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Convergence of formal Dulac series satisfying an algebraic ordinary differential equation

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Abstract. A sufficient condition is proposed which ensures that a Dulac series that formally satisfies an algebraic ordinary differential equation (ODE) is convergent. Such formal solutions of algebraic ODEs are quite common: in particular, the Painlevé III, V and VI equations have formal solutions given by Dulac series; they are convergent in view of the sufficient condition presented.

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§ 1. Introduction

Consider an *algebraic* ODE of order n

$$F(x, y, \delta y, \dots, \delta^n y) = 0, \tag{1.1}$$

where $F = F(x, y_0, y_1, \dots, y_n)$ is a polynomial in $n + 2$ complex variables and δ is the differentiation $x(d/dx)$. Suppose that (1.1) has a formal solution in the form of a *Dulac series*

$$\varphi = \sum_{k=0}^{\infty} p_k(\ln x)x^k, \qquad p_k \in \mathbb{C}[t].$$

Such series appeared in Dulac's papers from the 1920s (for instance, [1]) concerning limit cycles of a planar vector field; they were asymptotic expansions of the monodromy (first return) map in a neighbourhood of a hyperbolic polycycle. (More precisely, the series in Dulac's papers have a more general form, involving the power functions x^{λ_k} rather than x^k, where the real exponents λ_k increase to infinity.) Subsequently, in the 1980s Dulac series were important in completing the proofs of finiteness theorems for limit cycles (see [2] or [3]).

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Dulac series also appear as formal solutions of algebraic ODEs (such as, for example, Abel's equations, equations of Emden-Fowler type, Painlevé's equations and so on); in this context they are also known as *power-logarithmic* expansions (see [4] and [5]). Dulac series treated just as formal solutions of algebraic ODE are our subject in this paper. The first question, arising here in a natural way, concerns the convergence of such formal solutions. We propose the following sufficient condition for convergence.

Theorem. *Let φ be a series that formally satisfies* (1.1):

$$F(x, \Phi) := F(x, \varphi, \delta\varphi, \dots, \delta^n\varphi) = 0,$$

and assume that for $j = 0, \dots, n$

$$\frac{\partial F}{\partial y_j}(x, \Phi) = a_j x^m + b_j(\ln x)x^{m+1} + \cdots ,$$

where $a_j \in \mathbb{C}$, $b_j \in \mathbb{C}[t]$ and $m \in \mathbb{Z}_+$ is the same integer for all j. If $a_n \neq 0$, then φ is uniformly convergent in each open sector S of sufficiently small radius, with vertex at zero and opening less than 2π.

For instance, the Painlevé III, V and VI equations have formal solutions given by Dulac series. Using connections between these equations and isomonodromic deformations of certain systems of linear ODEs, Shimomura [6], [7] proved that for the Painlevé V and VI equations these series converge. Our theorem can be used to show that for all Painlevé equations the formal solutions expressed by Dulac series converge (see examples in § 6).

Note that when $p_k = \text{const}$ for all k we obtain a power series φ formally satisfying (1.1), and our result becomes the well-known sufficient condition for the convergence of this series which is due to Malgrange [8]. Note also that combining our techniques here with the ones in [9] concerning the convergence of generalized power series satisfying (1.1), a result similar to ours can be established for more general formal Dulac series (involving power functions x^{λ_k}, where $\lambda_k \in \mathbb{C}$, in place of x^k).

The paper is organized as follows. We prove our theorem in § 5, and precede this by several auxiliary constructions: in § 2, we go over from the original ODE to a reduced ODE of special form, in § 3 we explain how methods from linear algebra are used in the proof of the theorem, and in § 4 we construct an ODE which 'majorizes' the reduced ODE in § 2. The idea of applying the classical method of majorants to analyze the convergence of formal solutions of a general ODE (1.1) in a neighbourhood of a singular point of the equation (so that Cauchy's theorem cannot be applied) has been used before by a number of authors. In particular, an alternative proof of Malgrange's theorem, mentioned above, was set out in [10] and [5] on this basis (in Malgrange's original proof of a sufficient condition for the convergence of a formal power series satisfying (1.1) the implicit function theorem for Banach spaces is the main tool). In the final section, § 6, we give two examples when our theorem can be used.

§ 2. A reduced ODE of a special form

Lemma 1. *Under the assumptions of the theorem in* § 1 *there exists* $\ell' \in \mathbb{Z}_+$ *such that for each* $\ell \geqslant \ell'$ *the transformation*

$$y = \sum_{k=0}^{\ell} p_k(\ln x)x^k + x^\ell u$$

takes the original equation (1.1) *to the following form*:

$$L(\delta)u = xM(x, \ln x, u, \delta u, \dots, \delta^n u), \tag{2.1}$$

where

$$L(\delta) = \sum_{j=0}^{n} a_j(\delta+\ell)^j, \qquad a_n \neq 0,$$

and M *is a polynomial in* $n+3$ *variables. Furthermore, the polynomial* L *does not vanish in the open right-hand half-plane.*

Proof. The method used in the proof is standard: it is similar to the proof of the reduction lemma for an ODE having a formal solution in the form of a Taylor series (see [8]).

