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# On integral weight spectra of the MDS codes cosets of weight 1, 2, and 3

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**Abstract.** The weight of a coset of a code is the smallest Hamming weight of any vector in the coset. For a linear code of length $n$, we call *integral weight spectrum* the overall numbers of weight $w$ vectors, $0 \le w \le n$, in all the cosets of a fixed weight. For maximum distance separable (MDS) codes, we obtained new convenient formulas of integral weight spectra of cosets of weight 1 and 2. Also, we give the spectra for the weight 3 cosets of MDS codes with minimum distance 5 and covering radius 3.

Keywords: cosets weight distribution, MDS codes.

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## 1 Introduction

Let $\mathbb{F}_q$ be the Galois field with $q$ elements, $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Let $\mathbb{F}_q^n$ be the space of $n$-dimensional vectors over $\mathbb{F}_q$. We denote by $[n, k, d]_q R$ an $\mathbb{F}_q$-linear code of length $n$, dimension $k$, minimum distance $d$, and covering radius $R$. If $d = n - k + 1$, it is a maximum distance separable (MDS) code. For an introduction to coding theory see [2, 11, 16, 19].

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A *coset* of a code is a translation of the code. A coset $\mathcal{V}$ of an $[n,k,d]_qR$ code $\mathcal{C}$ can be represented as

$$\mathcal{V} = \{\mathbf{x} \in \mathbb{F}_q^n \,|\, \mathbf{x} = \mathbf{c} + \mathbf{v}, \mathbf{c} \in \mathcal{C}\} \subset \mathbb{F}_q^n \tag{1.1}$$

where $\mathbf{v} \in \mathcal{V}$ is a vector fixed for the given representation; see [2, 11, 16, 17, 19] and the references therein.

The weight distribution of code cosets is an important combinatorial property of a code. In particular, the distribution serves to estimate decoding results. There are many papers connected with distinct aspects of the weight distribution of cosets for codes over distinct fields and rings, see e.g. [1–7, 9, 10, 12–15, 20, 21], [8, Sect. 6.3], [11, Sect. 7], [16, Sections 5.5, 6.6, 6.9], [17, Sect. 10] and the references therein.

For a linear code of length $n$, we call *integral weight spectrum* the overall numbers of weight $w$ vectors, $0 \le w \le n$, in all the cosets of a fixed weight.

*In this work,* for MDS codes, using and developing the results of [5], we obtain new convenient formulas of integral weight spectra of cosets of weight 1 and 2. The obtained formulas for weight 1 and 2 cosets, seem to be simple and expressive.

This paper is organized as follows. Section 2 contains preliminaries. In Section 3, we consider the integral weight spectrum of the weight 1 cosets of MDS codes with minimum distance $d \ge 3$. In Section 4, we obtain the integral weight spectrum of the weight 2 cosets of MDS codes with minimum distance $d \ge 5$. In Section 5, we give the spectra for the weight 3 cosets of MDS codes with minimum distance 5 and covering radius 3.

# 2 Preliminaries

## 2.1 Cosets of a linear code

We give a few known definitions and properties connected with cosets of linear codes, see e.g. [2, 11, 16, 17, 19] and the references therein.

We consider a coset $\mathcal{V}$ of an $[n,k,d]_qR$ code $\mathcal{C}$ in the form (1.1). We have $\#\mathcal{V} = \#\mathcal{C} = q^k$. One can take as $\mathbf{v}$ any vector of $\mathcal{V}$. So, there are $\#\mathcal{V} = q^k$ formally distinct representations of the form (1.1); all they give the same coset $\mathcal{V}$. If $\mathbf{v} \in \mathcal{C}$, we have $\mathcal{V} = \mathcal{C}$. The distinct cosets of $\mathcal{C}$ partition $\mathbb{F}_q^n$ into $q^{n-k}$ sets of size $q^k$.

We remind that the *Hamming weight* of the vector $\mathbf{x} \in \mathbb{F}_q^n$ is the number of nonzero entries in $\mathbf{x}$.

