

Twisted cubic and orbits of lines in $\text{PG}(3, q)$

ALEXANDER A. DAVYDOV ¹

INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS (KHARKEVICH
INSTITUTE)

RUSSIAN ACADEMY OF SCIENCES
MOSCOW, 127051, RUSSIAN FEDERATION

E-mail address: adav@iitp.ru

STEFANO MARCUGINI ², FERNANDA PAMBIANCO ³

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, PERUGIA
UNIVERSITY,

PERUGIA, 06123, ITALY

E-mail address: {stefano.marcugini, fernanda.pambianco}@unipg.it

Abstract. In the projective space $\text{PG}(3, q)$, we consider the orbits of lines under the stabilizer group of the twisted cubic. It is well known that the lines can be partitioned into classes every of which is a union of line orbits. All types of lines forming a unique orbit are found. For the rest of the line types (apart from one of them) it is proved that they form exactly two or three orbits; sizes and structures of these orbits are determined. Problems remaining open for one type of lines are formulated. For $5 \leq q \leq 37$ and $q = 64$, they are solved.

Keywords: Twisted cubic, Projective space, Orbits of lines

Mathematics Subject Classification (2010). 51E21, 51E22

¹A.A. Davydov ORCID <https://orcid.org/0000-0002-5827-4560>

²S. Marcugini ORCID <https://orcid.org/0000-0002-7961-0260>

³F. Pambianco ORCID <https://orcid.org/0000-0001-5476-5365>

1 Introduction

Let $\text{PG}(N, q)$ be the N -dimensional projective space over the Galois field \mathbb{F}_q with q elements. An n -arc in $\text{PG}(N, q)$, with $n \geq N + 1 \geq 3$, is a set of n points such that no $N + 1$ points belong to the same hyperplane of $\text{PG}(N, q)$. An n -arc is complete if it is not contained in an $(n + 1)$ -arc, see [1] and the references therein. For an introduction to projective geometry over finite fields see [13, 15, 16].

In $\text{PG}(N, q)$, $2 \leq N \leq q - 2$, a normal rational curve is any $(q + 1)$ -arc projectively equivalent to the arc $\{(t^N, t^{N-1}, \dots, t^2, t, 1) : t \in \mathbb{F}_q\} \cup \{(1, 0, \dots, 0)\}$. In $\text{PG}(3, q)$, the normal rational curve is called a *twisted cubic* [14, 16]. Twisted cubics have important connections with a number of other combinatorial objects. This prompted the twisted cubics to be widely studied, see e.g. [2, 3, 5, 6, 8–10, 12, 14–17, 19] and the references therein. In [14], the orbits of planes, lines and points under the group of the projectivities fixing the twisted cubic are considered. Also, in [2], the structure of the *point-plane* incidence matrix of $\text{PG}(3, q)$ using orbits under the stabilizer group of the twisted cubic is described.

In this paper, we consider the orbits of lines in $\text{PG}(3, q)$ under the stabilizer group G_q of the twisted cubic. We use the partitions of lines into unions of orbits (called *classes*) under G_q described in [14]. All types of lines forming a unique orbit are found. For the rest of the line types (apart from one of them) it is proved that they form exactly two or three orbits; sizes and structures of these orbits are determined. Problems remaining open for one type of lines are formulated. For $5 \leq q \leq 37$ and $q = 64$, they are solved.

The theoretic results hold for $q \geq 5$. For $q = 2, 3, 4$ we describe the orbits by computer search.

The results obtained increase our knowledge on the properties of lines in $\text{PG}(3, q)$. The new results can be useful for feature investigations, in particular, for considerations of the plane-line incidence matrix of $\text{PG}(3, q)$, see [11].

The paper is organized as follows. Section 2 contains preliminaries. In Section 3, the main results of this paper are summarized. In Sections 4–7, orbits of lines in $\text{PG}(3, q)$ under G_q are considered. In Section 8, the open problems are formulated and their solutions for $5 \leq q \leq 37$ and $q = 64$ are considered.

2 Preliminaries on the twisted cubic in $\text{PG}(3, q)$

We summarize the results on the twisted cubic of [14] useful in this paper.