For each integer $\ell \geqslant 0$ the formal Dulac series φ can be represented as

$$\varphi = \sum_{k=0}^{\ell} p_k(\ln x)x^k + x^\ell \sum_{k=1}^{\infty} p_{k+\ell}(\ln x)x^k =: \varphi_\ell + x^\ell \psi,$$

so that

$$\Phi = (\varphi, \delta\varphi, \dots, \delta^n\varphi) = \Phi_\ell + x^\ell \Psi,$$

where $\Phi_\ell = (\varphi_\ell, \delta\varphi_\ell, \dots, \delta^n\varphi_\ell)$ and $\Psi = (\psi, (\delta+\ell)\psi, \dots, (\delta+\ell)^n\psi)$. From Taylor's formula we obtain

$$\begin{aligned} 0 &= F(x, \Phi_\ell + x^\ell\Psi) \\ &= F(x, \Phi_\ell) + x^\ell \sum_{j=0}^{n} \frac{\partial F}{\partial y_j}(x, \Phi_\ell)\psi_j + \frac{x^{2\ell}}{2} \sum_{i,j=0}^{n} \frac{\partial^2 F}{\partial y_i\,\partial y_j}(x, \Phi_\ell)\psi_i\psi_j + \cdots, \end{aligned} \tag{2.2}$$

where $\psi_j = (\delta+\ell)^j\psi$.

We choose ℓ to satisfy the following two conditions:

1) $\ell > m$;
2) $L(\xi) := \sum_{j=0}^{n} a_j(\xi+\ell)^j \neq 0$ for each $\xi \in \{\operatorname{Re}\xi > 0\}$ (recall that the integer $m \geqslant 0$ is fixed in the assumptions of the theorem in § 1).

Definition. The *order of a Dulac series*

$$\varphi = \sum_{k=0}^{\infty} p_k(\ln x)x^k$$

is defined by $\operatorname{val}(\varphi) := \min\{k \mid p_k \not\equiv 0\}$.

By Taylor's formula

$$\frac{\partial F}{\partial y_j}(x,\Phi)-\frac{\partial F}{\partial y_j}(x,\Phi_\ell)=x^\ell\sum_{i=0}^{n}\frac{\partial^2 F}{\partial y_i\,\partial y_j}(x,\Phi_\ell)\psi_i+\cdots;$$

furthermore, $\operatorname{val}(\psi_i)\geqslant 1$ for all i, so that

$$\frac{\partial F}{\partial y_j}(x,\Phi_\ell)=a_jx^m+\widetilde{b}_j(\ln x)x^{m+1}+\cdots,\qquad \widetilde{b}_j\in\mathbb{C}[t],$$

for each $j=0,1,\dots,n$, and thus substituting the finite sum Φ_ℓ for Φ into $\partial F/\partial y_j$ preserves the leading coefficient a_j. Now it follows from (2.2) that

$$\operatorname{val}(F(x,\Phi_\ell))\geqslant m+\ell+1.$$

Dividing (2.2) by $x^{m+\ell}$ we obtain an equation of the required form (2.1); it has the formal solution

$$\psi=\sum_{k=1}^{\infty}p_{k+\ell}(\ln x)x^k=:\sum_{k=1}^{\infty}P_k(\ln x)x^k.$$

Lemma 1 is proved.

Lemma 2. *The formal series ψ is the unique Dulac series (in positive powers of x) satisfying (2.1). In addition, the degrees ν_k of the polynomials P_k have the estimate $\nu_k\leqslant kC$, where C is the degree of M in $t=\ln x$.*

Proof. We start by pointing out the following rule of differentiation:

$$\delta\colon P_k(\ln x)x^k\mapsto x^k\left(k+\frac{d}{dt}\right)P_k(t)|_{t=\ln x},$$

and therefore

$$(\delta+\ell)^j\colon P_k(\ln x)x^k\mapsto x^k\left(k+\ell+\frac{d}{dt}\right)^jP_k(t)|_{t=\ln x},\qquad j=0,1,\dots,n,$$
$$L(\delta)\colon P_k(\ln x)x^k\mapsto x^kL\left(k+\frac{d}{dt}\right)P_k(t)|_{t=\ln x}.$$

Hence, plugging $\psi=\sum_{k=1}^{\infty}P_k(\ln x)x^k$ into

$$L(\delta)u=xM(x,\ln x,u,\delta u,\dots,\delta^n u),$$

we obtain an equality between two Dulac series. Comparing the polynomials in $t=\ln x$ multiplying each power of x in these Dulac series, for the first power of x we obtain

$$L\left(1+\frac{d}{dt}\right)P_1(t)=M(0,t,0,\dots,0).$$

This is an inhomogeneous linear ODE with constant coefficients with respect to P_1. As $L(1)\neq 0$ by Lemma 1, zero is not a root of its characteristic equation. Hence this ODE has a unique polynomial solution, whose degree is equal to the degree of the polynomial on the right-hand side:

$$\deg P_1(t) = \deg M(0,t,0,\dots,0) \leqslant C.$$

Let $P_k^j(t)$ denote the polynomial $\left(k+\frac{d}{dt}\right)^j P_k(t)$, $j=0,1,\dots,n$ (in particular, $P_k^0 = P_k$). Then

