***76***	3137	78	***3163***	***77***
3167	77	3169	79	***3181***	***78***	3187	77	***3191***	***77***	3203	77
3209	77	3217	78	3221	80	3229	78	***3251***	***78***	3253	78
3257	78	3259	78	3271	79	3299	79	3301	79	3307	80
3313	79	3319	80	3323	79	***3329***	***79***	***3331***	***78***	3343	80
3347	***78***	3359	81	***3361***	***79***	3371	81	3373	79	3389	80
3391	79	3407	80	3413	81	3433	80	3449	80	3457	81
3461	80	3463	80	3467	80	3469	80	***3481***	***79***	3491	80
3499	80	3511	81	3517	80	3527	81	***3529***	***81***	3533	82
3539	82	3541	80	3547	81	***3557***	***81***	3559	82	***3571***	***80***
3581	***80***	3583	81	3593	81	***3607***	***81***	***3613***	***79***	3617	81
3623	82	3631	82	***3637***	***81***	3643	83	3659	82	***3671***	***81***
3673	***81***	3677	82	3691	83	***3697***	***82***	3701	83	***3709***	***82***
3719	82	3721	83	3727	82	3733	83	3739	84	3761	82
3767	83	3769	83	***3779***	***82***	3793	83	3797	84	3803	83
3821	***82***	3823	83	***3833***	***82***	3847	83	3851	84	3853	83
3863	84	***3877***	***83***	3881	84	3889	83	3907	85	***3911***	***83***
3917	83	3919	84	3923	85	3929	84	3931	85	3943	84
3947	84	3967	84	3989	85	4001	86	***4003***	***83***	***4007***	***84***
4013	85	***4019***	***85***	4021	84	4027	84	***4049***	***83***	***4051***	***84***
4057	85	4073	85	***4079***	***85***	4091	85	***4093***	***85***	***4096***	***85***
4099	***85***	***4111***	***85***	***4127***	***85***	***4129***	***85***	4133	86	***4139***	***85***
4153	***85***	4157	86	4159	86	***4177***	***86***	4201	85	***4211***	***85***
4217	87	4219	87	4229	86	***4231***	***86***	4241	87	4243	87
4253	87	***4259***	***87***	4261	87	4271	88	***4273***	***86***	***4283***	***86***
4289	87	4297	87	***4327***	***87***	***4337***	***87***	4339	88	***4349***	***87***
4357	87	4363	87	4373	88	4391	87	4397	88	4409	88
4421	88	***4423***	***88***	4441	89	4447	88	4451	88	4457	87
4463	89	4481	89	4483	89	4489	89	4493	88	***4507***	***88***
4513	89	4517	88	4519	89	4523	89	4547	89	***4549***	***89***