**Notation 2.1.** For an $[n,k,d]_qR$ code $\mathcal{C}$ and its coset $\mathcal{V}$ of the form (1.1), the following notation is used:

$t = \left\lfloor \dfrac{d-1}{2} \right\rfloor$ the number of correctable errors;

| | |
|---|---|
| $A_w(\mathcal{C})$ | the number of weight $w$ codewords of the code $\mathcal{C}$; |
| $A_w(\mathcal{V})$ | the number of weight $w$ vectors in the coset $\mathcal{V}$; |
| the weight of a coset | the smallest Hamming weight of any vector in the coset; |
| $\mathcal{V}^{(W)}$ | a coset of weight $W$; $A_w(\mathcal{V}^{(W)}) = 0$ if $w < W$; |
| a coset leader | a vector in the coset of the smallest Hamming weight; |
| $\mathcal{A}_w^\Sigma(\mathcal{V}^{(W)})$ | the overall number of weight $w$ vectors in all cosets of weight $W$; |
| $\mathcal{A}_w^\Sigma(\mathcal{V}^{\le W})$ | the overall number of weight $w$ vectors in all cosets of weight $\le W$. |

In cosets of weight $> t$, a vector of the minimal weight is not necessarily unique. Cosets of weight $\le t$ have a unique leader.

The code $\mathcal{C}$ is the coset of weight zero. The leader of $\mathcal{C}$ is the zero vector of $\mathbb{F}_q^n$.

**Definition 2.2.** Let $\mathcal{C}$ be an $[n,k,d]_qR$ code and let $\mathcal{V}^{(W)}$ be its coset of weight $W$. Let $\mathcal{A}_w^\Sigma(\mathcal{V}^{(W)})$ be the overall number of weight $w$ vectors in all cosets of weight $W$. For a fixed $W$, we call the set $\{\mathcal{A}_w^\Sigma(\mathcal{V}^{(W)})|w = 0,1,\dots,n\}$ *integral weight spectrum* of the code cosets of weight $W$.

Distinct representations of the integral weight spectra $\mathcal{A}_w^\Sigma(\mathcal{V}^{(W)})$ and values of $\mathcal{A}_w^\Sigma(\mathcal{V}^{\le W})$ are considered in the literature, see e.g. [2, Th. 14.2.2], [5, 6], [15, Lem. 2.14], [16, Th. 6.22]. For instance, in [5, Eqs. (11)–(13)], for an MDS code correcting $t$-fold errors, the value $D_u$ gives $\mathcal{A}_u^\Sigma(\mathcal{V}^{\le t})$.

## 2.2 Some useful relations

For $w \ge d$, the weight distribution $A_w(\mathcal{C})$ of an $[n,k,d = n-k+1]_q$ MDS code $\mathcal{C}$ has the following form, see e.g. [11, Th. 7.4.1], [16, Th. 11.3.6]:

$$A_w(\mathcal{C}) = \binom{n}{w}\sum_{j=0}^{w-d}(-1)^j\binom{w}{j}(q^{w-d+1-j}-1). \tag{2.1}$$

In $\mathbb{F}_q^n$, the volume of a sphere of radius $t$ is

$$V_n(t) = \sum_{i=0}^{t}(q-1)^i\binom{n}{i}. \tag{2.2}$$

The following combinatorial identities are well known, see e.g. [18, Sect. 1, Eqs. (I),(IV), Problem 9(a)]:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \tag{2.3}$$

$$\binom{n}{m}\binom{m}{p} = \binom{n}{p}\binom{n-p}{m-p} = \binom{n}{m-p}\binom{n-m+p}{p}. \tag{2.4}$$

$$\sum_{k=0}^{m}(-1)^k\binom{n}{k} = (-1)^m\binom{n-1}{m}. \tag{2.5}$$

In [5, Eqs. (11)–(13)], for an $[n,k,d \geq 2t+1]_q$ MDS code correcting $t$-fold errors, the following relations for $\mathcal{A}_u^{\Sigma}(\mathcal{V}^{\leq t})$ denoted by $D_u$ are given:

$$\mathcal{A}_u^{\Sigma}(\mathcal{V}^{\leq t}) = D_u = \binom{n}{u}\sum_{j=0}^{u-d+t}(-1)^j N_j, \;\; d-t \leq u \leq n, \tag{2.6}$$

where

$$N_j = \binom{u}{j}\left[q^{u-d+1-j}V_n(t) - \sum_{i=0}^{t}\binom{u-j}{i}(q-1)^i\right] \;\; \text{if} \;\; 0 \leq j \leq u-d, \tag{2.7}$$

$$N_j = \binom{u}{j}\left[\sum_{w=d-u+j}^{t}\binom{n-u+j}{w}\sum_{i=0}^{w-d+u-j}(-1)^i\binom{w}{i}(q^{w-d+u-j-i+1}-1)\right. \tag{2.8}$$

$$\left.\times\sum_{s=w}^{t}\binom{u-j}{s-w}(q-1)^{s-w}\right] \;\; \text{if} \;\; u-d+1 \leq j \leq u-d+t.$$

# 3 The integral weight spectrum of the weight 1 cosets of MDS codes with minimum distance $d \geq 3$

In Sections 3–5, we represent the values $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(W)})$ in distinct forms that can be convenient in distinct utilizations, e.g. for estimates of the decoder error probability, see [5, 6] and the references therein.