$$\delta^j \psi = \sum_{k=1}^{\infty} P_k^j(\ln x) x^k.$$

Assume that M is a linear combination of monomials of the form

$$x^{\mu}(\ln x)^{\nu} u^{q_0} (\delta u)^{q_1} \cdots (\delta^n u)^{q_n}.$$

Then for each $P_k(t)$ in succession, $k \geqslant 2$, we obtain inhomogeneous linear ODEs with constant coefficients

$$L\left(k+\frac{d}{dt}\right) P_k(t) = R_k(t), \tag{2.3}$$

where $R_k(t)$ is a linear combination of polynomials of the form

$$t^{\nu}(P_{k_1}^0 \cdots P_{k_{q_0}}^0)(P_{l_1}^1 \cdots P_{l_{q_1}}^1) \cdots (P_{m_1}^n \cdots P_{m_{q_n}}^n),$$

where

$$\nu \leqslant C \quad \text{and} \quad \sum_{i=1}^{q_0} k_i + \sum_{i=1}^{q_1} l_i + \cdots + \sum_{i=1}^{q_n} m_i \leqslant k-1.$$

By the natural induction assumption

$$\begin{aligned} \deg P_{k_1}^0 \cdots P_{k_{q_0}}^0 &\leqslant (k_1 + \cdots + k_{q_0})C, \\ \deg P_{l_1}^1 \cdots P_{l_{q_1}}^1 &\leqslant (l_1 + \cdots + l_{q_1})C, \\ &\cdots\cdots\cdots\cdots \\ \deg P_{m_1}^n \cdots P_{m_{q_n}}^n &\leqslant (m_1 + \cdots + m_{q_n})C, \end{aligned}$$

so that

$$\deg R_k(t) \leqslant C + (k-1)C = kC.$$

As $L(k) \neq 0$ by Lemma 1, zero is not a root of the characteristic polynomial of the linear ODE (2.3). Hence this ODE has a unique polynomial solution P_k, whose degree is equal to the degree of the polynomial on the right-hand side:

$$\deg P_k(t) = \deg R_k(t) \leqslant kC.$$

Lemma 2 is proved.

§ 3. From linear ODEs to linear algebra

We write the formal series as

$$\psi = \sum_{k=1}^{\infty} P_k(-\epsilon \ln x)x^k, \tag{3.1}$$

where we specify $\epsilon > 0$ in what follows (again, we denote the new polynomials by P_k). Then the operators δ and $L(\delta)$ act on the term $P_k(-\epsilon \ln x)x^k$ of this series as follows:

$$\delta\colon P_k(-\epsilon \ln x)x^k \mapsto x^k \left(k - \epsilon \frac{d}{dt}\right) P_k(t)|_{t=-\epsilon \ln x},$$
$$L(\delta)\colon P_k(-\epsilon \ln x)x^k \mapsto x^k L\left(k - \epsilon \frac{d}{dt}\right) P_k(t)|_{t=-\epsilon \ln x}.$$

These actions have natural coordinate representations in terms of vectors and matrices: let $\mathbf{b}_k \in \mathbb{C}^{\nu_k+1}$ be the coefficient column of P_k, and let $\mathbf{c}_k$ and $\mathbf{d}_k \in \mathbb{C}^{\nu_k+1}$ be the coefficient columns of $\bigl(k - \epsilon \frac{d}{dt}\bigr)P_k$ and $L\bigl(k - \epsilon \frac{d}{dt}\bigr)P_k$, respectively; then

$$\mathbf{c}_k = (kI - N_k)\mathbf{b}_k \quad \text{and} \quad \mathbf{d}_k = L(kI - N_k)\mathbf{b}_k,$$

where I is the identity matrix and N_k is a nilpotent matrix of the following form:

$$N_k = \begin{pmatrix} 0 & \epsilon & 0 & \dots & 0 & 0 \\ 0 & 0 & 2\epsilon & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 0 & \nu_k\epsilon \\ 0 & \dots & \dots & \dots & 0 & 0 \end{pmatrix}, \qquad N_k^{\nu_k+1} = 0.$$

We factor the polynomial

$$L(\xi) = a_0 + \dots + a_n(\xi + \ell)^n = a_n \prod_{j=1}^{n} (\xi + \lambda_j), \qquad \operatorname{Re} \lambda_j \geqslant 0.$$

Then the matrix $L(kI - N_k)$ has the representation

$$\begin{aligned} L(kI - N_k) &= a_n \prod_{j=1}^{n} \bigl((k + \lambda_j)I - N_k\bigr) \\ &= a_n \prod_{j=1}^{n} (k + \lambda_j) \prod_{j=1}^{n} \left(I - \frac{N_k}{k + \lambda_j}\right) = L(k) \prod_{j=1}^{n} \left(I - \frac{N_k}{k + \lambda_j}\right). \end{aligned}$$

The inverse matrix is

$$L(kI - N_k)^{-1} = \frac{1}{L(k)} \prod_{j=1}^{n} \left(I - \frac{N_k}{k + \lambda_j}\right)^{-1},$$

where

$$\left(I - \frac{N_k}{k+\lambda_j}\right)^{-1} = I + \frac{N_k}{k+\lambda_j} + \left(\frac{N_k}{k+\lambda_j}\right)^2 + \cdots + \left(\frac{N_k}{k+\lambda_j}\right)^{\nu_k}.$$

In what follows, we use the 1-norm $\|A\| = \|A\|_1 = \max_j \sum_i |a_{ij}|$ for the matrices, which corresponds to the 1-norm $\|x\|_1 = \sum_i |x_i|$ for vectors.