Let $\mathbf{P}(x_0, x_1, x_2, x_3)$ be a point of $\text{PG}(3, q)$ with homogeneous coordinates $x_i \in \mathbb{F}_q$. Let $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, $\mathbb{F}_q^+ = \mathbb{F}_q \cup \{\infty\}$. For $t \in \mathbb{F}_q^+$, let $P(t)$ be a point such that

$$P(t) = \mathbf{P}(t^3, t^2, t, 1) \text{ if } t \in \mathbb{F}_q; \quad P(\infty) = \mathbf{P}(1, 0, 0, 0). \quad (2.1)$$

Let $\mathcal{C} \subset \text{PG}(3, q)$ be the *twisted cubic* consisting of $q + 1$ points P_1, \dots, P_{q+1} no four of which are coplanar. We consider \mathcal{C} in the canonical form

$$\mathcal{C} = \{P_1, P_2, \dots, P_{q+1}\} = \{P(t) \mid t \in \mathbb{F}_q^+\}. \quad (2.2)$$

A *chord* of \mathcal{C} is a line through a pair of real points of \mathcal{C} or a pair of complex conjugate points. In the last case it is an *imaginary chord*. If the real points are distinct, it is a *real chord*; if they coincide with each other, it is a *tangent*.

Let $\pi(c_0, c_1, c_2, c_3) \subset \text{PG}(3, q)$, be the plane with equation

$$c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3 = 0, \quad c_i \in \mathbb{F}_q. \quad (2.3)$$

The *osculating plane* in the point $P(t) \in \mathcal{C}$ is as follows:

$$\pi_{\text{osc}}(t) = \pi(1, -3t, 3t^2, -t^3) \text{ if } t \in \mathbb{F}_q; \quad \pi_{\text{osc}}(\infty) = \pi(0, 0, 0, 1). \quad (2.4)$$

The $q + 1$ osculating planes form the osculating developable Γ to \mathcal{C} , that is a *pencil of planes* for $q \equiv 0 \pmod{3}$ or a *cubic developable* for $q \not\equiv 0 \pmod{3}$.

An *axis* of Γ is a line of $\text{PG}(3, q)$ which is the intersection of a pair of real planes or complex conjugate planes of Γ . In the last case it is a *generator* or an *imaginary axis*. If the real planes are distinct it is a *real axis*; if they coincide with each other it is a *tangent* to \mathcal{C} .

For $q \not\equiv 0 \pmod{3}$, the null polarity \mathfrak{A} [13, Sections 2.1.5, 5.3], [14, Theorem 21.1.2] is given by

$$\mathbf{P}(x_0, x_1, x_2, x_3)\mathfrak{A} = \pi(x_3, -3x_2, 3x_1, -x_0). \quad (2.5)$$

Notation 2.1. In future, we consider $q \equiv \xi \pmod{3}$ with $\xi \in \{-1, 0, 1\}$. Many values depend of ξ or have sense only for specific ξ . We note this by remarks or by superscripts “ (ξ) ”. If a value is the same for all q or a property holds for all q , or it is not relevant, or it is clear by the context, the remarks and superscripts “ (ξ) ” are not used. If a value is the same for $\xi = -1, 1$, then one may use “ $\neq 0$ ”. In superscripts, instead of “ \bullet ”, one can write “od” for odd q or “ev” for even q . If a value is the same for odd and even q , one may omit “ \bullet ”.

The following notation is used.

G_q	the group of projectivities in $\text{PG}(3, q)$ fixing \mathcal{C} ;
\mathbf{Z}_n	cyclic group of order n ;
\mathbf{S}_n	symmetric group of degree n ;
A^{tr}	the transposed matrix of A ;
$\#S$	the cardinality of a set S ;
\overline{AB}	the line through the points A and B ;
\triangleq	the sign “equality by definition”.

Types π of planes:

Γ -plane	an osculating plane of Γ ;
$d_{\mathcal{C}}$ -plane	a plane containing <i>exactly</i> d distinct points of \mathcal{C} , $d = 0, 2, 3$;
$\overline{1_{\mathcal{C}}}$ -plane	a plane not in Γ containing <i>exactly</i> 1 point of \mathcal{C} ;
\mathfrak{P}	the list of possible types π of planes, $\mathfrak{P} \triangleq \{\Gamma, 2_{\mathcal{C}}, 3_{\mathcal{C}}, \overline{1_{\mathcal{C}}}, 0_{\mathcal{C}}\}$;
π -plane	a plane of the type $\pi \in \mathfrak{P}$;
\mathcal{N}_{π}	the orbit of π -planes under G_q , $\pi \in \mathfrak{P}$.