We use (with some transformations) the results of [5, Eqs. (11)–(13)] where, for an MDS code correcting $t$-fold errors, the value $D_u$ gives the overall number $\mathcal{A}_u^{\Sigma}(\mathcal{V}^{\leq t})$ of weight $u$ vectors in all cosets of weight $\leq t$. We cite [5, Eqs. (11)–(13)] by formulas (2.6)–(2.8), respectively.

In the rest of the paper we put that a sum $\sum_{i=0}^{A}\ldots$ is equal to zero if $A < 0$.

**Lemma 3.1.** [5, Eqs. (11)–(13)] *Let $d-1 \leq w \leq n$. For an $[n,k,d=n-k+1]_q$ MDS code $\mathcal{C}$ of minimum distance $d \geq 3$, the overall number $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{\leq 1})$ of weight $w$ vectors in all cosets of weight $\leq 1$ is as follows:*

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{\leq 1}) = \binom{n}{w}\left[\sum_{j=0}^{w-d}(-1)^j\binom{w}{j}\left[q^{w-d+1-j}(1+n(q-1)) - 1 - (w-j)(q-1)\right]\right. \tag{3.1}$$

$$-(-1)^{w-d}\binom{w}{d-1}(n-d+1)(q-1)\Bigg].$$

*Proof.* In the relations for $D_u$ of [5] cited by (2.6)–(2.8), we put $t = 1$ and then use (2.2). In (2.8), we have $j = u - d + 1$ whence $w = 1$ in all terms. Finally, we change $u$ by $w$ to save the notations of this paper. □

**Lemma 3.2.** *The following holds:*

$$\sum_{j=0}^{m}(-1)^j\binom{w}{j}\binom{w-j}{v} = (-1)^m\binom{w}{v}\binom{w-v-1}{m}. \tag{3.2}$$

*Proof.* In (2.4), we put $n = w$, $p = j$, $m - p = v$, and obtain

$$\sum_{j=0}^{m}(-1)^j\binom{w}{j}\binom{w-j}{v} = \binom{w}{v}\sum_{j=0}^{m}(-1)^j\binom{w-v}{j}.$$

Now we use (2.5). □

**Lemma 3.3.** *Let $d - 1 \le w \le n$. The following holds:*

$$\sum_{j=0}^{w+1-d}(-1)^j\binom{w}{j}q^{w+1-d-j} = \sum_{j=0}^{w-d}(-1)^j\binom{w}{j}\left(q^{w+1-d-j}-1\right) - (-1)^{w-d}\binom{w-1}{d-2}.$$

*Proof.* We write the left sum of the assertion as

$$\sum_{j=0}^{w-d}(-1)^j\binom{w}{j}\left(q^{w+1-d-j}-1+1\right) - (-1)^{w-d}\binom{w}{d-1}.$$

By (2.5),

$$\sum_{j=0}^{w-d}(-1)^j\binom{w}{j} = (-1)^{w-d}\binom{w-1}{d-1}.$$

Finally, we apply (2.3). □

For an $[n,k,d]_q$ code $\mathcal{C}$, we denote

$$\Omega_w^{(j)}(\mathcal{C}) = (-1)^{w-d}\binom{n-j}{w-j}\binom{w-j-1}{d-j-2}. \tag{3.3}$$

Also, we denote

$$\Phi_w^{(j)} = (-1)^{w-5}\left[\binom{q+1}{w}\binom{w-1}{3} - \binom{q+1-j}{w-j}\binom{w-1-j}{3-j}\right]. \tag{3.4}$$

**Theorem 3.4. (integral weight spectrum 1)**

*Let $d-1 \le w \le n$. Let $\mathcal{C}$ be an $[n,k,d=n-k+1]_q$ MDS code of minimum distance $d \ge 3$.*

**(i)** *For the code $\mathcal{C}$, the overall number $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(1)})$ of weight $w$ vectors in all weight 1 cosets is as follows:*