Lemma 3. *If $\epsilon > 0$ is sufficiently small, then there exists ε, $0 < \varepsilon < 1$, such that for each polynomial L of degree n that does not vanish in the open right-hand half-plane*

$$\|L(kI - N_k)\| \leqslant (1+\varepsilon)^n |L(k)|$$

and

$$\left\|L(kI - N_k)^{-1}\right\| \leqslant \frac{1}{(1-\varepsilon)^n |L(k)|}.$$

In particular,

$$\|kI - N_k\| \leqslant (1+\varepsilon)k \quad \textit{and} \quad \left\|(kI - N_k)^{-1}\right\| \leqslant \frac{1}{(1-\varepsilon)k}.$$

Proof. Fix ε, $0 < \varepsilon < 1$, and let ϵ be sufficiently small so that $\|N_k\| = \nu_k \epsilon \leqslant \varepsilon k$. Then for each integer $k > 0$

$$\left\|I - \frac{N_k}{k+\lambda_j}\right\| \leqslant 1 + \varepsilon, \qquad j = 1, \ldots, n.$$

Therefore,

$$\begin{aligned}\|L(kI - N_k)\| &= \left\|L(k) \prod_{j=1}^{n} \left(I - \frac{N_k}{k+\lambda_j}\right)\right\| \\ &\leqslant |L(k)| \prod_{j=1}^{n} \left\|I - \frac{N_k}{k+\lambda_j}\right\| \leqslant |L(k)|(1+\varepsilon)^n.\end{aligned}$$

To estimate the norm of the inverse matrix we observe that

$$\begin{aligned}\left\|\left(I - \frac{N_k}{k+\lambda_j}\right)^{-1}\right\| &\leqslant \|I\| + \left\|\frac{N_k}{k+\lambda_j}\right\| + \cdots + \left\|\frac{N_k}{k+\lambda_j}\right\|^{\nu_k} \\ &\leqslant 1 + \varepsilon + \cdots + \varepsilon^{\nu_k} \leqslant \frac{1}{1-\varepsilon},\end{aligned}$$

so that

$$\left\|L(kI - N_k)^{-1}\right\| = \left\|\frac{1}{L(k)} \prod_{j=1}^{n} \left(I - \frac{N_k}{k+\lambda_j}\right)^{-1}\right\| \leqslant \frac{1}{|L(k)|(1-\varepsilon)^n}.$$

The proof is complete.

§ 4. A majorizing ODE

We write (2.1) as

$$L(\delta)u = xM(x, -\epsilon\ln x, u, \delta u, \dots, \delta^n u), \tag{4.1}$$

where M on the right-hand side denotes a new polynomial. We look at another equation, which (as we will show below) majorizes (4.1) in a certain sense:

$$\sigma\delta^n u = x\widetilde{M}(x, -\epsilon\ln x, \delta^n u), \tag{4.2}$$

where

$$\frac{1}{\sigma} := \sup_{k\geqslant 1}\bigl(\bigl\|L(kI-N_k)^{-1}\bigr\|\cdot\bigl\|(kI-N_k)^n\bigr\|\bigr) < +\infty$$

by Lemma 3 and the polynomial $\widetilde{M}$ is constructed as follows. Let M be the sum of monomials of the form

$$\alpha x^{\mu}(-\epsilon\ln x)^{\nu}u^{q_0}(\delta u)^{q_1}\cdots(\delta^n u)^{q_n}, \qquad \alpha\in\mathbb{C}. \tag{4.3}$$

Then we define $\widetilde{M}$ simply by replacing each such monomial in M by

$$|\alpha|x^{\mu}(-\epsilon\ln x)^{\nu}(c\delta^n u)^{q_0}(c\delta^n u)^{q_1}\cdots(c\delta^n u)^{q_n}, \qquad c=\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n. \tag{4.4}$$

Lemma 4. *There exists a unique formal Dulac series in positive powers of x*

$$\widetilde{\psi} = \sum_{k=1}^{\infty}\widetilde{P}_k(-\epsilon\ln x)x^k$$

that satisfies (4.2). Here the $\widetilde{P}_k\in\mathbb{R}_+[t]$ are polynomials of degree $\widetilde{\nu}_k=\deg\widetilde{P}_k\leqslant kC$ with nonnegative real coefficients.