Types λ of lines with respect to the twisted cubic \mathcal{C} :

RC-line	a real chord of \mathcal{C} ;
RA-line	a real axis of Γ for $\xi \neq 0$;
T-line	a tangent to \mathcal{C} ;
IC-line	an imaginary chord of \mathcal{C} ;
IA-line	an imaginary axis of Γ for $\xi \neq 0$;
U Γ	a non-tangent unisecant in a Γ -plane;
Un Γ -line	a unisecant not in a Γ -plane (it is always non-tangent);
E Γ -line	an external line in a Γ -plane (it cannot be a chord);

En Γ -line	an external line, other than a chord, not in a Γ -plane;
A-line	the axis of Γ for $\xi = 0$ (it is the single line of intersection of all the $q + 1$ Γ -planes);
EA-line	an external line meeting the axis of Γ for $\xi = 0$;
$\mathfrak{L}^{(\xi)}$	the list of possible types λ of lines, $\mathfrak{L}^{(\neq 0)} \triangleq \{\text{RC, RA, T, IC, IA, UT, Un}\Gamma, \text{E}\Gamma, \text{En}\Gamma\}$ for $\xi \neq 0$, $\mathfrak{L}^{(0)} \triangleq \{\text{RC, T, IC, UT, Un}\Gamma, \text{En}\Gamma, \text{A, EA}\}$ for $\xi = 0$;
λ -line	a line of the type $\lambda \in \mathfrak{L}^{(\xi)}$;
$L_{\Sigma}^{(\xi)}$	the total number of orbits of lines in $PG(3, q)$;
$L_{\lambda\Sigma}^{(\xi)\bullet}$	the total number of orbits of λ -lines, $\lambda \in \mathfrak{L}^{(\xi)}$;
\mathcal{O}_{λ}	the union (class) of all $L_{\lambda\Sigma}^{(\xi)\bullet}$ orbits of λ -lines under G_q , $\lambda \in \mathfrak{L}^{(\xi)}$.
Types of points with respect to the twisted cubic \mathcal{C}:	
\mathcal{C} -point	a point of \mathcal{C} ;
μ_{Γ} -point	a point off \mathcal{C} lying on <i>exactly</i> μ distinct osculating planes, $\mu_{\Gamma} \in \{0_{\Gamma}, 1_{\Gamma}, 3_{\Gamma}\}$ for $\xi \neq 0$, $\mu_{\Gamma} \in \{(q + 1)_{\Gamma}\}$ for $\xi = 0$;
T-point	a point off \mathcal{C} on a tangent to \mathcal{C} for $\xi \neq 0$;
TO-point	a point off \mathcal{C} on a tangent and one osculating plane for $\xi = 0$;
RC-point	a point off \mathcal{C} on a real chord;
IC-point	a point on an imaginary chord (it is always off \mathcal{C}).

The following theorem summarizes results from [14] useful in this paper.

Theorem 2.2. [14, Chapter 21] *The following properties of the twisted cubic \mathcal{C} of (2.2) hold:*

- (i) *The group G_q acts triply transitively on \mathcal{C} . Also,*
- (a) $G_q \cong PGL(2, q)$ for $q \geq 5$;
 $G_4 \cong \mathbf{S}_5 \cong P\Gamma L(2, 4) \cong \mathbf{Z}_2 PGL(2, 4)$, $\#G_4 = 2 \cdot \#PGL(2, 4) = 120$;
 $G_3 \cong \mathbf{S}_4 \mathbf{Z}_2^3$, $\#G_3 = 8 \cdot \#PGL(2, 3) = 192$;
 $G_2 \cong \mathbf{S}_3 \mathbf{Z}_2^3$, $\#G_2 = 8 \cdot \#PGL(2, 2) = 48$.
- (b) *The matrix \mathbf{M} corresponding to a projectivity of G_q has the general form*

$$\mathbf{M} = \begin{bmatrix} a^3 & a^2c & ac^2 & c^3 \\ 3a^2b & a^2d + 2abc & bc^2 + 2acd & 3c^2d \\ 3ab^2 & b^2c + 2abd & ad^2 + 2bcd & 3cd^2 \\ b^3 & b^2d & bd^2 & d^3 \end{bmatrix}, \quad a, b, c, d \in \mathbb{F}_q, \quad (2.6)$$

$ad - bc \neq 0$.

(ii) Under G_q , $q \geq 5$, there are five orbits of planes and five orbits of points.

(a) For all q , the orbits \mathcal{N}_i of planes are as follows:

$$\begin{aligned} \mathcal{N}_1 &= \mathcal{N}_\Gamma = \{\Gamma\text{-planes}\}, & \#\mathcal{N}_\Gamma &= q + 1; \\ \mathcal{N}_2 &= \mathcal{N}_{2_\mathcal{C}} = \{2_\mathcal{C}\text{-planes}\}, & \#\mathcal{N}_{2_\mathcal{C}} &= q^2 + q; \\ \mathcal{N}_3 &= \mathcal{N}_{3_\mathcal{C}} = \{3_\mathcal{C}\text{-planes}\}, & \#\mathcal{N}_{3_\mathcal{C}} &= (q^3 - q)/6; \\ \mathcal{N}_4 &= \mathcal{N}_{1_\mathcal{C}} = \{\overline{1}_\mathcal{C}\text{-planes}\}, & \#\mathcal{N}_{1_\mathcal{C}} &= (q^3 - q)/2; \\ \mathcal{N}_5 &= \mathcal{N}_{0_\mathcal{C}} = \{0_\mathcal{C}\text{-planes}\}, & \#\mathcal{N}_{0_\mathcal{C}} &= (q^3 - q)/3. \end{aligned} \quad (2.7)$$

