On cosets weight distributions of the doubly-extended Reed-Solomon codes of codimension 4

Alexander A. Davydov*

Institute for Information Transmission Problems (Kharkevich institute), Russian Academy of Sciences

Bol'shoi Karetnyi per. 19, Moscow, 127051, Russian Federation. E-mail: adav@iitp.ru

Stefano Marcugini† and Fernanda Pambianco†

Department of Mathematics and Computer Science, University of Perugia,

Via Vanvitelli 1, Perugia, 06123, Italy. E-mail: {stefano.marcugini,fernanda.pambianco}@unipg.it

Abstract. We consider the $[q+1, q-3, 5]_q3$ generalized doubly-extended Reed-Solomon code of codimension 4 as the code associated with the twisted cubic in the projective space $\mathrm{PG}(3, q)$. Basing on the point-plane incidence matrix of $\mathrm{PG}(3, q)$, we obtain the number of weight 3 vectors in all the cosets of the considered code. This allows us to classify the cosets by their weight distributions and to obtain these distributions. For the cosets of equal weight having distinct weight distributions, we prove that the difference between the w-th components, $3 < w \leq q+1$, of the distributions is unambiguously determined by the difference between the 3-rd components. This implies an interesting (and in some sense unexpected) symmetry of the obtained distributions. To obtain the property of differences we prove a useful relation for the Krawtchouck polynomials. Also, we describe an alternative way of obtaining the cosets weight distributions on the base of the integral weight spectra over all the cosets of the fixed weight.

Keywords: Reed-Solomon codes, cosets weight distributions, MDS codes, twisted cubic, projective spaces.

Mathematics Subject Classication (2010). 94B05, 51E21, 51E22

---

*The research of A.A. Davydov was done at IITP RAS and supported by the Russian Government (Contract No 14.W03.31.0019).

†The research of S. Marcugini and F. Pambianco was supported in part by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INDAM) (No U-UFMBAZ-2017-000532, 28.11.2017) and by University of Perugia (Project: Curve, codici e configurazioni di punti, Base Research Fund 2018, No 46459, 15.06.2018).

1 Introduction

Let $\mathbb{F}_q$ be the Galois field with q elements, $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, $\mathbb{F}_q^+ = \mathbb{F}_q \cup \{\infty\}$. Let $\mathbb{F}_q^n$ be the space of n-dimensional vectors over $\mathbb{F}_q$. We denote by $[n, k, d]_q R$ an $\mathbb{F}_q$-linear code of length n, dimension k, minimum distance d, and covering radius R. If $d = n - k + 1$, it is a maximum distance separable (MDS) code. Let $\mathrm{PG}(N, q)$ be the N-dimensional projective space over $\mathbb{F}_q$. For an introduction to coding theory see [1–3]. For an introduction to projective spaces over finite fields and connections between projective geometry and coding theory see [4–9].

In a parity check $(N+1) \times (q+1)$ matrix of a $[q+1, q-N, N+2]_q$ generalized doubly-extended Reed-Solomon (GDRS) code, the j-th column has the form $(v_j, v_j\alpha_j, v_j\alpha_j^2, \ldots, v_j\alpha_j^N)^{tr}$, where $j = 1, 2, \ldots, q$; $\alpha_1, \ldots, \alpha_q$ are distinct elements of $\mathbb{F}_q$; $v_1, \ldots, v_q$ are nonzero (not necessarily distinct) elements of $\mathbb{F}_q$. Also, this matrix contains one more column $(0, \ldots, 0, v)^{tr}$ with $v \neq 0$. The code, dual to a GDRS code, is a GDRS code too. Clearly, a GDRS code is MDS.

An n-arc in $\mathrm{PG}(N, q)$, with $n \geq N + 1 \geq 3$, is a set of n points such that no $N + 1$ points belong to the same hyperplane of $\mathrm{PG}(N, q)$. An n-arc is complete if it is not contained in an $(n + 1)$-arc. Arcs and MDS codes are equivalent objects, see e.g. [2, 8, 9].

In $\mathrm{PG}(N, q)$, $2 \leq N \leq q - 2$, a normal rational curve is any $(q + 1)$-arc projectively equivalent to the arc $\{(1, t, t^2, \ldots, t^{N-1}, t^N) : t \in \mathbb{F}_q\} \cup \{(0, \ldots, 0, 1)\}$. The points (in homogeneous coordinates) of a normal rational curve in $\mathrm{PG}(N, q)$ treated as columns define a parity check matrix of a $[q+1, q-N, N+2]_q$ GDRS code [3, 8, 9]. We say that this GDRS code is *associated* with the normal rational curve. In $\mathrm{PG}(3, q)$, the normal rational curve is called a *twisted cubic* [7, 10]. We denote the cubic by $\mathscr{C}$, see Section 2.1 for detail.

Let $\mathcal{C}_{\mathscr{C}}$ be the $[q + 1, q - 3, 5]_q 3$ GDRS code associated with the cubic $\mathscr{C}$.

Twisted cubics in $\mathrm{PG}(3, q)$ have been widely studied; see e.g. [10–13] and the references therein. In particular, in [10], the orbits of planes and points under the group G_q of the projectivities fixing a cubic are considered. In [13], the structure of the point-plane incidence matrix in $\mathrm{PG}(3, q)$ with respect to the orbits of points and planes under G_q is described.

Also in [13], it is shown that twisted cubics can be treated as multiple ρ-saturating sets with $\rho = 2$ which, in turn, give rise to asymptotically optimal non-binary linear multiple covering $[q+1, q-3, 5]_q 3$ codes of radius $R = 3$. This means that the code $\mathcal{C}_{\mathscr{C}}$ can be viewed as an asymptotically optimal multiple covering. For an introduction to multiple covering codes and multiple saturating sets see [14–16].

A *coset* of a code is a translation of the code. A coset $\mathcal{V}$ of an $[n, k, d]_q R$ code $\mathcal{C}$ can be represented as

$$\mathcal{V} = \{\mathbf{x} \in \mathbb{F}_q^n \,|\, \mathbf{x} = \mathbf{c} + \mathbf{v}, \mathbf{c} \in \mathcal{C}\} \subset \mathbb{F}_q^n \tag{1}$$

where $\mathbf{v} \in \mathcal{V}$ is a vector fixed for the given representation; see [1–3, 17, 18] and the references therein.

The weight distribution of code cosets, including the number of the cosets with distinct distributions, is interesting itself; it is an important combinatorial property of a code. In particular, the distribution serves to estimate decoding results. There are many papers connected with distinct aspects of the weight distribution of cosets for codes over distinct fields and rings, see e.g. [1], [2, Sec. 5.5, 6.6, 6.9], [17, Sec. 7], [18, Sec. 10], [19–33], [34, Sec. 6.3] and the references therein. Nevertheless, as far as it is known to the authors, the weight distribution of GDRS code cosets is an open problem.

In this paper, we consider the weight distribution of the cosets of the $[q+1, q-3, 5]_q 3$ GDRS code of codimension 4. We consider it as the code $\mathcal{C}_\mathscr{C}$ associated with the twisted cubic. By a geometrical way, basing on the PG$(3,q)$ point-plane incidence matrix of [13], we obtain the number of weight 3 vectors in all the cosets of $\mathcal{C}_\mathscr{C}$. This allows us to use the approach of [17, Sec. 7], [18, Sec. 10]. As a result, we classify the cosets of $\mathcal{C}_\mathscr{C}$ by their weight distributions and obtain the relations that fully describe the needed weight distributions.

For the cosets of equal weight having distinct weight distributions, we prove the following *property of differences*: the difference between the w-th components, $3 < w \leq q+1$, of the distributions is unambiguously determined by the difference between the 3-rd components. This implies an interesting (and in some sense unexpected) *symmetry of the weight distributions of the* $[q+1, q-3, 5]_q 3$ *code cosets*. To obtain the property of differences we prove an useful relation for the Krawtchouck polynomials, see Lemma 20.

Also, we propose an alternative way of obtaining the cosets weight distributions on the base of the integral weight spectra of the cosets. (For a linear code of length n, we call *integral weight spectrum* the overall numbers of weight w vectors, $0 \leq w \leq n$, in all the cosets of some fixed weight).

This paper is organized as follows. Section 2 contains preliminaries. In Section 3, we give the distribution of weight 3 vectors in the cosets of the code $\mathcal{C}_\mathscr{C}$ and classify the cosets. In Section 4, the relations that fully describe the weight distribution of the cosets of $\mathcal{C}_\mathscr{C}$ are obtained. In Section 5, we prove the property of differences and consider a symmetry in the weight distributions of the code $\mathcal{C}_\mathscr{C}$ cosets. An useful relation for the Krawtchouck polynomials is proven. Finally, in Section 6, we describe an alternative way of obtaining the cosets weight distributions on the base of the integral weight spectra.

Throughout the paper, we consider $q \geq 5$.

2 Preliminaries

2.1 Twisted cubic

In this subsection, we summarize some known results on twisted cubics from [10, Ch. 21].

Let $\mathbf{P}(x_0, x_1, x_2, x_3)$ be a point of PG$(3,q)$ with the homogeneous coordinates $x_i \in \mathbb{F}_q$;

the leftmost nonzero coordinate is equal to 1. For $t \in \mathbb{F}_q^+$, let $P(t)$ be a point such that

$$P(t) = \mathbf{P}(1, t, t^2, t^3) \text{ if } t \in \mathbb{F}_q,$$
$$P(\infty) = \mathbf{P}(0, 0, 0, 1).$$

Let $\mathscr{C} \subset \mathrm{PG}(3, q)$ be the *twisted cubic* consisting of $q+1$ points $P_1, \ldots, P_{q+1}$ no four of which are coplanar. We consider $\mathscr{C}$ in the canonical form

$$\mathscr{C} = \{P_1, P_2, \ldots, P_{q+1}\} = \{P(t) | t \in \mathbb{F}_q^+\}. \quad (2)$$

Let $\boldsymbol{\pi}(c_0, c_1, c_2, c_3) \subset \mathrm{PG}(3, q)$ with $c_i \in \mathbb{F}_q$ be the plane with equation $c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3 = 0$. In every point $P(t) \in \mathscr{C}$, there is an *osculating plane* $\pi_{\mathrm{osc}}(t)$ such that $\pi_{\mathrm{osc}}(t) = \boldsymbol{\pi}(-t^3, 3t^2, -3t, 1)$, $P(t) \in \pi_{\mathrm{osc}}(t)$.

A *chord* of $\mathscr{C}$ is a line through a pair of real points of $\mathscr{C}$ or a pair of complex conjugate points. In the last case it is an *imaginary chord*. If the real points are distinct, it is a *real chord*. If the real points coincide with each other, it is a *tangent*.