Table 3. Continue 3

q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$	q	$n_q^{\mathrm{IL}}(4,3)$
4561	90	4567	90	4583	89	***4591***	***88***	4597	89	***4603***	***88***
4621	***88***	***4637***	***88***	4639	89	***4643***	***89***	4649	90	4651	90
4657	90	***4663***	***89***	4673	90	***4679***	***90***	***4691***	***89***	4703	90
4721	90	4723	91	***4729***	***89***	***4733***	***89***	4751	90	4759	90
4783	90	4787	90	4789	90	4793	90	4799	91	***4801***	***89***
4813	***91***	4817	91	***4831***	***90***	***4861***	***89***	4871	92	4877	92
4889	92	4903	92	***4909***	***90***	4913	92	4919	92	***4931***	***90***
4933	92	4937	91	4943	91	4951	91	4957	91	4967	91
4969	91	***4973***	***90***	4987	91	4993	93	4999	92	***5003***	***91***
5009	92	***5011***	***92***	5021	92	***5023***	***91***	***5039***	***91***	5041	92
5051	93	5059	93	5077	93	5081	94	5087	92	***5099***	***93***
5101	94	5107	93	***5113***	***93***	5119	93	5147	95	5153	94
5167	***92***	5171	93	5179	93	5189	93	5197	94	***5209***	***92***
5227	93	***5231***	***93***	5233	94	5237	94	5261	93	***5273***	***93***
5279	***94***	***5281***	***93***	***5297***	***93***	***5303***	***94***	***5309***	***93***	5323	94
5329	94	***5333***	***93***	5347	95	***5351***	***94***	5381	95	***5387***	***93***
5393	95	5399	95	***5407***	***94***	5413	95	5417	95	5419	95
5431	***94***	5437	96	5441	94	***5443***	***93***	5449	95	5471	95
5477	95	5479	97	5483	96	***5501***	***95***	5503	96	5507	94
5519	96	5521	96	***5527***	***95***	5531	96	5557	96	5563	96
5569	95	5573	96	5581	97	5591	96	***5623***	***95***	***5639***	***95***
5641	***96***	***5647***	***96***	***5651***	***95***	***5653***	***96***	***5657***	***95***	5659	96
5669	***95***	***5683***	***96***	5689	96	5693	97	5701	96	5711	96
5717	98	5737	96	5741	97	***5743***	***95***	5749	98	5779	97
5783	97	***5791***	***96***	5801	99	5807	97	***5813***	***96***	5821	97
5827	98	***5839***	***96***	5843	98	5849	96	5851	97	5857	97
5861	97	5867	98	5869	98	5879	99	***5881***	***97***	5897	97
5903	98	5923	98	5927	97	***5939***	***97***	5953	98	***5981***	***97***
5987	***98***	6007	99	6011	99	6029	98	6037	98	6043	99
6047	98	***6053***	***98***	6067	99	6073	100	6079	98	6089	100
6091	***97***	6101	99								
6143	***98***	***6217***	***99***	***6287***	***100***	***6299***	***100***	***6529***	***100***	***6563***	***100***

Table 4. Lengths $\overline{n}_q(4,3)$ of the ***shortest known*** $[\overline{n}_q(4,3), \overline{n}_q(4,3)-4]_q3$ codes, $2 \le q \le 6607$. The cases $\overline{n}_q(4,3,5) = \overline{n}_q(4,3) + j$ are noted by the superscript "$+j$". For the rest of q we have $\overline{n}_q(4,3,5) = \overline{n}_q(4,3)$. The improvements of code distance up to $d = 5$ in comparison with [20, Tab. 1] are noted in bold italic font. For $q = 841$ the complete 42-arc of [33] is used. The cases $\ell_q(4,3) = \overline{n}_q(4,3)$ are noted by the subscript "$\bullet$"

q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$
2	$\boldsymbol{5}_\bullet$	3	$5_\bullet$	4	$5_\bullet$	5	$6_\bullet$	7	$7_\bullet^{+1}$	8	$7_\bullet$
9	$7_\bullet^{+1}$	11	$8_\bullet$	13	8	16	9	17	9	19	9
23	10	25	11	27	11	29	11	31	11^{+1}	32	12
37	12	41	13	43	13	47	14	49	14	53	15
59	15	61	15^{+1}	64	16	67	16	71	16	73	16^{+1}
79	17	81	17^{+1}	83	17^{+1}	89	18	97	19	101	19
103	19	107	19	109	20	113	20	121	20^{+1}	125	21
127	21	128	21	131	21	137	22	139	22	149	22
151	***22***	157	23	163	23	167	24	169	24	173	24
179	24	181	24^{+1}	191	25	193	25	197	25	199	25
211	26	223	27	227	27	229	27	233	27	239	27
241	28	243	28	251	28	256	28	257	28	263	28
269	29	271	29	277	29	281	29	283	29	289	29^{+1}
293	29^{+1}	307	30	311	30^{+1}	313	30^{+1}	317	30^{+1}	331	31
337	31^{+1}	343	31^{+1}	347	32	349	32	353	32	359	32
361	32^{+1}	367	32^{+1}	373	33	379	33	383	33	389	33^{+1}
397	34	401	34	409	34	419	34^{+1}	***421***	***34***	431	35
433	***35***	439	35	443	35	449	35	457	35^{+1}	461	36
463	***36***	***467***	***36***	***479***	***36***	487	36	491	36^{+1}	499	37
503	37	509	37	512	36^{+1}	***521***	***37***	***523***	***38***	529	38
541	38	***547***	***38***	557	39	563	39	569	39	571	39
577	39	587	39	593	39	599	40	601	40	607	40
613	40	617	40	619	40	625	41	631	41	641	41
643	41	647	41	653	41	659	41	661	41	673	41
677	42	683	42	691	42	701	42	709	43	719	43
727	42	729	40^{+3}	733	43	739	43	743	43	751	43
757	43	761	43	769	44	773	44	787	44	797	45
809	45	811	45	821	45	823	45	827	45	829	45
839	45^{+1}	*841*	*42*	853	45	857	46	859	46	863	46
877	46	881	46	883	46	887	46	907	47	911	47
919	47	929	47	937	47	941	48	947	48	953	48
961	48	967	48	971	48	977	48	983	48	991	48
997	48	1009	49	1013	49	1019	49	1021	49	1024	49
1031	49	1033	49	1039	49	1049	50	1051	49	1061	50
1063	49	1069	50	1087	50	1091	50	1093	50	1097	50