(b) For $q \not\equiv 0 \pmod{3}$, the orbits \mathcal{M}_j of points are as follows:

$$\begin{aligned} \mathcal{M}_1 &= \{\mathcal{C}\text{-points}\}, \quad \mathcal{M}_2 = \{\text{T-points}\}, \quad \mathcal{M}_3 = \{3_\Gamma\text{-points}\}, \\ \mathcal{M}_4 &= \{1_\Gamma\text{-points}\}, \quad \mathcal{M}_5 = \{0_\Gamma\text{-points}\}. \end{aligned}$$

Also, if $q \equiv 1 \pmod{3}$ then $\mathcal{M}_3 \cup \mathcal{M}_5 = \{\text{RC-points}\}$, $\mathcal{M}_4 = \{\text{IC-points}\}$;
if $q \equiv -1 \pmod{3}$ then $\mathcal{M}_3 \cup \mathcal{M}_5 = \{\text{IC-points}\}$, $\mathcal{M}_4 = \{\text{RC-points}\}$.

(c) For $q \equiv 0 \pmod{3}$, the orbits \mathcal{M}_j of points are as follows:

$$\begin{aligned} \mathcal{M}_1 &= \{\mathcal{C}\text{-points}\}, \quad \mathcal{M}_2 = \{(q+1)_\Gamma\text{-points}\}, \quad \mathcal{M}_3 = \{\text{TO-points}\}, \\ \mathcal{M}_4 &= \{\text{RC-points}\}, \quad \mathcal{M}_5 = \{\text{IC-points}\}. \end{aligned}$$

(iii) For $q \not\equiv 0 \pmod{3}$, the null polarity \mathfrak{A} (2.5) interchanges \mathcal{C} and Γ and their corresponding chords and axes.

(iv) The lines of $\text{PG}(3, q)$ can be partitioned into classes called \mathcal{O}_i and \mathcal{O}'_i , each of which is a union of orbits under G_q .

$$\begin{aligned} \text{(a)} \quad q \not\equiv 0 \pmod{3}, \quad q \geq 5, \quad \mathcal{O}'_i &= \mathcal{O}_i \mathfrak{A}, \quad \#\mathcal{O}'_i = \#\mathcal{O}_i, \quad i = 1, \dots, 6. \\ \mathcal{O}_1 &= \mathcal{O}_{\text{RC}} = \{\text{RC-lines}\}, \quad \mathcal{O}'_1 = \mathcal{O}_{\text{RA}} = \{\text{RA-lines}\}, \\ \#\mathcal{O}_{\text{RC}} &= \#\mathcal{O}_{\text{RA}} = (q^2 + q)/2; \\ \mathcal{O}_2 &= \mathcal{O}'_2 = \mathcal{O}_{\text{T}} = \{\text{T-lines}\}, \quad \#\mathcal{O}_{\text{T}} = q + 1; \end{aligned} \quad (2.8)$$

$$\begin{aligned}
\mathcal{O}_3 &= \mathcal{O}_{\text{IC}} = \{\text{IC-lines}\}, \quad \mathcal{O}'_3 = \mathcal{O}_{\text{IA}} = \{\text{IA-lines}\}, \\
\#\mathcal{O}_{\text{IC}} &= \#\mathcal{O}_{\text{IA}} = (q^2 - q)/2; \\
\mathcal{O}_4 &= \mathcal{O}'_4 = \mathcal{O}_{\text{U}\Gamma} = \{\text{U}\Gamma\text{-lines}\}, \quad \#\mathcal{O}_{\text{U}\Gamma} = q^2 + q; \\
\mathcal{O}_5 &= \mathcal{O}_{\text{Un}\Gamma} = \{\text{Un}\Gamma\text{-lines}\}, \quad \mathcal{O}'_5 = \mathcal{O}_{\text{E}\Gamma} = \{\text{E}\Gamma\text{-lines}\}, \\
\#\mathcal{O}_{\text{Un}\Gamma} &= \#\mathcal{O}_{\text{E}\Gamma} = q^3 - q; \\
\mathcal{O}_6 &= \mathcal{O}'_6 = \mathcal{O}_{\text{En}\Gamma} = \{\text{En}\Gamma\text{-lines}\}, \quad \#\mathcal{O}_{\text{En}\Gamma} = (q^2 - q)(q^2 - 1).
\end{aligned}$$

For $q > 4$ even, the lines in the regulus complementary to that of the tangents form an orbit of size $q + 1$ contained in $\mathcal{O}_4 = \mathcal{O}_{\text{U}\Gamma}$.