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(1)}) = \binom{n}{w}(q-1)\left[n\sum_{j=0}^{w+1-d}(-1)^j\binom{w}{j}q^{w+1-d-j} + (-1)^{w-d}w\binom{w-2}{d-3}\right] \tag{3.5}$$

$$= n(q-1)\left[\binom{n}{w}\sum_{j=0}^{w+1-d}(-1)^j\binom{w}{j}q^{w+1-d-j} + \Omega_w^{(1)}(\mathcal{C})\right] \tag{3.6}$$

$$= n(q-1)\left[\binom{n}{w}\sum_{j=0}^{w-d}(-1)^j\binom{w}{j}\left(q^{w+1-d-j}-1\right) - \Omega_w^{(0)}(\mathcal{C}) + \Omega_w^{(1)}(\mathcal{C})\right] \tag{3.7}$$

$$= n(q-1)\left[A_w(\mathcal{C}) - \Omega_w^{(0)}(\mathcal{C}) + \Omega_w^{(1)}(\mathcal{C})\right] \tag{3.8}$$

$$= n(q-1)\left[A_w(\mathcal{C}) - (-1)^{w-d}\left(\binom{n}{w}\binom{w-1}{d-2} - \binom{n-1}{w-1}\binom{w-2}{d-3}\right)\right]. \tag{3.9}$$

**(ii)** *Let the code $\mathcal{C}$ be a $[q+1,k,d=q+2-k]_q$ MDS code of length $n=q+1$ and minimum distance $d \ge 3$. For $\mathcal{C}$, the overall number $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(1)})$ of weight $w$ vectors in all weight 1 cosets is as follows*

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(1)}) = \binom{q+1}{w}(q-1)\left[q^{w+2-d} - \sum_{i=0}^{w-d}(-1)^i\left(\binom{w}{i+1} - \binom{w}{i}\right)q^{w+1-d-i}\right. \tag{3.10}$$

$$\left. -(-1)^{w-d}\left(\binom{w}{d-1} - w\binom{w-2}{d-3}\right)\right],\ d-1 \le w \le q+1.$$

**(iii)** *Let the code $\mathcal{C}$ be a $[q+1,q-3,5]_q$ MDS code of length $n=q+1$ and minimum distance $d=5$. For $\mathcal{C}$, the overall number $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(1)})$ of weight $w$ vectors in all weight 1 cosets is as follows*

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(1)}) = (q^2-1)\left[A_w(\mathcal{C}) - \Phi_w^{(1)}\right],\ 4 \le w \le q+1. \tag{3.11}$$

*Proof.* **(i)** By the definition of $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{\le 1})$, see Notation 2.1, for the code $\mathcal{C}$ of Lemma 3.1, we have

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(1)}) = \mathcal{A}_w^{\Sigma}(\mathcal{V}^{\le 1}) - A_w(\mathcal{C}). \tag{3.12}$$

We subtract (2.1) from (3.1) that gives

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(1)}) = \binom{n}{w}(q-1)\left[-(-1)^{w-d}\binom{w}{d-1}(n-d+1)\right.$$

$$+\sum_{j=0}^{w-d}(-1)^j\binom{w}{j}\left(q^{w-d+1-j}n-w+j\right)\Bigg]$$

$$=\binom{n}{w}(q-1)\left[n\sum_{j=0}^{w-d+1}(-1)^j\binom{w}{j}q^{w-d+1-j}-\sum_{j=0}^{w-d+1}(-1)^j\binom{w}{j}(w-j)\right].$$

Here some simple transformations are missed out. Now, for the 2-nd sum $\sum_{j=0}^{w-d+1}\ldots$, we use Lemma 3.2 and obtain (3.5).

To form (3.6) from (3.5), we change $w\binom{n}{w}$ by $n\binom{n-1}{w-1}$, see (2.4). To obtain (3.7) from (3.6), we apply Lemma 3.3. For (3.8), we use (2.1). Finally, (3.9) is (3.8) in detail.

**(ii)** We substitute $n = q + 1$ to (3.5) that implies (3.10) after simple transformations.

**(iii)** We substitute $n = q + 1$ and $d = 5$ to (3.9) that gives (3.11). □

For $\mathcal{A}_w^\Sigma(\mathcal{V}^{\le 1})$, we give a formula alternative to (3.1).