Notation 1. The following notation is used:

G_q	the group of projectivities in $\mathrm{PG}(3, q)$ fixing $\mathscr{C}$;
Γ	the osculating developable to $\mathscr{C}$;
Γ-plane	an osculating plane of Γ;
$d_{\mathscr{C}}$-plane	a plane containing *exactly* d distinct points of $\mathscr{C}$, $d = 0, 1, 2, 3$;
$1_{\mathscr{C}} \setminus \Gamma$-plane	a $1_{\mathscr{C}}$-plane not in Γ;
$\mathscr{C}$-point	a point of $\mathscr{C}$;
μ_Γ-point	a point off $\mathscr{C}$ lying on *exactly* μ osculating planes, $\mu_\Gamma = 0, 1, 3, q+1$;
T-point	a point off $\mathscr{C}$ on a tangent to $\mathscr{C}$ for $q \not\equiv 0 \pmod 3$;
TO-point	a point off $\mathscr{C}$ on a tangent and one osculating plane for $q \equiv 0 \pmod 3$;
RC-point	a point off $\mathscr{C}$ on a real chord;
IC-point	a point on an imaginary chord;
$\#S$	the cardinality of a set S.

Theorem 2. [10, Ch. 21] *Under G_q, there are five orbits $\mathscr{N}_i$ of planes and five orbits $\mathscr{M}_j$ of points. These orbits have the following properties:*

(i) *For all q, the orbits $\mathscr{N}_i$ of planes are as follows: $\mathscr{N}_1 = \{\Gamma$-planes$\}$, $\mathscr{N}_2 = \{2_{\mathscr{C}}$-planes$\}$, $\mathscr{N}_3 = \{3_{\mathscr{C}}$-planes$\}$, $\mathscr{N}_4 = \{1_{\mathscr{C}} \setminus \Gamma$-planes$\}$, $\mathscr{N}_5 = \{0_{\mathscr{C}}$-planes$\}$.*

(ii) *For $q \not\equiv 0 \pmod 3$, the orbits $\mathscr{M}_j$ of points are as follows:*

$$\mathscr{M}_1 = \mathscr{C}, \mathscr{M}_2 = \{\text{T-}points\}, \mathscr{M}_3 = \{3_\Gamma\text{-}points\}, \mathscr{M}_4 = \{1_\Gamma\text{-}points\}, \mathscr{M}_5 = \{0_\Gamma\text{-}points\}; \quad (3)$$

$\#\mathscr{M}_1 = q+1$, $\#\mathscr{M}_2 = q(q+1)$, $\#\mathscr{M}_3 = \dfrac{q^3-q}{6}$, $\#\mathscr{M}_4 = \dfrac{q^3-q}{2}$, $\#\mathscr{M}_5 = \dfrac{q^3-q}{3}$.

Also,

$$\text{if } q \equiv 1 \pmod 3 \text{ then } \mathscr{M}_3 \cup \mathscr{M}_5 = \{\text{RC-}points\},\ \mathscr{M}_4 = \{\text{IC-}points\}; \tag{4}$$

$$\text{if } q \equiv -1 \pmod 3 \text{ then } \mathscr{M}_3 \cup \mathscr{M}_5 = \{\text{IC-}points\},\ \mathscr{M}_4 = \{\text{RC-}points\}. \tag{5}$$

(iii) *For* $q \equiv 0 \pmod 3$, *the orbits* $\mathscr{M}_k$ *of points are as follows:*

$$\mathscr{M}_1 = \mathscr{C},\ \mathscr{M}_2 = \{(q+1)_\Gamma\text{-}points\},\ \mathscr{M}_3 = \{\text{TO-}points\},\ \mathscr{M}_4 = \{\text{RC-}points\}, \tag{6}$$

$$\mathscr{M}_5 = \{\text{IC-}points\};\ \#\mathscr{M}_1 = \#\mathscr{M}_2 = q+1;\ \#\mathscr{M}_3 = q^2-1;\ \#\mathscr{M}_4 = \#\mathscr{M}_5 = \frac{q^3-q}{2}.$$

(iv) *No two chords of* $\mathscr{C}$ *meet off* $\mathscr{C}$. *Every point off* $\mathscr{C}$ *lies on exactly one chord of* $\mathscr{C}$.

2.2 The number of $3_\mathscr{C}$-planes through points and lines of $\mathrm{PG}(3,q)$

Lemma 3. [13, Lem. 4.3] *The number of* $3_\mathscr{C}$*-planes and* $2_\mathscr{C}$*-planes through a real chord of* $\mathscr{C}$ *is equal to* $q-1$ *and* 2, *respectively.*

Lemma 4. [13, Lem. 4.12] *Through every point of the orbit* $\mathscr{M}_j$ *we have the same number of planes from the orbit* $\mathscr{N}_i$.

Throughout the paper, for $q \equiv \xi \pmod 3$, let $r_{3j}^{(\xi)}$ be the number of $3_\mathscr{C}$-planes from the orbit $\mathscr{N}_3$ through every point of the orbit $\mathscr{M}_j$.

Theorem 5. [13, Th. 3.1, Tab. 1, 2] *Let* $q \equiv \xi \pmod 3$. *Let* $r_{3j}^{(\xi)}$ *be as above. The following holds:*

(i) $r_{31}^{(1)} = r_{31}^{(-1)} = \dfrac{1}{2}(q^2-q)$, $r_{32}^{(1)} = r_{32}^{(-1)} = \dfrac{1}{6}(q^2-3q+2)$;

$r_{33}^{(1)} = \dfrac{1}{6}(q^2+q+4)$, $r_{34}^{(1)} = \dfrac{1}{6}(q^2-q)$, $r_{35}^{(1)} = \dfrac{1}{6}(q^2+q-2)$;

$r_{33}^{(-1)} = \dfrac{1}{6}(q^2-q+4)$, $r_{34}^{(-1)} = \dfrac{1}{6}(q^2+q)$, $r_{35}^{(-1)} = \dfrac{1}{6}(q^2-q-2)$.

(ii) $r_{31}^{(0)} = \dfrac{1}{2}(q^2-q)$, $r_{32}^{(0)} = r_{35}^{(0)} = \dfrac{1}{6}(q^2-q)$, $r_{33}^{(0)} = \dfrac{1}{6}(q^2-3q)$, $r_{34}^{(0)} = \dfrac{1}{6}(q^2+q)$.

2.3 Cosets of a linear code

We give a few known definitions and properties connected with cosets of linear codes, see e.g. [1–3, 17, 18] and the references therein.

We consider a coset $\mathcal{V}$ of an $[n,k,d]_qR$ code $\mathcal{C}$ in the form (1). We have $\#\mathcal{V} = \#\mathcal{C} = q^k$. One can take as $\mathbf{v}$ any vector of $\mathcal{V}$. So, there are $\#\mathcal{V} = q^k$ formally distinct representations of the form (1); all they give the same coset $\mathcal{V}$. If $\mathbf{v} \in \mathcal{C}$, we have $\mathcal{V} = \mathcal{C}$. The distinct cosets of $\mathcal{C}$ partition $\mathbb{F}_q^n$ into q^{n-k} sets of size q^k.

Notation 6. For an $[n,k,d]_qR$ code $\mathcal{C}$ and its coset $\mathcal{V}$ of the form (1), the following notation is used:

$t = \left\lfloor \dfrac{d-1}{2} \right\rfloor$	the number of correctable errors;
$wt(\mathbf{x})$	the Hamming weight of a vector $\mathbf{x} \in \mathbb{F}_q^n$;
$A_w(\mathcal{C})$	the number of weight w codewords of $\mathcal{C}$;
$S(\mathcal{C})$	the set of nonzero weights in $\mathcal{C}$; $S(\mathcal{C}) = \{w > 0 \vert A_w(\mathcal{C}) \neq 0\}$;
$s(\mathcal{C}) = \#S(\mathcal{C})$	the number of nonzero weights in $\mathcal{C}$;
$\mathcal{C}^\perp$	the $[n, n-k, d^\perp]_q R^\perp$ code dual to $\mathcal{C}$;
$\mathbf{v} + \mathcal{C}$	the coset of $\mathcal{C}$ of the form (1);
$A_w(\mathcal{V})$	the number of weight w vectors in $\mathcal{V}$;
the weight of a coset	the smallest Hamming weight of any vector in the coset;
$\mathcal{V}^{(W)}$	a coset of weight W; if $w < W$ then $A_w(\mathcal{V}^{(W)}) = 0$;
a coset leader	a vector in the coset of the smallest Hamming weight;
$H(\mathcal{C})$	a parity check $(n-k) \times n$ matrix of $\mathcal{C}$;
$\mathbf{x}^{tr}$	the transposed vector $\mathbf{x}$;
$H(\mathcal{C})\mathbf{x}^{tr}$	the *syndrome* of a vector $\mathbf{x} \in \mathbb{F}_q^n$, $H(\mathcal{C})\mathbf{x}^{tr} \in \mathbb{F}_q^{n-k}$;
a coset syndrome	the syndrome of any vector of the coset.

In cosets of weight $> t$, a vector of the minimal weight is not necessarily unique. Cosets of weight $\leq t$ have a unique leader.

The code $\mathcal{C}$ is the coset of weight zero. The leader of $\mathcal{C}$ is the zero vector of $\mathbb{F}_q^n$.

All vectors in a code coset have the same syndrome; it is called the *coset syndrome*. Thus, there is a one-to-one correspondence between cosets and syndromes. The syndrome of $\mathcal{C}$ is the zero vector of $\mathbb{F}_q^{n-k}$.

A linear $[n,k]_q$ code has covering radius R if every column of $\mathbb{F}_q^{n-k}$ is equal to a linear combination of at most R columns of a parity check matrix of the code, and R is the smallest value with such property.

The covering radius R of the code $\mathcal{C}$ is equal to the maximum weight of a coset of $\mathcal{C}$.

Theorem 7. (i) [17, Lem. 7.5.1] *For a code* $\mathcal{C}$*, the weight distributions of all cosets* $\alpha\mathbf{v}+\mathcal{C}$*,* $\alpha \in \mathbb{F}_q^*$*, are identical.*

(ii) [2, Th. 6.20], [17, Th. 7.5.2], [18, Th. 10.10] *For a code* $\mathcal{C}$*, the weight distribution of any coset of weight* $< s(\mathcal{C}^\perp)$ *is unambiguously determined if, in the coset, the numbers of vectors of weights* $1, 2, \ldots, s(\mathcal{C}^\perp) - 1$ *are known.*

As far as it is known to the authors, the weight distribution of cosets of GDRS codes, including the number of the cosets with distinct weight distributions, is an open combinatorial problem.