(b) $q \equiv 0 \pmod{3}$, $q > 3$.

Classes $\mathcal{O}_1, \dots, \mathcal{O}_6$ are as in (2.8); $\mathcal{O}_7 = \mathcal{O}_{\text{A}} = \{\text{A-line}\}$, $\#\mathcal{O}_{\text{A}} = 1$; (2.9)
 $\mathcal{O}_8 = \mathcal{O}_{\text{EA}} = \{\text{EA-lines}\}$, $\#\mathcal{O}_{\text{EA}} = (q + 1)(q^2 - 1)$.

(v) The following properties of chords and axes hold.

(a) For all q , no two chords of \mathcal{C} meet off \mathcal{C} .

Every point off \mathcal{C} lies on exactly one chord of \mathcal{C} .

(b) Let $q \not\equiv 0 \pmod{3}$.

No two axes of Γ meet unless they lie in the same plane of Γ .

Every plane not in Γ contains exactly one axis of Γ .

(vi) For $q > 2$, the unisecants of \mathcal{C} such that every plane through such a unisecant meets \mathcal{C} in at most one point other than the point of contact are, for q odd, the tangents, while for q even, the tangents and the unisecants in the complementary regulus.

3 The main results

Throughout the paper, we consider orbits of lines and planes under G_q .

From now on, we consider $q \geq 5$ apart from Theorem 3.2.

Theorem 3.1 summarizes the results of Sections 4–7.

Theorem 3.1. *Let $q \geq 5$, $q \equiv \xi \pmod{3}$. Let notations be as in Section 2 including Notation 2.1. For line orbits under G_q the following holds.*

(i) The following classes of lines consist of a single orbit:

$\mathcal{O}_1 = \mathcal{O}_{\text{RC}} = \{\text{RC-lines}\}$, $\mathcal{O}_2 = \mathcal{O}_{\text{T}} = \{\text{T-lines}\}$, and

$\mathcal{O}_3 = \mathcal{O}_{\text{IC}} = \{\text{IC-lines}\}$, for all q ;

$$\begin{aligned}
\mathcal{O}_4 &= \mathcal{O}_{\text{U}\Gamma} = \{\text{U}\Gamma\text{-lines}\}, \text{ for odd } q; \\
\mathcal{O}_5 &= \mathcal{O}_{\text{Un}\Gamma} = \{\text{Un}\Gamma\text{-lines}\} \text{ and } \mathcal{O}'_5 = \mathcal{O}_{\text{E}\Gamma} = \{\text{E}\Gamma\text{-lines}\}, \text{ for even } q; \\
\mathcal{O}'_1 &= \mathcal{O}_{\text{RA}} = \{\text{RA-lines}\} \text{ and } \mathcal{O}'_3 = \mathcal{O}_{\text{IA}} = \{\text{IA-lines}\}, \text{ for } \xi \neq 0; \\
\mathcal{O}_7 &= \mathcal{O}_{\text{A}} = \{\text{A-lines}\}, \text{ for } \xi = 0.
\end{aligned}$$