Table 4. Continue 1

q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$
1103	50	1109	51	1117	51	1123	51	1129	51	1151	51
1153	51	1163	51	1171	52	1181	52	1187	52	1193	52
1201	52	1213	52	1217	52	1223	52	1229	53	1231	53
1237	53	1249	52	1259	53	1277	53	1279	53	1283	53
1289	54	1291	54	1297	54	1301	54	1303	54	1307	54
1319	54	1321	54^{+1}	1327	55	1331	48^{+7}	1361	54	1367	54
1369	55	1373	55	1381	55	1399	55	1409	55	1423	55
1427	56	1429	56	1433	56	1439	56	1447	56	1451	56
1453	56	1459	56	1471	56	1481	57	1483	57	1487	57
1489	57	1493	57	1499	57	1511	57	1523	57^{+1}	1531	57
1543	57	1549	57	1553	58	1559	58	1567	57	1571	58
1579	58	1583	58	1597	58	1601	58	1607	58	1609	58
1613	58	1619	59	1621	59	1627	58	1637	58	1657	58
1663	59^{+1}	1667	59	1669	59	1681	59	1693	60	1697	59
1699	59	1709	60	1721	60	1723	60	1733	60^{+1}	1741	60
1747	60	1753	60	1759	60	1777	61	1783	61	1787	60
1789	61	1801	61	1811	61	1823	61	1831	61	1847	61
1849	62	1861	61	1867	61	1871	62	1873	62	1877	61
1879	62	1889	62	1901	62	1907	62	1913	62	1931	62
1933	63	1949	63	1951	63	1973	63	1979	63	1987	64
1993	64	1997	63	1999	63	2003	63	2011	63	2017	63
2027	63	2029	64	2039	64	2048	64	2053	64	2063	64
2069	64	2081	65	2083	65	2087	64	2089	64	2099	65
2111	65	2113	65	2129	65	2131	65	2137	66	2141	65
2143	65	2153	65	2161	65	2179	66	2187	66	2197	56^{+10}
2203	66	2207	66	2209	66	2213	66	2221	65	2237	67
2239	66	2243	67	2251	66	2267	66	2269	67	2273	66
2281	66	2287	66	2293	67	2297	67	2309	67	2311	68
2333	67	2339	68	2341	67	2347	68	2351	68	2357	68
2371	68	2377	68	2381	68	2383	68	2389	68	2393	68
2399	69	2401	68	2411	68	2417	69	2423	68	2437	69
2441	68	2447	68	2459	69	2467	69	2473	69	2477	69
2503	69	2521	70	2531	69	2539	70	2543	70	2549	69
2551	70	2557	70	2579	70	2591	70	2593	69	2609	70
2617	71	2621	70	2633	70	2647	70	2657	70	2659	70
2663	70	2671	70	2677	71	2683	71	2687	71	2689	71