**Corollary 3.5.** *Let $V_n(1)$ be as in (2.2). Let $\mathcal{C}$ be an $[n, k, d = n - k + 1]_q$ MDS code of minimum distance $d \ge 3$. Then for $\mathcal{C}$, the overall number $\mathcal{A}_w^\Sigma(\mathcal{V}^{\le 1})$ of weight $w$ vectors in all cosets of weight $\le 1$ is as follows:*

$$\mathcal{A}_w^\Sigma(\mathcal{V}^{\le 1}) = A_w(\mathcal{C})\cdot V_n(1) - (-1)^{w-d}n(q-1)\sum_{j=0}^{1}(-1)^j\binom{n-j}{w-j}\binom{w-j-1}{d-j-2}. \tag{3.13}$$

*Proof.* We use (3.12) and (3.9). □

# 4 The integral weight spectrum of the weight 2 cosets of MDS codes with minimum distance $d \ge 5$

As well as in Lemma 3.1, we use the results of [5] with some transformations.

**Lemma 4.1.** [5, Eqs. (11)–(13)] *Let $d - 2 \le w \le n$. Let $V_n(t)$ be as in (2.2). For an $[n, k, d = n - k + 1]_q$ MDS code $\mathcal{C}$ of minimum distance $d \ge 5$, the overall number $\mathcal{A}_w^\Sigma(\mathcal{V}^{\le 2})$ of weight $w$ vectors in all cosets of weight $\le 2$ is as follows:*

$$\mathcal{A}_w^\Sigma(\mathcal{V}^{\le 2}) = \binom{n}{w}\Bigg[\sum_{j=0}^{w-d}(-1)^j\binom{w}{j}\left[q^{w-d+1-j}\cdot V_n(2) - V_{w-j}(2)\right] \tag{4.1}$$

$$-(-1)^{w-d}\frac{(n-d+1)(q-1)}{2}\left(\binom{w}{d-1}[2+(q-1)(n+d-2)] - \binom{w}{d-2}(n-d+2)\right)\Bigg].$$

*Proof.* In the relations for $D_u$ of [5] cited by (2.6)–(2.8), we put $t = 2$ that gives, in (2.8), $j = u - d + 1$ and $j = u - d + 2$, whence $w = 1, 2$ and $w = 2$, respectively. Then we do simple transformations. Finally, we change $u$ by $w$ to save the notations of this paper. □

For an $[n,k,d]_q$ code $\mathcal{C}$, we denote

$$\Delta_w(\mathcal{C}) = (-1)^{w-d}\binom{n}{w}\binom{w}{d-2}\binom{n-d+2}{2}(q-1); \tag{4.2}$$
$$\Delta^\star_w(\mathcal{C}) = \frac{\Delta_w(\mathcal{C})}{\binom{n}{2}(q-1)^2}.$$

**Lemma 4.2.** *The following holds:*

$$\Delta^\star_w(\mathcal{C}) = (-1)^{w-d}\binom{n-d+2}{n-w}\binom{n-2}{d-2}\frac{1}{q-1}. \tag{4.3}$$

*Proof.* By (2.4), we have

$$\binom{n}{w}\binom{w}{d-2} = \binom{n}{d-2}\binom{n-d+2}{w-d-2} = \binom{n}{d-2}\binom{n-d+2}{n-w},$$
$$\binom{n}{d-2}\binom{n-d+2}{2} = \binom{n}{d}\binom{d}{d-2} = \binom{n}{d}\binom{d}{2} = \binom{n}{2}\binom{n-2}{d-2}.$$

□

**Theorem 4.3. (integral weight spectrum 2)**

*Let $d-2 \le w \le n$. Let $\mathcal{C}$ be an $[n,k,d=n-k+1]_q$ MDS code of minimum distance $d \ge 5$. Let $\Omega^{(j)}_w(\mathcal{C})$ and $\Phi^{(j)}_w$ be as in (3.3) and (3.4).*