Proof. Our argument is similar to the proof of Lemma 2: each $\widetilde{P}_k$ is obtained as a solution of an inhomogeneous linear ODE with constant coefficients. We start with the first polynomial, which solves the equation

$$\sigma\left(1-\epsilon\frac{d}{dt}\right)^n\widetilde{P}_1(t) = \widetilde{M}(0,t,0)\in\mathbb{R}_+[t],$$

and then find the other $\widetilde{P}_k$, $k\geqslant 2$, in succession as the unique polynomial solutions of the corresponding ODEs

$$\sigma\left(k-\epsilon\frac{d}{dt}\right)^n\widetilde{P}_k(t) = \widetilde{Q}_k(t)\in\mathbb{R}_+[t],$$

where we express the polynomial $\widetilde{Q}_k$ in terms of $\widetilde{Q}_1=\widetilde{M}(0,t,0), \widetilde{Q}_2,\dots,\widetilde{Q}_{k-1}$ (see the expressions in the proof of the next result, Lemma 5); it has degree at most kC. Thus the fact that each $\widetilde{P}_k$ has nonnegative coefficients follows because the corresponding $\widetilde{Q}_k$ has nonnegative coefficients and the matrix $(kI-N_k)^{-1}$ has nonnegative entries.

The proof is complete.

For an arbitrary polynomial $P \in \mathbb{C}[t]$ we set its norm $\|P\|$ to be the 1-norm of the column of its coefficients. Apart from the standard properties of norms, the norm defined in this way has the following properties, which are easy to verify:

1) for any $P, Q \in \mathbb{C}[t]$ we have $\|PQ\| \leqslant \|P\| \cdot \|Q\|$;

2) if $P, Q \in \mathbb{R}_+[t]$, then $\|P+Q\| = \|P\| + \|Q\|$ and $\|PQ\| = \|P\| \cdot \|Q\|$.

Now we show that equation (4.2) constructed above majorizes the original equation (4.1) in the following sense.

Lemma 5. *The formal Dulac series $\widetilde{\psi}$ satisfying (4.2) majorizes the formal Dulac series ψ in (3.1), which satisfies (4.1): $\|P_k\| \leqslant \|\widetilde{P}_k\|$ for all k.*

Proof. We have already mentioned that P_k and $\widetilde{P}_k$ are solutions of the corresponding inhomogeneous linear ODEs with constant coefficients

$$\begin{gathered} L\biggl(k - \epsilon \frac{d}{dt}\biggr) P_k(t) = Q_k(t), \\ \sigma \biggl(k - \epsilon \frac{d}{dt}\biggr)^n \widetilde{P}_k(t) = \widetilde{Q}_k(t), \end{gathered} \tag{4.5}$$

where $Q_1(t) = M(0, t, 0, \dots, 0)$ and $\widetilde{Q}_1(t) = \widetilde{M}(0, t, 0)$, so that $\|Q_1\| = \|\widetilde{Q}_1\|$. Hence

$$\begin{aligned} \|P_1\| &\leqslant \bigl\|L(I - N_1)^{-1}\bigr\| \cdot \|Q_1\| = \bigl\|L(I - N_1)^{-1}\bigr\| \cdot \|\widetilde{Q}_1\| \\ &\leqslant \sigma \bigl\|L(I - N_1)^{-1}\bigr\| \cdot \bigl\|(I - N_1)^n\bigr\| \cdot \|\widetilde{P}_1\| \leqslant \|\widetilde{P}_1\|. \end{aligned}$$

To obtain similar estimates for all $k \geqslant 2$ we look at the relations for the corresponding Q_k and $\widetilde{Q}_k$ more closely. We let $P_k^j(t)$ denote the polynomial $\bigl(k - \epsilon \frac{d}{dt}\bigr)^j P_k(t)$, $j = 0, 1, \dots, n$ (in particular, $P_k^0 = P_k$). Then

$$\delta^j \psi = \sum_{k=1}^{\infty} P_k^j(-\epsilon \ln x) x^k$$

and

$$\delta^n \widetilde{\psi} = \sum_{k=1}^{\infty} \frac{1}{\sigma} \widetilde{Q}_k(-\epsilon \ln x) x^k.$$

Returning to (4.3) and (4.4) we conclude that $Q_k(t)$ is a sum of polynomials of the following form:

$$\alpha t^\nu (P_{k_1}^0 \cdots P_{k_{q_0}}^0)(P_{l_1}^1 \cdots P_{l_{q_1}}^1) \cdots (P_{m_1}^n \cdots P_{m_{q_n}}^n), \tag{4.6}$$

where $\sum_{i=1}^{q_0} k_i + \sum_{i=1}^{q_1} l_i + \dots + \sum_{i=1}^{q_n} m_i \leqslant k - 1$, and $\widetilde{Q}_k(t)$ is the sum of the corresponding polynomials

$$|\alpha| t^\nu \biggl(\frac{c}{\sigma} \widetilde{Q}_{k_1} \cdots \frac{c}{\sigma} \widetilde{Q}_{k_{q_0}}\biggr) \biggl(\frac{c}{\sigma} \widetilde{Q}_{l_1} \cdots \frac{c}{\sigma} \widetilde{Q}_{l_{q_1}}\biggr) \cdots \biggl(\frac{c}{\sigma} \widetilde{Q}_{m_1} \cdots \frac{c}{\sigma} \widetilde{Q}_{m_{q_n}}\biggr). \tag{4.7}$$