Table 4. Continue 2

q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$
2693	71	2699	72	2707	72	2711	71	2713	71	2719	71
2729	71	2731	71	2741	72	2749	72	2753	71	2767	72
2777	72	2789	72	2791	72	2797	73	2801	72	2803	72
2809	73	2819	73	2833	73	2837	73	2843	73	2851	72
2857	72	2861	73	2879	72	2887	72	2897	73	2903	73
2909	74	2917	73	2927	74	2939	73	2953	73	2957	73
2963	74	2969	74	2971	74	2999	74	3001	73	3011	75
3019	75	3023	75	3037	74	3041	74	3049	75	3061	75
3067	74	3079	75	3083	74	3089	75	3109	75	3119	75
3121	76	3125	76	3137	75	3163	76	3167	75	3169	75
3181	75	3187	75	3191	75	3203	76	3209	76	3217	77
3221	75	3229	76	3251	76	3253	77	3257	76	3259	77
3271	76	3299	77	3301	76	3307	77	3313	78	3319	77
3323	77	3329	77	3331	78	3343	77	3347	78	3359	78
3361	77	3371	78	3373	77	3389	78	3391	77	3407	77
3413	78	3433	78	3449	77	3457	78	3461	79	3463	78
3467	78	3469	78	3481	79	3491	78	3499	78	3511	78
3517	79	3527	79	3529	78	3533	79	3539	80	3541	79
3547	79	3557	79	3559	80	3571	79	3581	79	3583	79
3593	80	3607	79	3613	79	3617	80	3623	80	3631	80
3637	79	3643	80	3659	80	3671	80	3673	80	3677	81
3691	80	3697	80	3701	81	3709	80	3719	80	3721	81
3727	81	3733	80	3739	80	3761	81	3767	80	3769	80
3779	81	3793	81	3797	81	3803	81	3821	82	3823	80
3833	82	3847	81	3851	82	3853	82	3863	82	3877	82
3881	82	3889	81	3907	83	3911	83	3917	82	3919	83
3923	82	3929	83	3931	83	3943	82	3947	82	3967	83
3989	83	4001	83	4003	83	4007	83	4013	83	4019	83
4021	83	4027	83	4049	83	4051	83	4057	83	4073	83
4079	84	4091	83	4093	83	4096	68^{+16}	4099	84	4111	84
4127	83	4129	84	4133	83	4139	84	4153	84	4157	84
4159	84	4177	84	4201	84	4211	85	4217	85	4219	85
4229	84	4231	85	4241	85	4243	85	4253	85	4259	85
4261	85	4271	85	4273	85	4283	86	4289	86	4297	85
4327	86	4337	85	4339	85	4349	85	4357	86	4363	86
4373	86	4391	87	4397	87	4409	87	4421	85	4423	87
4441	86	4447	86	4451	86	4457	87	4463	88	4481	87
4483	88	4489	87	4493	88	4507	88	4513	88	4517	88

Table 4. Continue 3

q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$	q	$\overline{n}_q(4,3)$
4519	89	4523	89	4547	88	4549	89	4561	89	4567	89
4583	89	4591	88	4597	89	4603	88	4621	88	4637	88
4639	89	4643	89	4649	89	4651	89	4657	90	4663	89
4673	90	4679	88	4691	89	4703	89	4721	90	4723	90
4729	89	4733	89	4751	90	4759	90	4783	90	4787	89
4789	89	4793	89	4799	91	4801	89	4813	91	4817	89
4831	90	4861	89	4871	90	4877	90	4889	90	4903	91
4909	90	4913	72^{+17}	4919	90	4931	90	4933	91	4937	90
4943	91	4951	91	4957	90	4967	91	4969	91	4973	90
4987	90	4993	92	4999	92	5003	91	5009	92	5011	92
5021	91	5023	91	5039	91	5041	91	5051	91	5059	92
5077	91	5081	92	5087	92	5099	93	5101	92	5107	93
5113	93	5119	91	5147	92	5153	93	5167	92	5171	93
5179	93	5189	93	5197	93	5209	92	5227	92	5231	93
5233	93	5237	93	5261	93	5273	93	5279	94	5281	93
5297	93	5303	94	5309	93	5323	93	5329	94	5333	93
5347	94	5351	94	5381	94	5387	93	5393	95	5399	95
5407	94	5413	94	5417	94	5419	95	5431	94	5437	93
5441	94	5443	93	5449	93	5471	94	5477	94	5479	95
5483	95	5501	95	5503	95	5507	94	5519	96	5521	95
5527	95	5531	95	5557	94	5563	95	5569	95	5573	95
5581	94	5591	96	5623	95	5639	95	5641	96	5647	96
5651	95	5653	96	5657	95	5659	96	5669	95	5683	96
5689	96	5693	97	5701	96	5711	96	5717	97	5737	96
5741	95	5743	95	5749	97	5779	96	5783	96	5791	96
5801	94	5807	96	5813	96	5821	97	5827	97	5839	96
5843	97	5849	96	5851	97	5857	97	5861	97	5867	97
5869	98	5879	97	5881	97	5897	97	5903	97	5923	97
5927	97	5939	97	5953	98	5981	97	5987	98	6007	98
6011	98	6029	97	6037	98	6043	99	6047	98	6053	98
6067	99	6073	99	6079	98	6089	99	6091	97	6101	98
6113	99	6121	99	6131	98	6133	97	6143	98	6151	98
6163	99	6173	99	6197	100	6199	100	6203	98	6211	100
6217	99	6221	100	6229	99	6241	100	6247	99	6257	100
6263	100	6269	100	6271	100	6277	98	6287	100	6299	100
6301	99	6311	99	6317	100	6323	100	6329	100	6337	101
6343	100	6353	100	6359	100	6361	99	6367	101	6373	100
6379	100	6389	101	6397	101	6421	101	6427	101	6449	101
6451	101	6469	100	6473	101	6481	101	6491	101	6521	101
6529	100	6547	102	6551	102	6553	101	6561	102	6563	100
6569	101	6571	102	6577	101	6581	101	6599	101	6607	101