- (ii) Let $q \geq 8$ be even. The non-tangent unisecants in a Γ -plane (i.e. $\text{U}\Gamma$ -lines, class $\mathcal{O}_4 = \mathcal{O}_{\text{U}\Gamma}$) form two orbits of size $q + 1$ and $q^2 - 1$. The orbit of size $q + 1$ consists of the lines in the regulus complementary to that of the tangents. Also, the $(q + 1)$ -orbit and $(q^2 - 1)$ -orbit can be represented in the form $\{\ell_1\varphi | \varphi \in G_q\}$ and $\{\ell_2\varphi | \varphi \in G_q\}$, respectively, where ℓ_j is a line such that $\ell_1 = \overline{P_0\mathbf{P}(0, 1, 0, 0)}$, $\ell_2 = \overline{P_0\mathbf{P}(0, 1, 1, 0)}$, $P_0 = \mathbf{P}(0, 0, 0, 1) \in \mathcal{C}$.
- (iii) Let $q \geq 5$ be odd. The non-tangent unisecants not in a Γ -plane (i.e. $\text{Un}\Gamma$ -lines, class $\mathcal{O}_5 = \mathcal{O}_{\text{Un}\Gamma}$) form two orbits of size $\frac{1}{2}(q^3 - q)$. These orbits can be represented in the form $\{\ell_j\varphi | \varphi \in G_q\}$, $j = 1, 2$, where ℓ_j is a line such that $\ell_1 = \overline{P_0\mathbf{P}(1, 0, 1, 0)}$, $\ell_2 = \overline{P_0\mathbf{P}(1, 0, \rho, 0)}$, $P_0 = \mathbf{P}(0, 0, 0, 1) \in \mathcal{C}$, ρ is not a square.
- (iv) Let $q \geq 5$ be odd. Let $q \not\equiv 0 \pmod{3}$. The external lines in a Γ -plane (class $\mathcal{O}'_5 = \mathcal{O}_{\text{E}\Gamma}$) form two orbits of size $(q^3 - q)/2$. These orbits can be represented in the form $\{\ell_j\varphi | \varphi \in G_q\}$, $j = 1, 2$, where $\ell_j = \mathbf{p}_0 \cap \mathbf{p}_j$ is the intersection line of planes \mathbf{p}_0 and \mathbf{p}_j such that $\mathbf{p}_0 = \boldsymbol{\pi}(1, 0, 0, 0) = \pi_{\text{osc}}(0)$, $\mathbf{p}_1 = \boldsymbol{\pi}(0, -3, 0, -1)$, $\mathbf{p}_2 = \boldsymbol{\pi}(0, -3\rho, 0, -1)$, ρ is not a square, cf. (2.3), (2.4).
- (v) Let $q \equiv 0 \pmod{3}$, $q \geq 9$. The external lines meeting the axis of Γ (i.e. EA -lines, class $\mathcal{O}_8 = \mathcal{O}_{\text{EA}}$) form three orbits of size $q^3 - q$, $(q^2 - 1)/2$, $(q^2 - 1)/2$. The $(q^3 - q)$ -orbit and the two $(q^2 - 1)/2$ -orbits can be represented in the form $\{\ell_1\varphi | \varphi \in G_q\}$ and $\{\ell_j\varphi | \varphi \in G_q\}$, $j = 2, 3$, respectively, where ℓ_j are lines such that $\ell_1 = \overline{P_0^{\text{A}}\mathbf{P}(0, 0, 1, 1)}$, $\ell_2 = \overline{P_0^{\text{A}}\mathbf{P}(1, 0, 1, 0)}$, $\ell_3 = \overline{P_0^{\text{A}}\mathbf{P}(1, 0, \rho, 0)}$, $P_0^{\text{A}} = \mathbf{P}(0, 1, 0, 0)$, ρ is not a square.

Theorem 3.2 is obtained by an exhaustive computer search using the symbol calculation system Magma [4].

Theorem 3.2. *Let notations be as in Section 2 including Notation 2.1. For line orbits under G_q the following holds.*

- (i) Let $q = 2$. The group $G_2 \cong \mathbf{S}_3\mathbf{Z}_2^3$ contains 8 subgroups isomorphic to $PGL(2, 2)$ divided into two conjugacy classes. For one of these subgroups, the matrices corresponding to the projectivities of the subgroup assume the form described by (2.6). For this subgroup (and only for it) the line orbits under it are the same as in Theorem 3.1 for $q \equiv -1 \pmod{3}$.
- (ii) Let $q = 3$. The group $G_3 \cong \mathbf{S}_4\mathbf{Z}_2^3$ contains 24 subgroups isomorphic to $PGL(2, 3)$ divided into four conjugacy classes. For one of these subgroups, the matrices corresponding to the projectivities of the subgroup assume the form described by (2.6). For this subgroup (and only for it) the line orbits under it are the same as in Theorem 3.1 for $q \equiv 0 \pmod{3}$.
- (iii) Let $q = 4$. The group $G_4 \cong \mathbf{S}_5 \cong P\Gamma L(2, 4)$ contains one subgroup isomorphic to $PGL(2, 4)$. The matrices corresponding to the projectivities of this subgroup assume the form described by (2.6) and for this subgroup the line orbits under it are the same as in Theorem 3.1 for $q \equiv 1 \pmod{3}$.