3 The distribution of weight 3 vectors in the cosets of the code $\mathcal{C}_{\mathscr{C}}$. Classification of the cosets

Notation 8. Let $\mathcal{C}_{\mathscr{C}}$ be the $[q+1, q-3, 5]_q3$ GDRS code *associated* with the twisted cubic $\mathscr{C}$ of (2) so that columns of its parity check matrix are points of $\mathscr{C}$ in homogeneous coordinates.

In $PG(3,q)$, every point off $\mathscr{C}$ lies in a few $3_{\mathscr{C}}$-planes, see Section 2.2; therefore, the covering radius of $\mathcal{C}_{\mathscr{C}}$ is equal to 3 and the cosets of $\mathcal{C}_{\mathscr{C}}$ have weight ≤ 3.

Remark 9. Every point P of $\mathrm{PG}(3,q)$ gives rise to $q-1$ nonzero syndromes of cosets of $\mathcal{C}_{\mathscr{C}}$; the syndromes can be generated by multiplying P (in homogeneous coordinates) by elements of $\mathbb{F}_q^*$. Points of each orbit $\mathscr{M}_j$ generate $\#\mathscr{M}_j \cdot (q-1)$ syndromes corresponding to a set of cosets with the same weight distribution.

Theorem 10. *Let $q \equiv \xi \pmod 3$. The distribution of weight 3 vectors in all the cosets of the code $\mathcal{C}_{\mathscr{C}}$ is given by Table 1.*

Proof. The columns of the parity check matrix $H(\mathcal{C}_{\mathscr{C}})$ are points of $\mathscr{C}$ in homogeneous coordinates. By Remark 9, the number of cosets in Table 1 is equal to the cardinality of the corresponding orbits $\mathscr{M}_j$ multiplied by $\#\mathbb{F}_q^* = q-1$.

• Let a coset leader weight be 0.

The coset is the code $\mathcal{C}_{\mathscr{C}}$ of minimum distance 5; it does not contain weight 3 words. We obtain row 1 of Table 1.

• Let a coset leader weight be 1.

In this case, the syndrome of the coset is a column of $H(\mathcal{C}_{\mathscr{C}})$ multiplied by an element of $\mathbb{F}_q^*$; it is a $\mathscr{C}$-point. These cosets do not contain weight 3 vectors as minimum distance of $\mathcal{C}_{\mathscr{C}}$ is equal to 5. We have $\#\mathscr{M}_1 = q+1$, see (3); therefore the number of cosets is $\#\mathscr{M}_1 \cdot (q-1) = (q+1)(q-1)$. We obtain row 2 of Table 1.

• Let a coset leader weight be 2.

In this case, the syndrome of the coset can be represented as a linear combination of two columns of $H(\mathcal{C}_{\mathscr{C}})$; it is an RC-point. Since every RC-point lies in a few $3_{\mathscr{C}}$-planes, the syndrome also can be represented (in a few manners) as a linear combination of three columns of $H(\mathcal{C}_{\mathscr{C}})$; such a combination gives a weight 3 vector in the coset. Thus, a weight 3 vector in the coset corresponds to an *intersection off* $\mathscr{C}$ of a real chord and a $3_{\mathscr{C}}$-plane. To calculate the number $\mathbb{T}$ of the intersections, we should take the total number of $3_{\mathscr{C}}$-planes through an RC-point and subtract from it the number $q-1$ of $3_{\mathscr{C}}$-planes in which a real chord lies (see Lemma 3).

Table 1: The distribution of weight 3 words in the cosets of the $[q+1, q-3, 5]_q 3$ GDRS code $\mathcal{C}_{\mathscr{C}}$ associated with the twisted cubic $\mathscr{C}$ of (2), $q \equiv \xi \pmod 3$. {RC-p} = {RC-points}, {IC-p} = {IC-points}, $\{(q+1)\text{-p}\} = \{(q+1)_\Gamma\text{-points}\}$

no.	ξ	Coset leader weight	Coset notation	The number $A_3(\mathcal{V})$ of weight 3 vectors in a coset	The number of cosets of the given type	Orbits generating coset syndromes	$A_3(\mathcal{V})$
1	any	0	$\mathcal{C}_{\mathscr{C}}$	0	1		
2	any	1	$\mathcal{V}^{(1)}$	0	$(q+1)(q-1)$	$\mathscr{M}_1 = \{\mathscr{C}\text{-points}\}$	
3	1	2	$\mathcal{V}_a^{(2)}$	$\frac{1}{6}(q^2-5q+4)$	$\frac{1}{3}(q^3-q)(q-1)$	$\mathscr{M}_5 = \{\text{RC-p}\} \setminus \mathscr{M}_3$	$r_{35}^{(1)} - (q-1)$
4	1	2	$\mathcal{V}_b^{(2)}$	$\frac{1}{6}(q^2-5q+10)$	$\frac{1}{6}(q^3-q)(q-1)$	$\mathscr{M}_3 = \{\text{RC-p}\} \setminus \mathscr{M}_5$	$r_{33}^{(1)} - (q-1)$
5	1	3	$\mathcal{V}_a^{(3)}$	$\frac{1}{6}(q^2-3q+2)$	$(q^2+q)(q-1)$	$\mathscr{M}_2 = \{\text{T-points}\}$	$r_{32}^{(1)}$
6	1	3	$\mathcal{V}_b^{(3)}$	$\frac{1}{6}(q^2-q)$	$\frac{1}{2}(q^3-q)(q-1)$	$\mathscr{M}_4 = \{\text{IC-points}\}$	$r_{34}^{(1)}$
7	-1	2	$\mathcal{V}^{(2)}$	$\frac{1}{6}(q^2-5q+6)$	$\frac{1}{2}(q^3-q)(q-1)$	$\mathscr{M}_4 = \{\text{RC-points}\}$	$r_{34}^{(-1)} - (q-1)$
8	-1	3	$\mathcal{V}_a^{(3)}$	$\frac{1}{6}(q^2-3q+2)$	$(q^2+q)(q-1)$	$\mathscr{M}_2 = \{\text{T-points}\}$	$r_{32}^{(-1)}$
9	-1	3	$\mathcal{V}_b^{(3)}$	$\frac{1}{6}(q^2-q-2)$	$\frac{1}{3}(q^3-q)(q-1)$	$\mathscr{M}_5 = \{\text{IC-p}\} \setminus \mathscr{M}_3$	$r_{35}^{(-1)}$
10	-1	3	$\mathcal{V}_c^{(3)}$	$\frac{1}{6}(q^2-q+4)$	$\frac{1}{6}(q^3-q)(q-1)$	$\mathscr{M}_3 = \{\text{IC-p}\} \setminus \mathscr{M}_5$	$r_{33}^{(-1)}$
11	0	2	$\mathcal{V}^{(2)}$	$\frac{1}{6}(q^2-5q+6)$	$\frac{1}{2}(q^3-q)(q-1)$	$\mathscr{M}_4 = \{\text{RC-points}\}$	$r_{34}^{(0)} - (q-1)$
12	0	3	$\mathcal{V}_a^{(3)}$	$\frac{1}{6}(q^2-3q)$	$(q^2-1)(q-1)$	$\mathscr{M}_3 = \{\text{TO-points}\}$	$r_{33}^{(0)}$
13	0	3	$\mathcal{V}_b^{(3)}$	$\frac{1}{6}(q^2-q)$	$\frac{1}{2}(q+1)(q^2-q+2)(q-1)$	$\mathscr{M}_2 \cup \mathscr{M}_5 = \{(q+1)\text{-p}\} \cup \{\text{IC-p}\}$	$r_{32}^{(0)} + r_{35}^{(0)}$

If $\xi = 1$, the syndromes are points of $\mathscr{M}_3 \cup \mathscr{M}_5 = \{\text{RC-points}\}$, see (4). We take $r_{3,5}^{(1)} = \frac{1}{6}(q^2+q-2)$ and $r_{3,3}^{(1)} = \frac{1}{6}(q^2+q+4)$ from Theorem 5(i), subtract $q-1$, and obtain the values $\mathbb{T} = \frac{1}{6}(q^2-5q+4)$ and $\mathbb{T} = \frac{1}{6}(q^2-5q+10)$ for rows 3 and 4 of Table 1, respectively. The number of the corresponding cosets is $\#\mathscr{M}_5 \cdot (q-1)$ and $\#\mathscr{M}_3 \cdot (q-1)$, respectively. The values $\#\mathscr{M}_5$ and $\#\mathscr{M}_3$ can be taken from (3).

If $\xi \in \{-1, 0\}$, the syndromes are points of $\mathscr{M}_4 = \{\text{RC-points}\}$, see (4), (6). Therefore, the total number of $3_{\mathscr{C}}$-planes through an RC-point is the value $r_{34}^{(-1)} = r_{34}^{(0)} = \frac{1}{6}(q^2+q)$ of Theorem 5 whence $\mathbb{T} = \frac{1}{6}(q^2+q) - (q-1) = \frac{1}{6}(q^2-5q+6)$. The number of the cosets is $\#\mathscr{M}_4 \cdot (q-1)$, see (3). We obtain rows 7 and 11 of Table 1.

• Let a coset leader weight be 3.

In this case, as every point off $\mathscr{C}$ lies on exactly one chord of $\mathscr{C}$ (see Theorem 2(iv)), the syndrome of the coset belongs to $\mathrm{PG}(3,q) \setminus (\{\mathscr{C}\text{-points}\} \cup \{\text{RC-points}\})$. In other words, the syndrome belongs to $\{\text{T-points}\} \cup \{\text{IC-points}\}$ for $\xi \neq 0$ (see Theorem 2(ii)) or $\{\text{TO-points}\} \cup$

$\{(q+1)_\Gamma\text{-points}\} \cup \{\text{IC-points}\}$ for $\xi = 0$ (see Theorem 2(iii)). Note that, for $\xi = 0$, we have $\{\text{T-points}\} = \{\text{TO-points}\} \cup \{(q+1)_\Gamma\text{-points}\}$. Thus, a weight 3 vector in the coset corresponds to an intersection of a tangent or an imaginary chord with a $3_{\mathscr{C}}$-plane. No tangent or imaginary chord lies in a $3_{\mathscr{C}}$-plane, otherwise we have an intersection of a tangent or an imaginary chord with a real chord lying in the $3_{\mathscr{C}}$-plane. Therefore, to calculate the number of the intersections, we directly use the needed values of $r_{3j}^{(\xi)}$ from Theorem 5 without subtracting.