The norm of the polynomial (4.6) does not exceed the product

$$|\alpha| \cdot \|P_{k_1}^0\| \cdots \|P_{k_{q_0}}^0\| \cdot \|P_{l_1}^1\| \cdots \|P_{l_{q_1}}^1\| \cdots \|P_{m_1}^n\| \cdots \|P_{m_{q_n}}^n\|.$$

Here, using the inductive assumption and (4.5), we can find estimates for each factor $\|P_s^j\|$, $s<k$, with the help of Lemma 3:

$$\begin{aligned}\|P_s^j\| &\leqslant \big\|(sI-N_s)^j\big\|\cdot\|P_s\| \leqslant \big\|(sI-N_s)^j\big\|\cdot\|\widetilde{P}_s\| \\ &\leqslant \big\|(sI-N_s)^j\big\|\cdot\big\|(sI-N_s)^{-n}\big\|\cdot\left\|\frac{\widetilde{Q}_s}{\sigma}\right\| \\ &\leqslant \|sI-N_s\|^j\cdot\big\|(sI-N_s)^{-1}\big\|^n\cdot\left\|\frac{\widetilde{Q}_s}{\sigma}\right\| \leqslant \frac{(1+\varepsilon)^j s^j}{(1-\varepsilon)^n s^n}\left\|\frac{\widetilde{Q}_s}{\sigma}\right\| \leqslant \frac{c}{\sigma}\|\widetilde{Q}_s\|.\end{aligned}$$

Hence the norm of the polynomial (4.6) does not exceed that of (4.7) (recall that the norm of a product of polynomials with nonnegative real coefficients is the product of the norms of the factors) and therefore $\|Q_k\|\leqslant\|\widetilde{Q}_k\|$ (again, the norm of a sum of polynomials with nonnegative coefficients is the sum of their norms).

Finally, we conclude that

$$\begin{aligned}\|P_k\| &\leqslant \big\|L(kI-N_k)^{-1}\big\|\cdot\|Q_k\| \leqslant \big\|L(kI-N_k)^{-1}\big\|\cdot\|\widetilde{Q}_k\| \\ &\leqslant \sigma\big\|L(kI-N_k)^{-1}\big\|\cdot\big\|(kI-N_k)^n\big\|\cdot\|\widetilde{P}_k\| \leqslant \|\widetilde{P}_k\|.\end{aligned}$$

Lemma 5 is proved.

§ 5. Proof of the convergence theorem

As $\ln x$ is a transcendental function, the majorizing equation (4.2) can be regarded as the algebraic equation

$$\sigma U = x\widetilde{M}(x,t,U) \tag{5.1}$$

(with two independent variables x and t and unknown $U=\delta^n u$), which has a formal solution

$$\widehat{U}=\sum_{k=1}^{\infty}\frac{1}{\sigma}\widetilde{Q}_k(t)x^k, \qquad \widetilde{Q}_k\in\mathbb{R}_+[t].$$

Expanding the products in $\widehat{U}$ we obtain a power series in two variables

$$\widehat{U}_{\mathrm{pow}}=\sum_{k=1}^{\infty}\sum_{l=0}^{\widetilde{\nu}_k}c_{kl}t^l x^k, \qquad c_{kl}\in\mathbb{R}_+,$$

which also solves (5.1) formally (expanding the products in $\widehat{U}$ and plugging the resulting series $\widehat{U}_{\mathrm{pow}}$ into both sides of (5.1) is the same as the result of plugging $\widehat{U}$ into (5.1) and then multiplying out on both sides). By the implicit function theorem the series $\widehat{U}_{\mathrm{pow}}$ is absolutely convergent for small t and x; however, this is not the result we need, because t corresponds to $-\epsilon\ln x$ in the Dulac series, and $\ln x$ is unbounded for small x. So we fix an integer $r>C$ and consider an open sector S with vertex at the origin and opening less than 2π such that

$$|\epsilon\ln x|<|x|^{-1/r}\quad \forall\, x\in S,$$

which yields

$$\widetilde{Q}_k(|\epsilon \ln x|) < \widetilde{Q}_k\big(|x|^{-1/r}\big) \quad \forall\, x \in S. \tag{5.2}$$

Consider the formal Puiseux series

$$\phi = \sum_{k=1}^{\infty} \sum_{l=0}^{\widetilde{\nu}_k} c_{kl} x^{k-l/r}$$

obtained by setting $t = x^{-1/r}$ in the power series of two variables $\widehat{U}_{\rm pow}$. This formal series is well defined because, for fixed k and l such that $l \leqslant kC$, there exists only finitely many pairs (k_i, l_i), $l_i \leqslant k_i C$, such that $k_i - l_i/r = k - l/r$. In fact, otherwise there would exist two sequences $k_i \to \infty$ and $l_i \to \infty$ such that

$$r - \frac{l_i}{k_i} = \frac{kr - l}{k_i},$$

and this is impossible because the left-hand side of this relation is at least $r - C$ for each i, which is positive, and the right-hand side tends to zero as $i \to \infty$.