4 The null polarity \mathfrak{A} and orbits under G_q of lines in $PG(3, q)$

Lemma 4.1. *Let \mathbf{M} be the general form of the matrix corresponding to a projectivity of G_q given by (2.6). Then its inverse matrix \mathbf{M}^{-1} has the form*

$$\mathbf{M}^{-1} = \begin{bmatrix} d^3A^{-1} & cd^2A^{-1} & c^2dA^{-1} & c^3A^{-1} \\ 3bd^2A^{-1} & d(ad+2bc)A^{-1} & c(2ad+bc)A^{-1} & 3ac^2B^{-1} \\ 3b^2dB^{-1} & b(2ad+bc)B^{-1} & a(ad+2bc)B^{-1} & 3a^2cA^{-1} \\ b^3A^{-1} & ab^2A^{-1} & a^2bA^{-1} & a^3A^{-1} \end{bmatrix}, \quad (4.1)$$

$$A = a^3d^3 - b^3c^3 + 3ab^2c^2d - 3a^2bcd^2, \quad B = (a^2d^2 - 2abcd + b^2c^2)(ad - bc).$$

Proof. The assertion is obtained with the help of the system of symbolic computation Maple [18]. Note that by (2.6), we have $ad - bc \neq 0$. \square

Lemma 4.2. *Let $q \not\equiv 0 \pmod{3}$. Let \mathfrak{A} be the null polarity [14, Theorem 21.1.2] given by (2.5). Let $P = \mathbf{P}(x_0, x_1, x_2, x_3)$ be a point of $PG(3, q)$, $P\mathfrak{A}$*

be its polar plane, and Ψ be a projectivity belonging to G_q . Then

$$(P\mathfrak{A})\Psi = (P\Psi)\mathfrak{A}. \quad (4.2)$$

Proof. Let “ \times ” note the matrix multiplication. Using the matrices \mathbf{M} and \mathbf{M}^{-1} of (2.6) and (4.1), respectively, we define x'_i and \overline{c}_i as follows:

$$[x'_0, x'_1, x'_2, x'_3] = [x_0, x_1, x_2, x_3] \times \mathbf{M}, \quad [\overline{c}_0, \overline{c}_1, \overline{c}_2, \overline{c}_3]^{tr} = \mathbf{M}^{-1} \times [c_0, c_1, c_2, c_3]^{tr}.$$

Then it is well known (see e.g. [7, Chapter 4, Note 23]) that:

$$\pi(c_0, c_1, c_2, c_3)\Psi = \pi(\overline{c}_0, \overline{c}_1, \overline{c}_2, \overline{c}_3).$$

By above and by (2.5), (2.6), (4.1), we have $P\Psi = \mathbf{P}(x'_0, x'_1, x'_2, x'_3)$;

$$(P\Psi)\mathfrak{A} = \pi(x'_3, -3x'_2, 3x'_1, -x'_0); \quad P\mathfrak{A} = \pi(x_3, -3x_2, 3x_1, -x_0);$$

$$(P\mathfrak{A})\Psi = \pi(v_0, v_1, v_2, v_3), \quad [v_0, v_1, v_2, v_3]^{tr} = \mathbf{M}^{-1} \times [x_3, -3x_2, 3x_1, -x_0]^{tr}.$$

By direct symbolic computation using the system Maple, we verified that

$$\mathbf{M}^{-1} \times [x_3, -3x_2, 3x_1, -x_0]^{tr} = [x'_3, -3x'_2, 3x'_1, -x'_0]^{tr}. \quad \square$$

Theorem 4.3. *Let $q \not\equiv 0 \pmod{3}$. Let \mathcal{L} be an orbit of lines under G_q . Then $\mathcal{L}\mathfrak{A}$ also is an orbit of lines under G_q .*

Proof. We take the line ℓ_1 through the points P_1 and P_2 of $\text{PG}(3, q)$ and a projectivity $\Psi \in G_q$. Let ℓ_2 be the line through $Q_1 = P_1\Psi$ and $Q_2 = P_2\Psi$. Then ℓ_1 and ℓ_2 belong to the same orbit and $\ell_2 = \ell_1\Psi$.

We show that $\ell_2\mathfrak{A} = (\ell_1\mathfrak{A})\Psi$. Let $\mathfrak{p}_i = P_i\mathfrak{A}$, $\mathfrak{p}'_i = Q_i\mathfrak{A}$, $i = 1, 2$. By (4.2),

$$\mathfrak{p}'_1 = Q_1\mathfrak{A} = (P_1\Psi)\mathfrak{A} = (P_1\mathfrak{A})\Psi = \mathfrak{p}_1\Psi,$$

$$\mathfrak{p}'_2 = Q_2\mathfrak{A} = (P_2\Psi)\mathfrak{A} = (P_2\mathfrak{A})\Psi = \mathfrak{p}_2\Psi.$$

So, we have $\ell_2\mathfrak{A} = \mathfrak{p}'_1 \cap \mathfrak{p}'_2 = \mathfrak{p}_1\Psi \cap \mathfrak{p}_2\Psi = (\mathfrak{p}_1 \cap \mathfrak{p}_2)\Psi = (\ell_1\mathfrak{A})\Psi$. \square