If $\xi \in \{-1, 1\}$, we have $\mathscr{M}_2 = \{\text{T-points}\}$, see (3). For rows 5 and 8 of Table 1, we take $r_{32}^{(1)} = r_{32}^{(-1)}$ from Theorem 5 and use $\#\mathscr{M}_2 \cdot (q-1)$.

If $\xi = -1$, we have $\mathscr{M}_3 \cup \mathscr{M}_5 = \{\text{IC-points}\}$, $\mathscr{M}_3 = \{3_\Gamma\text{-points}\}$, $\mathscr{M}_5 = \{0_\Gamma\text{-points}\}$, see (4), (3). For rows 9 and 10 of Table 1, we take, respectively, $r_{35}^{(-1)} = \frac{1}{6}(q^2 - q - 2)$ and $r_{33}^{(-1)} = \frac{1}{6}(q^2 - q + 4)$ from Theorem 5 and use $\#\mathscr{M}_3 \cdot (q-1)$, $\#\mathscr{M}_5 \cdot (q-1)$, see (3).

If $\xi = 1$, we have $\mathscr{M}_4 = \{\text{IC-points}\}$, see (4). For row 6 of Tab. 1, we take $r_{34}^{(1)} = \frac{1}{6}(q^2 - q)$ from Theorem 5 and use $\#\mathscr{M}_4 \cdot (q-1)$, see (3).

If $\xi = 0$, we have $\mathscr{M}_3 = \{\text{TO-points}\}$, $\mathscr{M}_2 = \{(q+1)_\Gamma\text{-points}\}$, $\mathscr{M}_5 = \{\text{IC-points}\}$, see (6). For rows 12 and 13 of Table 1, we take, respectively, $r_{33}^{(0)}$ and $r_{32}^{(0)} + r_{35}^{(0)}$ from Theorem 5(ii) and use $\#\mathscr{M}_3 \cdot (q-1)$, $\#\mathscr{M}_2 \cdot (q-1)$, $\#\mathscr{M}_5 \cdot (q-1)$, see (3). $\square$

Theorem 11. *The weight distribution of any coset of $\mathcal{C}_{\mathscr{C}}$ is unambiguously determined by Table 1.*

Proof. The code $\mathcal{C}_{\mathscr{C}}^\perp$ dual to $\mathcal{C}_{\mathscr{C}}$ is an $[q+1, 4, q-2]_q R$ MDS code, $R = q-4$. By [35, Th. 10], $\mathcal{C}_{\mathscr{C}}^\perp$ has 4 distinct nonzero weights, i.e. $s(\mathcal{C}_{\mathscr{C}}^\perp) = 4$. By Theorem 10 we know the numbers of vectors of weights 1, 2, and $3 = s(\mathcal{C}_{\mathscr{C}}^\perp) - 1$ for all the cosets of $\mathcal{C}_{\mathscr{C}}$, see Table 1 for weight 3 vectors. Recall also, that the cosets of weight ≤ 2 have a unique leader; the weight 1 cosets do not contain weight 2 vectors. Now the assertion follows from Theorem 7(ii). $\square$

Theorem 12. (classification of the cosets) *Let $q \equiv \xi \pmod 3$. For the code $\mathcal{C}_{\mathscr{C}}$, the following holds:*

(i) *All weight 1 cosets have the same weight distribution.*

(ii) *If $\xi \neq 1$, all weight 2 cosets have the same weight distribution.*

For $\xi = 1$, there are 2 distinct weight distributions of weight 2 cosets. The numbers of the corresponding cosets are $\frac{1}{3}(q^3 - q)(q-1)$ and $\frac{1}{6}(q^3 - q)(q-1)$.

(iii) *For $\xi = 1$, the are 2 distinct weight distributions of weight 3 cosets. The numbers of the corresponding cosets are $(q^2 + q)(q-1)$ and $\frac{1}{2}(q^3 - q)(q-1)$.*

For $\xi = -1$, there are 3 distinct weight distributions of weight 3 cosets. The numbers of the corresponding cosets are $(q^2 + q)(q-1)$, $\frac{1}{3}(q^3 - q)(q-1)$, and $\frac{1}{6}(q^3 - q)(q-1)$.

For $\xi = 0$, there are 2 distinct weight distributions of weight 3 cosets. The numbers of the corresponding cosets are $(q^2 - 1)(q-1)$ and $\frac{1}{2}(q+1)(q^2 - q + 2)(q-1)$.

Proof. The assertion directly follows from Table 1 and Theorem 11. □

4 On the weight distribution of the cosets of the code $\mathcal{C}_{\mathscr{C}}$

We use the approach of [17, Th. 7.3.1, 7.5.2, Lem. 7.5.1], [18, Th. 10.7, 10.10].

For a code $\mathcal{C}$ and its coset $\mathcal{V}$, the approach includes the following stages:

- **(A)** Defining a special code $\mathcal{D}$ on the base of $\mathcal{C}$ and its cosets; searching of the needed properties of $\mathcal{D}$.
- **(B)** Obtaining the weight distribution of the code $\mathcal{D}^{\perp}$ dual to $\mathcal{D}$ using the values of $A_i(\mathcal{V})$, $i = 1, 2, \ldots, s(\mathcal{C}^{\perp}) - 1$.
- **(C)** Obtaining the weight distribution of the code $\mathcal{D}$ using the Krawtchouck polynomials. Obtaining the weight distribution of the coset $\mathcal{V}$ using the weight distributions of $\mathcal{D}$ and $\mathcal{C}$.

It is well known, see e.g. [2, Th. 11.3.6], [17, Th. 7.4.1], that for $w \geq d$, the weight distribution $A_w(\mathcal{C})$ of an $[n, k, d = n - k + 1]_q$ MDS code $\mathcal{C}$ has the form

$$A_w(\mathcal{C}) = \binom{n}{w} \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} (q^{w-d+1-j} - 1). \tag{7}$$

Remind that $\mathcal{C}_{\mathscr{C}}$ is a $[q + 1, q - 3, 5]_q 3$ GDRS code. Its weight distribution $A_w(\mathcal{C}_{\mathscr{C}})$ is given by (7) with $n = q + 1$, $d = 5$. The code $\mathcal{C}_{\mathscr{C}}^{\perp}$ dual to $\mathcal{C}_{\mathscr{C}}$ is a $[q + 1, 4, q - 2]_q R$ MDS code, $R = q - 4$. By [35, Th. 10], $\mathcal{C}_{\mathscr{C}}^{\perp}$ has 4 distinct nonzero weights, i.e.

$$S(\mathcal{C}_{\mathscr{C}}^{\perp}) = \{q - 2, q - 1, q, q + 1\}, \quad s(\mathcal{C}_{\mathscr{C}}^{\perp}) = 4. \tag{8}$$

Remind also that for a coset $\mathcal{V}^{(W)}$ of $\mathcal{C}_{\mathscr{C}}$, $\mathcal{V} \neq \mathcal{C}_{\mathscr{C}}$, the relation $1 \leq W \leq 3$ holds; so the coset can be written as $\mathcal{V}^{(W)}_{\mathbf{v}_W} = \mathbf{v}_W + \mathcal{C}_{\mathscr{C}}$, where $\mathbf{v}_W \in \mathbb{F}_q^{q+1} \setminus \mathcal{C}_{\mathscr{C}}$ is a weight W vector, $W \in \{1, 2, 3\}$. Therefore, *in the rest of the paper, we assume* $W \in \{1, 2, 3\}$.

For $\alpha \in \mathbb{F}_q^*$, the vector $\alpha \mathbf{v}_W$ is also a weight W vector. For $\alpha \in \mathbb{F}_q^*$, we denote by $\mathcal{V}^{(W)}_{\alpha \mathbf{v}_W}$ the weight W coset of $\mathcal{C}_{\mathscr{C}}$ such that $\mathcal{V}^{(W)}_{\alpha \mathbf{v}_W} = \alpha \mathbf{v}_W + \mathcal{C}_{\mathscr{C}}$. Thus, $\alpha \mathbf{v}_W$ is a leader of $\mathcal{V}^{(W)}_{\alpha \mathbf{v}_W}$. By Theorem 7(i), all the cosets $\mathcal{V}^{(W)}_{\alpha \mathbf{v}_W}$ have the same weight distribution.

We execute the stages **(A)** – **(C)**.

- **(A)**

Let $\mathcal{D}_W$ be the union of $\mathcal{C}_{\mathscr{C}}$ and all the cosets $\alpha \mathbf{v}_W + \mathcal{C}_{\mathscr{C}}$ with $\alpha \in \mathbb{F}_q^*$, i.e.

$$\mathcal{D}_w = \bigcup_{\alpha \in \mathbb{F}_q} (\alpha \mathbf{v}_W + \mathcal{C}_{\mathscr{C}}). \tag{9}$$

Lemma 13. *For the code* $\mathcal{D}_W$ *of* (9) *the following holds:*

(i) $\mathcal{D}_W$ *is a linear* $[q + 1, q - 2, W]_q$ *code.*

(ii) $\mathcal{D}_W^\perp$ *is a* $[q+1,3,d_{\mathcal{D}_W^\perp}]_q$ *code;* $\mathcal{D}_W^\perp\subset\mathcal{C}_{\mathscr{C}}^\perp$*;* $d_{\mathcal{D}_W^\perp}\geq q-2$.

(iii) $A_1(D_1)=q-1,\ A_2(D_1)=A_3(D_1)=0.$

(iv) $A_1(D_2)=0,\ A_2(D_2)=q-1,\ A_3(D_2)=(q-1)A_3(\mathcal{V}_{\mathbf{v}_2}^{(2)}).$

(v) $A_1(D_3)=A_2(D_3)=0,\ A_3(D_3)=(q-1)A_3(\mathcal{V}_{\mathbf{v}_3}^{(3)}).$

Proof. **(i)** By (9), $\mathcal{D}_W$ is a subspace of $\mathbb{F}_q^{q+1}$ with size $\#\mathcal{D}_W=q\cdot\#\mathcal{C}_{\mathscr{C}}=q\cdot q^{q-3}$. As the code distance of $\mathcal{C}_{\mathscr{C}}$ is equal to 5, minimum weight codewords of $\mathcal{D}_W$ are weight w leaders of the cosets $\alpha\mathbf{v}_W+\mathcal{C}_{\mathscr{C}}$ with $\alpha\neq 0$.

(ii) The codewords of $\mathcal{C}_{\mathscr{C}}^\perp$ are orthogonal to $\mathcal{C}_{\mathscr{C}}$ [3, Sec. 2.3]. But some of these codewords can be non-orthogonal to the cosets $\alpha\mathbf{v}_W+\mathcal{C}_{\mathscr{C}}$ with $\alpha\neq 0$. Therefore, $\mathcal{D}_W^\perp\subset\mathcal{C}_{\mathscr{C}}^\perp$. Then we use (8).