The Puiseux series ϕ is a formal solution of the equation

$$\sigma U = x\widetilde{M}(x, x^{-1/r}, U) \tag{5.3}$$

obtained from (5.1) by the corresponding substitution $t = x^{-1/r}$. In fact, if we set $t = x^{-1/r}$ in $\widehat{U}_{\rm pow}$ and plug the resulting Puiseux series ϕ into both sides of (5.3), then the result is the same as if we first substitute $\widehat{U}_{\rm pow}$ into (5.1) and then set $t = x^{-1/r}$ on both sides of the resulting identity. Hence ϕ is absolutely convergent in S for sufficiently small x (to see this it is sufficient to make the change of variable $x = z^r$ in (5.3) and use the implicit function theorem).

Now we observe that the series

$$\phi^{\circ}(|x|) = \sum_{k=1}^{\infty} \frac{1}{\sigma} \widetilde{Q}_k\big(|x|^{-1/r}\big)|x|^k$$

is just another way of writing down the real Puiseux series

$$\phi(|x|) = \sum_{k=1}^{\infty} \sum_{l=0}^{\widetilde{\nu}_k} c_{kl} |x|^{k-l/r},$$

so that it also converges in S if x is sufficiently small (for instance, if $|x| < \rho$) by the corresponding property of convergent positive series (see [11], Ch. VIII). In view of (5.2) the series

$$\sum_{k=1}^{\infty} \frac{1}{\sigma} \widetilde{Q}_k(|\epsilon \ln x|) x^k \tag{5.4}$$

is absolutely convergent in S for $|x| < \rho$; now we show that, as a consequence, the series

$$\sum_{k=1}^{\infty} \widetilde{P}_k(|\epsilon \ln x|) x^k$$

is also convergent.

Lemma 6. *The series* $\sum_{k=1}^{\infty} \widetilde{P}_k(|\epsilon \ln x|)x^k$ *is absolutely convergent in the sector*

$$S_\rho = S \cap \{|x|^{1-C/r} < \rho\}.$$

Proof. This follows because the series (5.4) converges and

$$\begin{aligned}\widetilde{P}_k(|\epsilon \ln x|)|x^k| &\leqslant \|\widetilde{P}_k\| \cdot |\epsilon \ln x|^{\widetilde{\nu}_k} \cdot |x|^k \leqslant \|\widetilde{P}_k\| \cdot |x|^{(1-C/r)k} \\ &\leqslant \big\|(kI - N_k)^{-1}\big\|^n \cdot \left\|\frac{\widetilde{Q}_k}{\sigma}\right\| \cdot |x|^{(1-C/r)k} \\ &\leqslant \frac{1}{(1-\varepsilon)^n k^n} \left\|\frac{\widetilde{Q}_k}{\sigma}\right\| \cdot |x|^{(1-C/r)k} \\ &\leqslant \frac{1}{(1-\varepsilon)^n k^n \sigma} \widetilde{Q}_k\big(\big|\epsilon \ln x^{1-C/r}\big|\big)|x|^{(1-C/r)k},\end{aligned}$$

in view of the second relation in (4.5) and Lemma 3. The lemma is proved.

We complete the proof of the theorem in § 1 by showing that the series

$$\psi = \sum_{k=1}^{\infty} P_k(-\epsilon \ln x)x^k$$

is convergent. Taking Lemma 5 into account, we obtain

$$\begin{aligned}\big|P_k(-\epsilon \ln x)x^k\big| &\leqslant \|P_k\| \cdot |\epsilon \ln x|^{\nu_k} \cdot |x|^k \leqslant \|P_k\| \cdot |x|^{(1-C/r)k} \\ &\leqslant \|\widetilde{P}_k\| \cdot |x|^{(1-C/r)k} \leqslant \widetilde{P}_k\big(\big|\epsilon \ln x^{1-C/r}\big|\big) \cdot |x|^{(1-C/r)k}.\end{aligned}$$

In view of Lemma 6 this ensures that the series ψ converges for $x^{1-C/r} \in S_\rho$. Now the proof of the theorem in § 1 is complete.

Remark. Using the techniques developed here and in [9] we can prove the following result, which is similar to the theorem in § 1, but is related to a more general form of Dulac series (the corresponding proof is technically more complicated).

Let the series

$$\varphi = \sum_{k=0}^{\infty} p_k(\ln x)x^{\lambda_k}, \qquad \lambda_k \in \mathbb{C}, \quad 0 \leqslant \operatorname{Re}\lambda_0 \leqslant \operatorname{Re}\lambda_1 \leqslant \cdots \to +\infty,$$

satisfy equation (1.1) *formally*:

$$F(x, \Phi) := F(x, \varphi, \delta\varphi, \ldots, \delta^n\varphi) = 0,$$

and suppose that for $j = 0, \ldots, n$

$$\frac{\partial F}{\partial y_j}(x, \Phi) = a_j x^\alpha + b_j(\ln x)x^{\alpha_j} + \cdots, \qquad \operatorname{Re}\alpha < \operatorname{Re}\alpha_j,$$

where $a_j \in \mathbb{C}$, $b_j \in \mathbb{C}[t]$ *and* $\alpha \in \mathbb{C}$ *is the same for all* j.