Table 5. Lengths $\overline{n}_q(5,3)$ of the shortest ***known*** $[\overline{n}_q(5,3), \overline{n}_q(5,3)-5]_q3$ codes obtained by the leximatrix and Rand-Greedy algorithms; $\overline{n}_q(5,3) = \min\{n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{G}}(5,3)\}$ for $3 \le q \le 401$; $\overline{n}_q(5,3) = \overline{n}_q(5,3,5) = n_q^{\mathrm{L}}(5,3)$ for $401 < q \le 839$. The improvements of code length in comparison with [20, Tab. 2] are noted in bold italic font. The cases $\ell_q(5,3) = \overline{n}_q(5,3)$ are noted by the subscript "$\bullet$"

q	$\overline{n}_q(5,3)$	q	$\overline{n}_q(5,3)$	q	$\overline{n}_q(5,3)$	q	$\overline{n}_q(5,3)$	q	$\overline{n}_q(5,3)$	q	$\overline{n}_q(5,3)$	q	$\overline{n}_q(5,3)$
3	$8_\bullet$	4	$9_\bullet$	5	$10_\bullet$	7	13	8	14	9	16	11	18
13	21	16	24	17	25	19	27	23	32	25	34	27	36
29	38	31	40	32	41	***37***	***47***	***41***	***51***	***43***	***52***	47	56
49	58	53	61	59	67	61	68	64	71	67	73	71	76
73	78	79	83	81	85	83	85	89	90	97	96	101	99
103	100	107	103	109	104	113	108	121	113	125	116	127	117
128	119	131	119	137	123	139	125	149	131	151	132	157	136
163	140	167	142	169	145	173	146	179	150	181	151	191	157
193	158	197	161	199	162	211	169	223	177	227	179	229	180
233	183	239	185	241	188	243	188	251	193	256	195	257	197
263	200	269	203	271	204	277	208	281	209	283	211	289	213
293	216	307	224	311	226	313	227	317	230	331	237	337	240
343	243	347	245	349	245	353	248	359	251	361	254	367	256
373	261	379	262	383	264	389	267	397	273	401	274	409	284
419	292	421	290	431	297	433	299	439	301	443	304	449	309
457	311	461	311	463	309	467	314	479	320	487	324	491	324
499	328	503	330	509	334	512	334	521	339	523	341	529	344
541	348	547	349	557	353	563	360	569	364	571	362	577	365
587	371	593	374	599	375	601	376	607	376	613	380	617	384
619	382	625	385	631	387	641	393	643	398	647	396	653	399
659	402	661	401	673	407	677	407	683	411	691	416	701	417
709	424	719	427	727	430	729	430	733	429	739	431	743	436
751	439	757	440	761	443	769	447	773	450	787	453	797	458
809	464	811	464	821	467	823	468	827	471	829	473	839	475