(iii) In $\mathcal{D}_1$, there are $q-1$ weight 1 cosets of $\mathcal{C}_{\mathscr{C}}$; it gives $A_1(D_1)=q-1$. The equality $A_2(D_1)=A_3(D_1)=0$ is obvious.

(iv) By part (i), the minimum distance of $\mathcal{D}_2$ is 2 whence $A_1(D_2)=0$. The coset $\mathcal{V}_{\mathbf{v}_2}^{(2)}$ contains one weight 2 vector and $A_3(\mathcal{V}_{\mathbf{v}_2}^{(2)})$ weight 3 vectors. By Theorem 7(i), all the cosets $\mathcal{V}_{\alpha\mathbf{v}_2}^{(2)}$, $\alpha\in\mathbb{F}_q^*$, have the same weight distribution.

(v) By part (i), the minimum distance of $\mathcal{D}_3$ is 3 whence $A_1(D_3)=A_2(D_3)=0$. The coset $\mathcal{V}_{\mathbf{v}_3}^{(3)}$ contains $A_3(\mathcal{V}_{\mathbf{v}_3}^{(3)})$ weight 3 vectors. By Theorem 7(i), all the cosets $\mathcal{V}_{\alpha\mathbf{v}_3}^{(3)}$, $\alpha\in\mathbb{F}_q^*$, have the same weight distribution. □

- **(B)**

Theorem 14. *The weight distribution* $A_j(\mathcal{D}_W^\perp)$ *of the* $[q+1,3,d_{\mathcal{D}_W^\perp}]_q$ *code* $\mathcal{D}_W^\perp$ *dual to* $\mathcal{D}_W$ *is as follows:*

(i) $A_0(\mathcal{D}_W^\perp)=1$*;* $A_i(\mathcal{D}_W^\perp)=0,\ i=1,2,\dots,q-3$.

(ii) *The values of* $A_j(\mathcal{D}_W^\perp)$*,* $j=q-2,q-1,q,q+1$*, are given by the system of four linear equations*

$$\sum_{j=q-2}^{q+1} j^u A_j(\mathcal{D}_W^\perp)=G_{W,u}(q),\quad u=0,1,2,3, \tag{10}$$

where

$$\begin{aligned}
&G_{W,0}(q)=q^3-1, \\
&G_{W,1}(q)=q^2(q^2-1-A_1(\mathcal{D}_W)),\\
&G_{W,2}(q)=q(q^2(q^2-1)-q(2q-1)A_1(\mathcal{D}_W)+2A_2(\mathcal{D}_W)),\\
&G_{W,3}(q)=(q^2-1)q(q^3+q-1)-q(3q^3-3q^2+4q-3)A_1(\mathcal{D}_W)+6(q^2-q+1)A_2(\mathcal{D}_W)\\
&\qquad\qquad -6A_3(\mathcal{D}_W).
\end{aligned} \tag{11}$$

Proof. The part (i) follows from Lemma 13(ii). Also, the length of $\mathcal{D}_W^\perp$ is $q+1$. Now we can obtain the system (10), (11) from [17, Eq. (7.7), p. 259–260]. □

Corollary 15. *Let $q \equiv \xi \pmod 3$. Let $G_{W,u}(q)$ be given by (11). The values of $G_{W,u}(q)$, $u = 1, 2, 3$, are as follows:*

$$\textbf{(i)}\ W=1.\ \ G_{1,1}(q) = q^3(q-1), \qquad (12)$$
$$G_{1,2}(q) = q^2(q-1)(q^2-q+1),$$
$$G_{1,3}(q) = q(q-1)(q^4-2q^3+4q^2-4q+2).$$
$$\textbf{(ii)}\ W=2.\ \ G_{2,1}(q) = q^2(q^2-1), \qquad (13)$$
$$G_{2,2}(q) = q(q-1)(q^3+q^2+2),$$
$$G_{2,3}(q) = (q-1)\left[q^5+q^4+q^3+6q^2-7q+6-6A_3(\mathcal{V}^{(2)}_{\mathbf{v}_2})\right]$$
$$\textit{where } 6A_3(\mathcal{V}^{(2)}_{\mathbf{v}_2}) = \begin{cases} q^2-5q+10 \ \textit{or}\ q^2-5q+4 & \textit{if}\ \ \xi=1 \\ q^2-5q+6 & \textit{if}\ \ \xi=-1,0 \end{cases}.$$
$$\textbf{(iii)}\ W=3.\ \ G_{3,1}(q) = q^2(q^2-1), \qquad (14)$$
$$G_{3,2}(q) = q^3(q^2-1),$$
$$G_{3,3}(q) = (q-1)\left[q^5+q^4+q^3-q-6A_3(\mathcal{V}^{(3)}_{\mathbf{v}_3})\right]$$
$$\textit{where } 6A_3(\mathcal{V}^{(3)}_{\mathbf{v}_3}) = \begin{cases} q^2-3q+2 \ \textit{or}\ q^2-q & \textit{if}\ \ \xi=1 \\ q^2-3q+2 \ \textit{or}\ q^2-q+4,\ \textit{or}\ q^2-q-2 & \textit{if}\ \ \xi=-1 \\ q^2-q \ \textit{or}\ q^2-3q & \textit{if}\ \ \xi=0 \end{cases}.$$

Proof. We use (11), Lemma 13(iii)–(v), and Table 1. □

Corollary 16. *Let $A_3(\mathcal{V}^{(W)}_{\mathbf{v}_W})$ be as in Table 1 and (13),(14); in particular, let $A_3(\mathcal{V}^{(1)}_{\mathbf{v}_1}) = 0$. The weight distribution $A_j(\mathcal{D}^{\perp}_W)$, of the $[q+1, 3, d_{\mathcal{D}^{\perp}_W}]_q$ code $\mathcal{D}^{\perp}_W$ dual to $\mathcal{D}_W$ is as follows:*

$$A_j(\mathcal{D}^{\perp}_W) = (q-1)\left[\Theta^{(W)}_j - (-1)^{q+1-j}\binom{3}{q+1-j} A_3\left(\mathcal{V}^{(W)}_{\mathbf{v}_W}\right)\right] \qquad (15)$$

where

$j = q-2, q-1, q, q+1,$

$\Theta^{(1)}_{q-2} = \frac{1}{2}q(q-1),\ \Theta^{(1)}_{q-1} = 2q,\ \Theta^{(1)}_{q} = \frac{1}{2}(q^2-q+2),\ \Theta^{(1)}_{q+1} = 0;$

$\Theta^{(2)}_{q-2} = q-1,\ \Theta^{(2)}_{q-1} = \frac{1}{2}(q^2-3q+6),\ \Theta^{(2)}_{q} = 2(q-1),\ \Theta^{(2)}_{q+1} = \frac{1}{2}(q^2-q+2);$

$\Theta^{(3)}_{q-2} = 0,\ \Theta^{(3)}_{q-1} = \frac{1}{2}q(q+1),\ \Theta^{(3)}_{q} = q+1,\ \Theta^{(3)}_{q+1} = \frac{1}{2}q(q-1).$

Proof. By the standard methods, we directly obtain the following solution of the linear system (10), (11):

$$A_{q-2}(\mathcal{D}^{\perp}_W) = \frac{1}{6}\left[q(q^2-1)G_{W,0}(q) - (3q^2-1)G_{W,1}(q) + 3qG_{W,2}(q) - G_{W,3}(q)\right],$$

$$A_{q-1}(\mathcal{D}_W^\perp) = \frac{1}{2}\left[-q(q-2)(q+1)G_{W,0}(q) + (3q^2-2q-2)G_{W,1}(q) - (3q-1)G_{W,2} + G_{W,3}(q)\right],$$

$$A_q(\mathcal{D}_W^\perp) = \frac{1}{2}\left[(q^2-1)(q-2)G_{W,0}(q) - (3q^2-4q-1)G_{W,1}(q) + (3q-2)G_{W,2}(q) - G_{W,3}(q)\right],$$

$$A_{q+1}(\mathcal{D}_W^\perp) = \frac{1}{6}\left[-q(q-1)(q-2)G_{W,0}(q) + (3q^2-6q+2)G_{W,1}(q) - 3(q-1)G_{W,2}(q) \right.$$
$$\left. + G_{W,3}(q)\right].$$

Then we substitute the values $G_{W,u}(q)$ from (11)–(14) and do simple transformations. □

- **(C)**
  Let

$$K_w^{n,q}(x) = \sum_{i=0}^{w}(-1)^i(q-1)^{w-i}\binom{x}{i}\binom{n-x}{w-i} \tag{16}$$

be the Krawtchouck polynomial [2, 17].

Theorem 17. *Let the weight distribution $A_j(\mathcal{D}_W^\perp)$ of the code dual to $\mathcal{D}_W$ be given by Corollary 16. The weight distribution $A_w(\mathcal{D}_W)$ of the $[q+1, q-2, W]_q$ code $\mathcal{D}_W$ is as follows:*

$$A_w(\mathcal{D}_W) = \frac{1}{q^3}\left(K_w^{q+1,q}(0) + \sum_{j=q-2}^{q+1} A_j(\mathcal{D}_W^\perp)K_w^{q+1,q}(j)\right), \quad w = W, W+1, \ldots, q+1. \tag{17}$$

Proof. We use the well known relations to obtain the weight distribution of $\mathcal{D}_W$ from one of $\mathcal{D}_W^\perp$, see e.g. [3, Sec. 4.4], [17, Eq. (K), p. 257]. □

Theorem 18. *Let the weight distribution $A_w(\mathcal{D}_W)$ of the code $\mathcal{D}_W$ be given by (17). The weight distribution $A_w(\mathbf{v}_W + \mathcal{C}_\mathscr{C})$ of the coset $v_W + \mathcal{C}_\mathscr{C}$ is as follows:*

$$A_w(\mathbf{v}_W + \mathcal{C}_\mathscr{C}) = \frac{A_w(\mathcal{D}_W) - A_w(\mathcal{C}_\mathscr{C})}{q-1}, \quad w = W+1, W+2, \ldots, q+1, \tag{18}$$

where $A_w(\mathcal{C}_\mathscr{C})$ is given by (7) with $n = q+1$, $d = 5$.