If $a_n \neq 0$, *then the series* φ *is uniformly convergent in each open sector* S *of sufficiently small radius, with vertex at zero and opening less than* 2π.

§ 6. Examples

In this section we present two examples, Abel's equation and the Painlevé III equation, and we use our theorem in § 1 to prove that formal solutions to these, given by Dulac series, converge.

6.1. Abel's equation. Consider the Abel equation of the second kind

$$w\,\frac{dw}{dx}=-w-x^m, \qquad m=-3$$

(see [12], § 1.3). We can show that it has a one-parameter family of formal solutions given by Dulac series[1]

$$\widehat{w}=\frac{1}{x}\biggl(1+(c-\ln x)x^2+\sum_{k=2}^{\infty}P_k(\ln x)x^{2k}\biggr), \qquad c\in\mathbb{C}.$$

Making the power transformation $w=y/x$ here, and introducing the operator $\delta=x(d/dx)$ we can write the result in the form of the equation

$$F(x,y,\delta y):=y\delta y-y^2+x^2y+1=0$$

which has a family of formal solutions given by Dulac series

$$\varphi=1+(c-\ln x)x^2+\sum_{k=2}^{\infty}P_k(\ln x)x^{2k}.$$

We use the theorem in § 1 to show that these series are convergent. As $F(x,y_0,y_1)=y_0y_1-y_0^2+x^2y_0+1$, we have

$$\frac{\partial F}{\partial y_0}=y_1-2y_0+x^2 \quad\text{and}\quad \frac{\partial F}{\partial y_1}=y_0.$$

Plugging in

$$y_0=\varphi=1+(c-\ln x)x^2+\cdots$$

and

$$y_1=\delta\varphi=(2c-1-2\ln x)x^2+\cdots,$$

we obtain

$$\frac{\partial F}{\partial y_0}=-2+\cdots \quad\text{and}\quad \frac{\partial F}{\partial y_1}=1+(c-\ln x)x^2+\cdots.$$

Hence the assumptions of the theorem in § 1 are fulfilled and the series φ converges in each open sector $S\subset\mathbb{C}$ of sufficiently small radius, with vertex at zero and opening less than 2π.

[1] For $m=-2,-1,0,1$ it is shown in [12], § 1.3, that this equation is integrable in quadratures or in terms of special functions. We can show that for $m\geqslant 0$ this equation has formal solutions only in the form of Taylor series, while for $m\leqslant -2$ it has formal solutions in the form of Dulac series.

6.2. The Painlevé III equation. The Painlevé III equation has the following form:

$$y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{ay^2+b}{x} + cy^3 + \frac{d}{y}, \tag{6.1}$$

where $a,b,c,d\in\mathbb{C}$. If $b,d\neq 0$ and $b/\sqrt{-d} = 2k_0 \in 2\mathbb{N}$, then (6.1) has a one-parameter family of formal solutions given by the Dulac series (see [13])

$$\varphi = -\frac{d}{b}x + \sum_{k=1}^{\infty} P_{2k+1}(\ln x)x^{2k+1}, \tag{6.2}$$

where $P_{2k+1} \equiv \text{const}$ for $2k+1 < 2k_0+1$, P_{2k_0+1} is a polynomial of degree one whose free term is an arbitrary parameter, and the polynomials P_{2k+1} with indices $2k+1 > 2k_0+1$ are uniquely defined.

In terms of the operator δ we can write (6.1) as

$$F(x,y,\delta y,\delta^2 y) := -y\delta^2 y + (\delta y)^2 + axy^3 + bxy + cx^2y^4 + dx^2 = 0, \tag{6.3}$$

so that

$$F(x,y_0,y_1,y_2) = -y_0y_2 + y_1^2 + axy_0^3 + bxy_0 + cx^2y_0^4 + dx^2.$$

We show that a formal solution (6.2) of (6.3) is convergent using the theorem in § 1. Plugging the formal Dulac series

$$y_0 = \varphi = -\frac{d}{b}x + \cdots, \qquad y_1 = \delta\varphi = -\frac{d}{b}x + \cdots \quad \text{and} \quad y_2 = \delta^2\varphi = -\frac{d}{b}x + \cdots,$$

into the expressions for the partial derivatives

$$\frac{\partial F}{\partial y_0} = -y_2 + 3axy_0^2 + bx + 4cx^2y_0^3, \qquad \frac{\partial F}{\partial y_1} = 2y_1 \quad \text{and} \quad \frac{\partial F}{\partial y_2} = -y_0$$

we obtain

$$\frac{\partial F}{\partial y_0} = \left(b + \frac{d}{b}\right)x + \cdots, \qquad \frac{\partial F}{\partial y_1} = -\frac{2d}{b}x + \cdots \quad \text{and} \quad \frac{\partial F}{\partial y_2} = \frac{d}{b}x + \cdots.$$

Since $d \neq 0$, the assumptions of the theorem in § 1 are fulfilled and the series φ converges in each open sector $S \subset \mathbb{C}$ of sufficiently small radius, with vertex at zero and opening less than 2π. This is apparently the first time that the convergence of solutions of the Painlevé III equations given by Dulac series has been established.

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