**(i)** *For the code $\mathcal{C}$, the overall number $\mathcal{A}^{\Sigma}_w(\mathcal{V}^{(2)})$ of weight $w$ vectors in all weight 2 cosets is as follows:*

$$\mathcal{A}^{\Sigma}_w(\mathcal{V}^{(2)}) = \binom{n}{w}(q-1)^2\left[\binom{n}{2}\sum_{j=0}^{w+1-d}(-1)^j\binom{w}{j}q^{w+1-d-j} + (-1)^{w-d}\binom{w}{2}\binom{w-3}{d-4}\right] \tag{4.4}$$
$$+ \Delta_w(\mathcal{C}).$$
$$= \binom{n}{2}(q-1)^2\left[\binom{n}{w}\sum_{j=0}^{w+1-d}(-1)^j\binom{w}{j}q^{w+1-d-j} + \Omega^{(2)}_w(\mathcal{C})\right] + \Delta_w(\mathcal{C}). \tag{4.5}$$
$$= \binom{n}{2}(q-1)^2\left[\binom{n}{w}\sum_{j=0}^{w-d}(-1)^j\binom{w}{j}\left(q^{w+1-d-j}-1\right) - \Omega^{(0)}_w(\mathcal{C}) + \Omega^{(2)}_w(\mathcal{C})\right] + \Delta_w(\mathcal{C}) \tag{4.6}$$
$$= \binom{n}{2}(q-1)^2\left[A_w(\mathcal{C}) - \Omega^{(0)}_w(\mathcal{C}) + \Omega^{(2)}_w(\mathcal{C})\right] + \binom{n}{2}(q-1)^2\Delta^\star_w(\mathcal{C}) \tag{4.7}$$

$$= \binom{n}{2}(q-1)^2\left[A_w(\mathcal{C}) - (-1)^{w-d}\left(\binom{n}{w}\binom{w-1}{d-2} - \binom{n-2}{w-2}\binom{w-3}{d-4}\right)\right] \tag{4.8}$$
$$+ (-1)^{w-d}\binom{n}{2}(q-1)\binom{n-d+2}{n-w}\binom{n-2}{d-2}.$$

**(ii)** *Let the code $\mathcal{C}$ be a $[q+1, q-3, 5]_q$ MDS code of length $n = q+1$ and minimum distance $d = 5$. For $\mathcal{C}$, the overall number $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(1)})$ of weight $w$ vectors in all weight 1 cosets is as follows*

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(2)}) = \binom{q+1}{2}(q-1)^2\left[A_w(\mathcal{C}) - \Phi_w^{(2)} + (-1)^{w-5}\frac{1}{3}\binom{q-2}{w-3}\binom{q-2}{2}\right], \tag{4.9}$$
$$3 \le w \le q+1.$$

*Proof.* **(i)** By the definition of $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{\le 2})$, see Notation 2.1, for the code $\mathcal{C}$ of Lemma 4.1, we have

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(2)}) = \mathcal{A}_w^{\Sigma}(\mathcal{V}^{\le 2}) - \mathcal{A}_w^{\Sigma}(\mathcal{V}^{\le 1}). \tag{4.10}$$

We subtract (3.1) from (4.1) that gives

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(2)}) = \binom{n}{w}\left[\sum_{j=0}^{w-d}(-1)^j\binom{w}{j}\left(q^{w+1-d-j}\binom{n}{2}(q-1)^2 - \binom{w-j}{2}(q-1)^2\right)\right.$$
$$\left. +(-1)^{w+1-d}\binom{w}{d-1}\frac{1}{2}(n-d+1)(q-1)^2(n+d-2)\right] + \Delta_w(\mathcal{C})$$
$$= \binom{n}{w}(q-1)^2\left[\binom{n}{2}\sum_{j=0}^{w-d}(-1)^j\binom{w}{j}q^{w+1-d-j} - \sum_{j=0}^{w-d}(-1)^j\binom{w}{j}\binom{w-j}{2}\right.$$
$$\left. -(-1)^{w-d}\binom{w}{d-1}\left(\frac{1}{2}(n-d+1)(n+d-2) + \binom{n}{2} - \binom{n}{2}\right)\right] + \Delta_w(\mathcal{C}).$$

Applying Lemma 3.2 to the 2-nd sum $\sum_{j=0}^{w-d}\ldots$, after simple transformations we obtain

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(2)}) = \binom{n}{w}(q-1)^2\left[\binom{n}{2}\sum_{j=0}^{w+1-d}(-1)^j\binom{w}{j}q^{w+1-d-j} - (-1)^{w-d}\binom{w}{2}\binom{w-3}{w-d}\right.$$
$$\left. +(-1)^{w-d}\binom{w}{d-1}\binom{d-1}{2}\right] + \Delta_w(\mathcal{C}).$$

Due to (2.4) and (2.3), we have

$$\binom{w}{d-1}\binom{d-1}{2} = \binom{w}{2}\binom{w-2}{d-3} = \binom{w}{2}\left[\binom{w-3}{d-4} + \binom{w-3}{d-3}\right].$$

Also, $\binom{w-3}{w-d} = \binom{w-3}{d-3}$. Now we can obtain (4.4). Moreover, by (2.4), we have

$$\binom{n}{w}\binom{w}{2} = \binom{n}{2}\binom{n-2}{w-2}$$

that gives (4.5).