Proof. By [17, Lem. 7.5.1], we have $A_w(\mathbf{v}_W + \mathcal{C}_\mathscr{C}) = A_w(\mathcal{D}_W \setminus \mathcal{C}_\mathscr{C})/(q-1)$. Also, by the construction (9), it is clear that $A_w(\mathcal{D}_W \setminus \mathcal{C}_\mathscr{C}) = A_w(\mathcal{D}_W) - A_w(\mathcal{C}_\mathscr{C})$. □

Example 19. Examples of the weight distribution of the cosets of the code $\mathcal{C}_\mathscr{C}$ are given in Table 19. Notation of the cosets are taken from Table 1. The distributions are obtained by the methods described above.

Table 2: Examples of the weight distribution of the cosets $\mathcal{V}^{(W)}$ of the $[q+1, q-3, 5]_q 3$ GDRS code $\mathcal{C}_\mathscr{C}$. $A_w = A_w(\mathcal{V}^{(W)})$, N is the number of the cosets of the given type

q	coset	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	N
5	$\mathcal{C}_\mathscr{C}$	0	0	0	0	24	0							1
	$\mathcal{V}^{(1)}$	1	0	0	5	15	4							24
	$\mathcal{V}^{(2)}$	0	1	1	6	11	6							240
	$\mathcal{V}_a^{(3)}$	0	0	2	9	6	8							120
	$\mathcal{V}_b^{(3)}$	0	0	3	6	9	7							160
	$\mathcal{V}_c^{(3)}$	0	0	4	3	12	6							80
7	$\mathcal{C}_\mathscr{C}$	0	0	0	0	336	336	1056	672					1
	$\mathcal{V}^{(1)}$	1	0	0	35	217	490	966	692					48
	$\mathcal{V}_a^{(2)}$	0	1	3	40	182	541	935	699					672
	$\mathcal{V}_b^{(2)}$	0	1	4	35	192	531	940	698					336
	$\mathcal{V}_a^{(3)}$	0	0	5	45	162	566	921	702					336
	$\mathcal{V}_b^{(3)}$	0	0	7	35	182	546	931	700					1008
8	$\mathcal{C}_\mathscr{C}$	0	0	0	0	882	1764	7812	12411	9898				1
	$\mathcal{V}^{(1)}$	1	0	0	70	588	2268	7372	12606	9863				63
	$\mathcal{V}^{(2)}$	0	1	5	75	523	2399	7251	12661	9853				1764
	$\mathcal{V}_a^{(3)}$	0	0	7	84	483	2464	7197	12684	9849				504
	$\mathcal{V}_b^{(3)}$	0	0	9	72	513	2424	7227	12672	9851				1176
	$\mathcal{V}_c^{(3)}$	0	0	10	66	528	2404	7242	12666	9852				588
9	$\mathcal{C}_\mathscr{C}$	0	0	0	0	2016	6720	40320	113760	205040	163584			1
	$\mathcal{V}^{(1)}$	1	0	0	126	1386	8064	38760	114795	204669	163640			80
	$\mathcal{V}^{(2)}$	0	1	7	133	1267	8365	38389	115048	204577	163654			2880
	$\mathcal{V}_a^{(3)}$	0	0	9	147	1197	8505	38235	115146	204543	163659			640
	$\mathcal{V}_b^{(3)}$	0	0	12	126	1260	8400	38340	115083	204564	163656			2960
11	$\mathcal{C}_\mathscr{C}$	0	0	0	0	7920	55440	554400	3366000	15037000	45074040	81962880	68301200	1
	$\mathcal{V}^{(1)}$	1	0	0	330	5742	61908	543180	3378375	15028145	45078044	81961836	68301320	120
	$\mathcal{V}^{(2)}$	0	1	12	342	5424	63042	541080	3380763	15026408	45078837	81961628	68301344	6600
	$\mathcal{V}_a^{(3)}$	0	0	15	360	5292	63420	540450	3381435	15025940	45079044	81961575	68301350	1320
	$\mathcal{V}_b^{(3)}$	0	0	18	333	5400	63168	540828	3381057	15026192	45078936	81961602	68301347	4400
	$\mathcal{V}_c^{(3)}$	0	0	19	324	5436	63084	540954	3380931	15026276	45078900	81961611	68301346	2200

5 The property of differences and a symmetry in the weight distributions of the code $\mathcal{C}_{\mathscr{C}}$ cosets

Lemma 20. *Let $n \geq 0$. For the Krawtchouck polynomials $K_w^{n,q}(x)$ of (16) the following holds:*

$$\sum_{x=n-m}^{n}(-1)^{n-x}\binom{m}{n-x}K_w^{n,q}(x) = (-1)^w q^m \binom{n-m}{w-m},\ 0 \leq m \leq w \leq n. \tag{19}$$

Proof. We write the left part of (19) directly by (16). In the product $(-1)^{n-x}\binom{m}{n-x}K_w^{n,q}(x)$ with $x = n-j-k$, the factor $(q-1)^j$, $j = 0, 1, \ldots, m-k$, belongs to the term

$$(-1)^{j+k}\binom{m}{j+k}\left[(-1)^{w-j}(q-1)^j\binom{n-j-k}{w-j}\binom{j+k}{j}\right],\ k = 0, 1, \ldots, m-j.$$

By [36, Sec. 1.2, Eq. (IV)], $\binom{m}{j+k}\binom{j+k}{j} = \binom{m}{j}\binom{m-j}{k}$. Grouping together terms with the same factor $(q-1)^j$, we transform the left part of (19) into the form

$$\begin{aligned}(-1)^w &\sum_{j=0}^{m}(q-1)^j\binom{m}{j}\sum_{k=0}^{m-j}(-1)^k\binom{n-j-k}{w-j}\binom{m-j}{k}\\ &= (-1)^w\binom{n-m}{w-m}\sum_{j=0}^{m}(q-1)^j\binom{m}{j} = (-1)^w\binom{n-m}{w-m}((q-1)+1)^m.\end{aligned}$$

For the sum $\sum_{k=0}^{m-j}\ldots$, we apply the combinatorial identity [36, Sec. 1.3, Eq. (5a)]

$$\sum_{k=0}^{p}(-1)^k\binom{n-k}{m}\binom{p}{k} = \binom{n-p}{m-p}.$$

The cases $n, m, w \in \{0, 1\}$ can be checked directly. □

In the next theorem, for the code $\mathcal{C}_{\mathscr{C}}$, we give the *property of differences* for the cosets of equal weight having distinct weight distributions. The property asserts that, for $4 \leq w \leq q+1$, the difference $A_w(\mathcal{V}_a^{(W)}) - A_w(\mathcal{V}_b^{(W)})$ between the w-th components of the distinct weight distributions is unambiguously determined by the difference $A_3(\mathcal{V}_a^{(W)}) - A_3(\mathcal{V}_b^{(W)})$ between the 3-rd components.

This property allows us to prove a symmetry of the coset weight distributions and propose an alternative way to obtain distinct distributions, see Section 6, Lemma 27, Theorem 28.

Theorem 21. (property of differences) *Let $W = 2, 3$. Let $3 \le w \le q+1$. Let $\mathcal{V}_a^{(W)}, \mathcal{V}_b^{(W)}$ be two weight W cosets of $\mathcal{C}_\mathscr{C}$ with distinct weight distributions according to Table 1. We have*

$$A_w(\mathcal{V}_a^{(W)}) - A_w(\mathcal{V}_b^{(W)}) = -(-1)^w \left[A_3(\mathcal{V}_a^{(W)}) - A_3(\mathcal{V}_b^{(W)})\right] \binom{q-2}{w-3}. \tag{20}$$

Proof. Let $\mathcal{D}_W^a$ (resp. $\mathcal{D}_W^b$) be the code $\mathcal{D}_W$ of (9) such that the vector $\mathbf{v}_W$ is a leader of a coset $\mathcal{V}_a^{(W)}$ (resp. $\mathcal{V}_b^{(W)}$) and let $\mathcal{D}_W^{a\perp}$ and $\mathcal{D}_W^{b\perp}$ be the corresponding dual codes.

By Theorems 18 and 17, we have

$$\begin{aligned} A_w(\mathcal{V}_a^{(W)}) - A_w(\mathcal{V}_b^{(W)}) &= \frac{A_w(\mathcal{D}_W^a) - A_w(\mathcal{D}_W^b)}{q-1} \\ &= \frac{1}{q^3(q-1)} \sum_{j=q-2}^{q+1} \left[A_j(\mathcal{D}_W^{a\perp}) - A_j(\mathcal{D}_W^{b\perp})\right] K_w^{q+1,q}(j). \end{aligned} \tag{21}$$

In Corollary 16, the values of $\Theta_j^{(W)}$ are the same for the codes $\mathcal{D}_W^{a\perp}$ and $\mathcal{D}_W^{b\perp}$. Therefore, from (21) and Corollary 16 with (15), it follows that

$$\begin{aligned} &A_w(\mathcal{V}_a^{(W)}) - A_w(\mathcal{V}_b^{(W)}) \\ &= -\frac{(q-1)\left[A_3(\mathcal{V}_a^{(W)}) - A_3(\mathcal{V}_b^{(W)})\right]}{q^3(q-1)} \sum_{j=q-2}^{q+1} (-1)^{q+1-j} \binom{3}{q+1-j} K_w^{q+1,q}(j) \end{aligned}$$

whence we obtain the assertion using Lemma 20 with $m = 3$. □

Theorem 22. (symmetry) *Let $w = 3, \ldots, \lfloor (q+3)/2 \rfloor$. Let $q \equiv \xi \pmod 3$. Let the notation of the cosets be as in Table 1. There is the following symmetry in the weight distributions of the code $\mathcal{C}_\mathscr{C}$.*

(i) $W = 2, \xi = 1;\ W = 3, \xi = 1;\ W = 3, \xi = 0:$

$A_{q+4-w}(\mathcal{V}_a^{(W)}) - (-1)^q A_w(\mathcal{V}_a^{(W)}) = A_{q+4-w}(\mathcal{V}_b^{(W)}) - (-1)^q A_w(\mathcal{V}_b^{(W)})$.

(ii) $\xi = -1:$

$A_{q+4-w}(\mathcal{V}_a^{(3)}) - (-1)^q A_w(\mathcal{V}_a^{(3)}) = A_{q+4-w}(\mathcal{V}_b^{(3)}) - (-1)^q A_w(\mathcal{V}_b^{(3)})$

$= A_{q+4-w}(\mathcal{V}_c^{(3)}) - (-1)^q A_w(\mathcal{V}_c^{(3)})$.

Proof. It is obvious that $\binom{q-2}{q+4-w-3} = \binom{q-2}{w-3}$ and

$$(-1)^w = \begin{cases} (-1)^{q+4-w} & \text{if } q \text{ is even} \\ -(-1)^{q+4-w} & \text{if } q \text{ is odd} \end{cases}.$$

Therefore, by Theorem 21 and (20), we have

$$A_{q+4-w}(\mathcal{V}_a^{(W)}) - A_{q+4-w}(\mathcal{V}_b^{(W)}) = \begin{cases} -\left[A_w(\mathcal{V}_a^{(W)}) - A_w(\mathcal{V}_b^{(W)})\right] & \text{if} \quad q \text{ is even} \\ A_w(\mathcal{V}_a^{(W)}) - A_w(\mathcal{V}_b^{(W)}) & \text{if} \quad q \text{ is odd} \end{cases}.$$

We apply this relation in all cases of Table 1 (see also Theorem 12) when $\mathcal{C}_{\mathscr{C}}$ cosets of the same weight have distinct weight distributions. This implies the assertions. □

6 The weight distributions of the $\mathcal{C}_{\mathscr{C}}$ code cosets on the base of the integral weight spectra

Let $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(W)})$ (resp. $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{\leq W})$) be the overall numbers of weight w vectors in all cosets of weight W (resp. weight $\leq W$).

Definition 23. For a fixed W, the set $\{\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(W)})|w = 0, 1, \ldots, n\}$ is an *integral weight spectrum* of the all code cosets of weight W.

Distinct representations of the integral weight spectra $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(W)})$ and values of $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{\leq W})$ are considered in the literature, see e.g. [1, Th. 14.2.2], [2, Th. 6.22], [22–24], [31, Lem. 2.14]. For instance, in [22, Eq. (11)–(13)], for an MDS code correcting t-fold errors, the value D_u gives $\mathcal{A}_u^{\Sigma}(\mathcal{V}^{\leq t})$. In [24], the results of [22] are modified and Lemma 24 is obtained.

We denote

$$\Phi_w^{(j)} = (-1)^{w-5}\left[\binom{q+1}{w}\binom{w-1}{3} - \binom{q+1-j}{w-j}\binom{w-1-j}{3-j}\right]. \tag{22}$$

Lemma 24. [24, Th. 3.4, 4.3, 5.1] **(integral weight spectra)** *Let $\mathcal{C}$ be an $[q+1, q-3, 5]_q 3$ MDS code of minimum distance $d = 5$ and covering radius $R = 3$. For the integral weight spectra $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(W)})$ of the code $\mathcal{C}$ cosets, the following holds:*

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(1)}) = (q^2-1)\left[A_w(\mathcal{C}) - \Phi_w^{(1)}\right], \ 4 \leq w \leq q+1.$$

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(2)}) = \binom{q+1}{2}(q-1)^2\left[A_w(\mathcal{C}) - \Phi_w^{(2)} + (-1)^{w-5}\frac{1}{3}\binom{q-2}{2}\binom{q-2}{w-3}\right],$$
$$3 \leq w \leq q+1.$$

$$\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(3)}) = \binom{q+1}{w}(q-1)^w - \left[A_w(\mathcal{C}) + \mathcal{A}_w^{\Sigma}(\mathcal{V}^{(1)}) + \mathcal{A}_w^{\Sigma}(\mathcal{V}^{(2)})\right], \ 3 \leq w \leq q+1.$$

Thus, for the $[q+1, q-3, 5]_q 3$ code $\mathcal{C}_{\mathscr{C}}$, the values of $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(W)})$, $W = 1, 2, 3$, are known and easy to calculate.

Now we give direct formulas of the weight distributions of the cosets of $\mathcal{C}_{\mathscr{C}}$ on the base of the integral weight spectra $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(W)})$, $W = 1, 2, 3$.

Let $\mathcal{V}_{\Sigma}^{(W)}(\mathcal{C})$ be the total number of weight W cosets of a linear code $\mathcal{C}$.

The following lemma is obvious.

Lemma 25. *If all weight W cosets $\mathcal{V}^{(W)}$ of a linear code $\mathcal{C}$ have the same weight distribution, then the weight distribution $A_w(\mathcal{V}^{(W)})$ of any coset $\mathcal{V}^{(W)}$ has the form*

$$A_w(\mathcal{V}^{(W)}) = \frac{\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(W)})}{\mathcal{V}_{\Sigma}^{(W)}(\mathcal{C})}.$$

Theorem 26. *Let the weight distribution $A_w(\mathcal{C}_{\mathscr{C}})$ of the code $\mathcal{C}_{\mathscr{C}}$ be as in (7) with $n = q+1$, $d = 5$. Let $\Phi_w^{(j)}$ be as in (22). The following holds:*

(i) *For all q, the weight distribution $A_w(\mathcal{V}^{(1)})$ of any weight 1 coset $\mathcal{V}^{(1)}$ of the code $\mathcal{C}_{\mathscr{C}}$ is as follows*

$$A_w(\mathcal{V}^{(1)}) = A_w(\mathcal{C}_{\mathscr{C}}) - \Phi_w^{(1)}, \ 4 \le w \le q+1.$$

(ii) *Let $q \not\equiv 1 \pmod 3$. The weight distribution $A_w(\mathcal{V}^{(2)})$ of any weight 2 coset $\mathcal{V}^{(2)}$ of the code $\mathcal{C}_{\mathscr{C}}$ is as follows*

$$A_w(\mathcal{V}^{(2)}) = A_w(\mathcal{C}_{\mathscr{C}}) - \Phi_w^{(2)} + (-1)^{w-5}\frac{1}{3}\binom{q-2}{2}\binom{q-2}{w-3}, \ 3 \le w \le q+1.$$

Proof. By Table 1 and Theorem 12, for all q, all the code $\mathcal{C}_{\mathscr{C}}$ cosets of weight 1 have the same weight distributions. If $q \not\equiv 1 \pmod 3$, all the code $\mathcal{C}_{\mathscr{C}}$ cosets of weight 2 also have the same weight distributions. So, we may apply Lemma 25 where we put $\mathcal{V}_{\Sigma}^{(1)}(\mathcal{C}_{\mathscr{C}}) = (q+1)(q-1)$, $\mathcal{V}_{\Sigma}^{(2)}(\mathcal{C}_{\mathscr{C}}) = \binom{q+1}{2}(q-1)^2$. Finally, we use Lemma 24. Note that for $q \not\equiv 1 \pmod 3$, the term $\frac{1}{3}\binom{q-2}{2}$ is an integer. □

In Lemma 27, we denote the constant terms of equations by Υ_j, Λ_j, and Γ_j.

Lemma 27. *Let $q \equiv \xi \pmod 3$. Let $4 \le w \le q+1$. Let $W = 2, 3$. Let the integral weight spectra $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(W)})$ be as in Lemma 24. Let $\mathcal{V}_a^{(W)}$, $\mathcal{V}_b^{(W)}$, and $\mathcal{V}_c^{(W)}$ be the weight W cosets of $\mathcal{C}_{\mathscr{C}}$ with distinct weight distributions $A_w(\mathcal{V}_a^{(W)})$, $A_w(\mathcal{V}_b^{(W)})$, and $A_w(\mathcal{V}_c^{(W)})$ according to Table 1. Then these weight distributions can be obtained from the following systems of linear equations.*

(i) $\xi = 1:$
$$\begin{cases} 2A_w(\mathcal{V}_a^{(2)}) + A_w(\mathcal{V}_b^{(2)}) = 6\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(2)})/[(q^3-q)(q-1)] := \Upsilon_1 \\ A_w(\mathcal{V}_a^{(2)}) - A_w(\mathcal{V}_b^{(2)}) = (-1)^w\binom{q-2}{w-3} := \Lambda_1 \end{cases}$$

(ii) $\xi = 1:$ $\begin{cases} 2A_w(\mathcal{V}_a^{(3)}) + (q-1)A_w(\mathcal{V}_b^{(3)}) = 2\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(3)})/(q^3-q) := \Upsilon_2 \\ A_w(\mathcal{V}_a^{(3)}) - A_w(\mathcal{V}_b^{(3)}) = (-1)^w \frac{1}{3}(q-1)\binom{q-2}{w-3} := \Lambda_2 \end{cases}$

(iii) $\xi = 0:$ $\begin{cases} 2(q-1)A_w(\mathcal{V}_a^{(3)}) + (q^2-q+2)A_w(\mathcal{V}_b^{(3)}) = 2\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(3)})/(q^2-1) := \Upsilon_3 \\ A_w(\mathcal{V}_a^{(3)}) - A_w(\mathcal{V}_b^{(3)}) = (-1)^w \frac{1}{3}q\binom{q-2}{w-3} := \Lambda_3 \end{cases}$

(iv) $\xi = -1:$ $\begin{cases} 6A_w(\mathcal{V}_a^{(3)}) + 2(q-1)A_w(\mathcal{V}_b^{(3)}) + (q-1)A_w(\mathcal{V}_c^{(3)}) \\ = 6\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(3)})/(q^3-q) := \Upsilon_4 \\ A_w(\mathcal{V}_a^{(3)}) - A_w(\mathcal{V}_b^{(3)}) = (-1)^w \frac{1}{3}(q-2)\binom{q-2}{w-3} := \Lambda_4 \\ A_w(\mathcal{V}_b^{(3)}) - A_w(\mathcal{V}_c^{(3)}) = (-1)^w \binom{q-2}{w-3} := \Gamma_4 \end{cases}$

Proof. In each of the systems, the 1-st equation directly follows from Table 1 and the definition of $\mathcal{A}_w^{\Sigma}(\mathcal{V}^{(W)})$, see also Theorem 12. From Table 1, we take the number of the cosets of each type, e.g. $N_a^{(W)}$ and $N_b^{(W)}$ for $\mathcal{V}_a^{(W)}$ and $\mathcal{V}_b^{(W)}$, respectively, and obtain the equation

$$N_a^{(W)} A_w(\mathcal{V}_a^{(W)}) + N_b^{(W)} A_w(\mathcal{V}_b^{(W)}) = \mathcal{A}_w^{\Sigma}(\mathcal{V}^{(W)}).$$

For example, in the part (i), we have

$$\frac{1}{3}(q^3-q)(q-1)A_w(\mathcal{V}_a^{(a)}) + \frac{1}{6}(q^3-q)(q-1)A_w(\mathcal{V}_b^{(W)}) = \mathcal{A}_w^{\Sigma}(\mathcal{V}^{(W)}).$$

The 2-nd and 3-rd equations of the systems are based on Theorem 21 and relation (20) where the values $A_3(\mathcal{V}_a^{(W)})$, $A_3(\mathcal{V}_b^{(W)})$, and $A_3(\mathcal{V}_c^{(W)})$ are given in Table 1. For example, in the part (i), we have

$$A_w(\mathcal{V}_a^{(2)}) - A_w(\mathcal{V}_b^{(2)}) = -(-1)^w \left[\frac{1}{6}(q^2-5q+4) - \frac{1}{6}(q^2-5q+10)\right]\binom{q-2}{w-3}.$$

Finally, in the equations, we do natural transformations. □

Theorem 28. *Let* $q \equiv \xi \pmod 3$. *Let* Υ_j, Λ_j, Γ_j, $j = 1,2,3,4$, *be the constant terms of the linear systems given in Lemma 27. Then the weight distributions* $A_w(\mathcal{V}_a^{(W)})$, $A_w(\mathcal{V}_b^{(W)})$, $A_w(\mathcal{V}_c^{(W)})$ *of the weight* W *cosets* $\mathcal{V}_a^{(W)}$, $\mathcal{V}_b^{(W)}$, $\mathcal{V}_c^{(W)}$ *of* $\mathcal{C}_{\mathscr{C}}$ *are as follows:*

(i) $\xi = 1:$ $A_w(\mathcal{V}_a^{(2)}) = \dfrac{\Upsilon_1 + \Lambda_1}{3}$, $A_w(\mathcal{V}_b^{(2)}) = \dfrac{\Upsilon_1 - 2\Lambda_1}{3}$.