To obtain (4.6) from (4.5), we apply Lemma 3.3. For (4.7), we use (2.1). Finally, (4.8) is (4.7) in detail.

**(ii)** We substitute $n = q+1$ and $d = 5$ to (4.8) that gives (4.9). □

# 5 The integral weight spectrum of the weight 3 cosets of MDS codes with minimum distance $d = 5$ and covering radius $R = 3$

**Theorem 5.1. (integral weight spectrum 3)**

*Let $d-2 \le w \le n$. Let $\mathcal{C}$ be an $[n, n-4, 5]_q3$ MDS code of minimum distance $d = 5$ and covering radius $R = 3$. Let $V_n(t)$, $\Phi_w^{(j)}$, $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{\le 2})$, and $\Delta_w(\mathcal{C})$ be as in (2.2), (3.4), (4.1), and (4.2), respectively. Let $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(1)})$ and $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(2)})$ be as in Theorems 3.4 and 4.3, respectively.*

**(i)** *For the code $\mathcal{C}$, the overall number $\mathcal{A}_w^{\Sigma}(\mathcal{V}^3)$ of weight $w$ vectors in all cosets of weight 3 is as follows:*

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(3)}) = \binom{n}{w}(q-1)^w - \mathcal{A}_w^{\Sigma}(\mathcal{V}^{\le 2}) \tag{5.1}$$

$$= \binom{n}{w}(q-1)^w - \left[A_w(\mathcal{C}) + \mathcal{A}_w^{\Sigma}(\mathcal{V}^{(1)}) + \mathcal{A}_w^{\Sigma}(\mathcal{V}^{(2)})\right] \tag{5.2}$$

$$= \binom{n}{w}(q-1)^w - \left[\binom{n}{w}\sum_{j=0}^{w-5}(-1)^j\binom{w}{j}\left[q^{w-4-j}\cdot V_n(2) - V_{w-j}(2)\right]\right. \tag{5.3}$$

$$\left. -(-1)^{w-5}\frac{(n-4)(q-1)}{2}\left(\binom{w}{4}[2+(q-1)(n+3)] - \binom{w}{3}(n-3)\right)\right].$$

**(ii)** *Let the code $\mathcal{C}$ be a $[q+1, q-3, 5]_q3$ MDS code of length $n = q+1$, minimum distance $d = 5$, and covering radius $R = 3$. For $\mathcal{C}$, the overall number $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(3)})$ of weight $w$ vectors* *in all weight* 3 *cosets is as follows*

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(3)}) = \binom{q+1}{w}(q-1)^w - \left[\binom{q+1}{w}\sum_{j=0}^{w-5}(-1)^j\binom{w}{j}\left[q^{w-4-j}\cdot V_{q+1}(2) - V_{w-j}(2)\right]\right. \quad (5.4)$$

$$\left. -(-1)^{w-5}\frac{(q-3)(q-1)}{2}\left(\binom{w}{4}(q^2+3q-2) - \binom{w}{3}(q-2)\right)\right]$$

$$= \binom{q+1}{w}(q-1)^w - \left[V_{q+1}(2)A_w(\mathcal{C}) - (q^2-1)\Phi_w^{(1)} - \binom{q+1}{2}(q-1)^2\Phi_w^{(2)} - \Delta_w(\mathcal{C})\right]. \quad (5.5)$$

*Proof.* **(i)** Due to covering radius 3, in $\mathcal{C}$ there are not cosets of weight $> 3$; therefore for $\mathcal{C}$ we have (5.1) where $\binom{n}{w}(q-1)^w$ is the total number of weight $w$ vectors in $\mathbb{F}_q^n$.

The relation (5.2) follows from (5.1), (3.12), and (4.10).

To form (5.3), we substitute (4.1) to (5.1) with $d = 5$.

**(ii)** We substitute $n = q + 1$ to (5.3) and obtain (5.4).

To obtain (5.5) from (5.2), we use (3.11), (4.9), (4.2), and (4.3) with $n = q+1$, $d = 5$. □

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