(ii) $\xi = 1:$ $A_w(\mathcal{V}_a^{(3)}) = \dfrac{\Upsilon_2 + (q-1)\Lambda_2}{q+1}$, $A_w(\mathcal{V}_b^{(3)}) = \dfrac{\Upsilon_2 - 2\Lambda_2}{q+1}$.

(iii) $\xi = 0:$ $A_w(\mathcal{V}_a^{(3)}) = \dfrac{\Upsilon_3 + (q^2-q+2)\Lambda_3}{q^2+q}$, $A_w(\mathcal{V}_b^{(3)}) = \dfrac{\Upsilon_3 - 2(q-1)\Lambda_3}{q^2+q}$.

(iv) $\xi = -1:$ $A_w(\mathcal{V}_a^{(3)}) = \dfrac{\Upsilon_4 + 3(q-1)\Lambda_4 + (q-1)\Gamma_4}{3(q+1)}$, $A_w(\mathcal{V}_b^{(3)}) = \dfrac{\Upsilon_4 - 6\Lambda_4 + (q-1)\Gamma_4}{3(q+1)}$,

$$A_w(\mathcal{V}_c^{(3)}) = \frac{\Upsilon_4 - 6\Lambda_4 - 2(q+2)\Gamma_4}{3(q+1)}.$$

Proof. The assertions are the obvious solutions of the linear systems of Lemma 27. □

References

[1] R.E. Blahut, Theory and Practice of Error Control Codes, Addison Wesley, Reading, 1984

[2] F.J. MacWilliams, N.J.A. Sloane, The Theory of Error-Correcting Codes, third ed., North- Holland, Amsterdam, The Netherlands, 1981

[3] R.M. Roth, Introduction to Coding Theory, Cambridge, Cambridge Univ. Press, 2007

[4] J.W.P. Hirschfeld, Projective Geometries over Finite Fields, second ed., Oxford University Press, Oxford, 1998.

[5] J.W.P. Hirschfeld, L. Storme, The packing problem in statistics, coding theory and finite projective spaces. J. Statist. Planning Inference **72**(1), 355–380 (1998) https : //doi.org/10.1016/S0378 – 3758(98)00043 – 3

[6] J.W.P. Hirschfeld, L. Storme, The packing problem in statistics, coding Theory and finite projective spaces: Update 2001, in: A. Blokhuis, J.W.P. Hirschfeld, D. Jungnickel, J.A. Thas (Eds.), Finite Geometries (Proc. 4th Isle of Thorns Conf., July 16-21, 2000), Dev. Math., vol. 3, Dordrecht: Kluwer, 2001, pp. 201–246 https : //doi.org/10.1007/978 – 1 – 4613 – 0283 – 4_13

[7] J.W.P. Hirschfeld, J.A. Thas, Open problems in finite projective spaces, Finite Fields Appl. **32**(1), 44–81 (2015) https : //doi.org/10.1016/j.ffa.2014.10.006

[8] I. Landjev, L. Storme, Galois geometry and coding theory. In: *Current Research Topics in Galois geometry,* J. De. Beule and L. Storme, Eds., Chapter 8, pp. 187–214, NOVA Academic, New York, 2011

[9] T. Etzion, L. Storme, Galois geometries and coding theory. Des. Codes Crypt. **78**(1), 311–350 (2016) https : //doi.org/10.1007/s10623 – 015 – 0156 – 5

[10] J.W.P. Hirschfeld, Finite Projective Spaces of Three Dimensions, Oxford University Press, Oxford, 1985

[11] A.A. Bruen, J.W.P. Hirschfeld, Applications of line geometry over finite fields I: The twisted cubic, Geom. Dedicata 6 (4), 495–509 (1977) https : //doi.org/10.1007/BF00147786

[12] M. Giulietti, R. Vincenti, Three-level secret sharing schemes from the twisted cubic, Discrete Math. 310 (22), 3236–3240 (2010) https : //doi.org/10.1016/j.disc.2009.11.040

[13] D. Bartoli, A.A. Davydov, S. Marcugini, F. Pambianco, On planes through points off the twisted cubic in PG(3,q) and multiple covering codes, Finite Fields Appl. **67** (Oct. 2020) https://doi.org/10.1016/j.ffa.2020.101710; see also arXiv:1909.00207 [math.CO] (2020) https://arxiv.org/pdf/1909.00207.pdf

[14] G. Cohen, I. Honkala, S. Litsyn, A. Lobstein, Covering codes, Vol. 54 of North-Holland Mathematical Library, Elsevier, Amsterdam, The Netherlands, 1997.

[15] D. Bartoli, A. A. Davydov, M. Giulietti, S. Marcugini, F. Pambianco, Multiple coverings of the farthest-off points with small density from projective geometry, Advan. Math. Commun. 9 (1), 63–85 (2015) https://doi.org/10.3934/amc.2015.9.63

[16] D. Bartoli, A. A. Davydov, M. Giulietti, S. Marcugini, F. Pambianco, Further results on multiple coverings of the farthest-off point, Advan. Math. Commun. 10 (3), 613–632 (2016) https://doi.org/10.3934/amc.2016030

[17] W.C. Huffman, V.S. Pless, Fundamentals of Error-Correcting Codes, Cambridge University Press, 2003

[18] V.S. Pless, W.C. Huffman, R.A. Brualdi, An Introductiom to Algebraic Codes, In: V.S. Pless, W.C. Huffman, R.A. Brualdi, (eds.) Handbook of Coding Theory, Chapter I, pp. 3–139, Elsevier, Amsterdam, The Netherlands, 1998

[19] E.F. Assmus, Jr., H.F. Mattson, Jr., The Weight-Distribution of a Coset of a Linear Code, IEEE Trans. Inform. Theory **24**(4), 497 (1978) https://doi.org/10.1109/tit.1978.1055903

[20] A. Bonnecaze, I.M. Duursma, Translates of linear codes over $\mathbf{Z}_4$, IEEE Trans. Inform. Theory **43**(4), 1218–1230 (1997) https://doi.org/10.1109/18.605585

[21] P. Charpin, T. Helleseth, V. Zinoviev, The Coset Distribution of Triple-Error-Correcting Binary Primitive BCH Codes. IEEE Trans. Inform. Theory, **52**(4), 1727–1732 (2006) https://doi.org/10.1109/TIT.2006.871605

[22] K.-M. Cheung, More on the decoder error probability for Reed-Solomon codes. IEEE Trans. Inform. Theory **35**(4), 895-900 (1989) https://doi.org/10.1109/18.32169

[23] K.-M. Cheung, On the decoder error probability of block codes. IEEE Trans. Commun. **40**(5), 857-859 (1992) https://doi.org/10.1109/26.141450

[24] A.A. Davydov, S. Marcugini, F. Pambianco, On integral weight spectra of the MDS codes cosets of weight 1, 2, and 3, arXiv:2007.02405 [cs.IT] (2020) https://arxiv.org/abs/2007.02405

[25] P. Delsarte, Four Fundamental Parameters of a Code and Their Combinatorial Significance. Inform. Control **23**, 407–438 (1973) https : //doi.org/10.1016/s0019 − 9958(73)80007 − 5

[26] P. Delsarte, V.I. Levenshtein, Association Schemes and Coding Theory. IEEE Trans. Inform. Theory **44**(6), 2477–2504 (1998) https : //doi.org/10.1109/18.720545

[27] T. Helleseth, The weight distribution of the coset leaders of some classes of codes with related parity-check matrices, Discrete Math. 28, 161–171 (1979) https : //doi.org/10.1016/0012 − 365X(79)90093 − 1

[28] R. Jurrius, R. Pellikaan, The coset leader and list weight enumerator. In: G. Kyureghyan, G.L. Mullen, A. Pott, Eds., Contemporary Math. vol. 632, Topics in Finite Fields (2015) 229–251 https : //doi.org/10.1090/conm/632/12631, corrected version (2019) https : //www.win.tue.nl/˜ruudp/paper/71.pdf

[29] K. Kaipa, Deep Holes and MDS Extensions of ReedSolomon Codes, IEEE Trans. Inform. Theory 63 (8) (2017) 4940 – 4948. https : //doi.org/10.1109/TIT.2017.2706677

[30] T. Kasami, S. Lin, On the Probability of Undetected Error for the Maximum Distance Separable Codes, IEEE Trans. Commun. **32**(9), 998–1006 (1984) https : //doi.org/10.1109/TCOM.1984.1096175

[31] F.J. MacWilliams, A theorem on the distribution of weights in a systematic code, Bell Syst. Tech. J. **42**(1), 79-94 (1963) https : //doi.org/10.1002/j.1538 − 7305.1963.tb04003.x

[32] J.R. Schatz, On the Weight Distributions of Cosets of a Linear Code. American Math. Month. **87**(7), 548-551 (1980) https : //doi.org/10.1080/00029890.1980.11995087

[33] J. Zhang, D. Wan, K. Kaipa, Deep Holes of Projective Reed-Solomon Codes. IEEE Trans. Inform. Theory **66**(4), 2392–2401 (2019) https : //doi.org/10.1109/TIT.2019.2940962

[34] P. Delsarte, An Algebraic Approach to the Association Schemes of Coding Theory. Philips Res. Repts. Suppl. **10**, 1973.

[35] M.F. Ezerman, M. Grassl, P. Sole, The Weights in MDS Codes. IEEE Trans. Inform. Theory **57**(1), 392–396 (2011) https : //doi.org/10.1109/TIT.2010.2090246

[36] J. Riordan, Combinatorial Identities, Willey, New York, 1968