5 Orbits under G_q of chords of the cubic \mathcal{C} and axes of the osculating developable Γ (orbits of RC-, T-, IC-, RA-, and IA-lines)

Theorem 5.1. *For any $q \geq 5$, the real chords (i.e. RC-lines, class $\mathcal{O}_1 = \mathcal{O}_{\text{RC}}$) of the twisted cubic \mathcal{C} (2.2) form an orbit under G_q .*

Proof. We consider real chords $\mathcal{RC}_1 = \overline{P(t_1)P(t_2)}$ and $\mathcal{RC}_2 = \overline{P(t_3)P(t_4)}$ through the real points of \mathcal{C} , respectively, $P(t_1), P(t_2)$ and $P(t_3), P(t_4)$ such that $t_1 \neq t_2, t_3 \neq t_4, \{t_1, t_2\} \neq \{t_3, t_4\}$. The group G_q acts triply transitively on \mathcal{C} , see Theorem 2.2(i). So, there is a projectivity $\Psi \in G_q$ such that $\{P(t_1), P(t_2)\}\Psi = \{P(t_3), P(t_4)\}$. This projectivity maps also \mathcal{RC}_1 to \mathcal{RC}_2 , i.e. $\mathcal{RC}_1\Psi = \mathcal{RC}_2$. So, the real chords form an orbit under G_q . \square

Corollary 5.2. *Let $q \not\equiv 0 \pmod{3}$. In $\text{PG}(3, q)$, for the osculating developable Γ of the twisted cubic \mathcal{C} (2.2), the real axes (i.e. RA-lines, class $\mathcal{O}'_1 = \mathcal{O}_{\text{RA}}$) form an orbit under G_q .*

Proof. The assertion follows from Theorems 2.2(iv)(a), 4.3, and 5.1. \square

Theorem 5.3. *For any $q \geq 5$, the tangents (i.e. T-lines, class $\mathcal{O}_2 = \mathcal{O}_{\text{T}}$) to the twisted cubic \mathcal{C} (2.2) form an orbit under G_q . Moreover, the group G_q acts triply transitively on this orbit.*

Proof. We consider two tangents $\mathcal{T}_{t_1} = \overline{P(t_1)P(t_1)}$ and $\mathcal{T}_{t_2} = \overline{P(t_2)P(t_2)}$ through the real points $P(t_1), P(t_1)$ and $P(t_2), P(t_2)$ such that $t_1 \neq t_2$. As the points of \mathcal{C} form an orbit under G_q , there is a projectivity $\Psi \in G_q$ such that $P(t_1)\Psi = P(t_2)$. This projectivity maps also \mathcal{T}_{t_1} to \mathcal{T}_{t_2} , i.e. $\mathcal{T}_{t_1}\Psi = \mathcal{T}_{t_2}$. Thus, the tangents form an orbit under G_q . On this orbit, G_q acts triply transitively since G_q acts triply transitively on \mathcal{C} . \square

Theorem 5.4. *For any $q \geq 5$, in $\text{PG}(3, q)$, the imaginary chords (i.e. IC-lines, class $\mathcal{O}_3 = \mathcal{O}_{\text{IC}}$) of the twisted cubic \mathcal{C} (2.2) form an orbit under G_q .*

Proof. Let $q \equiv \xi \pmod{3}$. By Theorem 2.2(ii)(b)(c), for $\xi = 1$ (resp. $\xi = 0$), points on imaginary chords form the orbit \mathcal{M}_4 (resp. \mathcal{M}_5). If $\xi = -1$, points on IC-lines are divided into two orbits $\mathcal{M}_3 = \{\text{points on three osculating planes}\}$ and $\mathcal{M}_5 = \{\text{points on no osculating plane}\}$. As in $\text{PG}(3, q)$ a plane and a line always meet, for $\xi = -1$ every imaginary chord contains a point belonging to an osculating plane and therefore to \mathcal{M}_3 .

Now, for any q , suppose that there exist at least two orbits $\overline{\mathcal{O}}_1$ and $\overline{\mathcal{O}}_2$ of imaginary chords. Consider IC-lines $\ell_1 \in \overline{\mathcal{O}}_1$ and $\ell_2 \in \overline{\mathcal{O}}_2$. By Theorem 2.2(v)(a), no two chords of \mathcal{C} meet off \mathcal{C} . Thus, $\ell_1 \cap \ell_2 = \emptyset$ and there exist at least two points $P_1 \in \ell_1$ and $P_2 \in \ell_2$ belonging to the same orbit; it is $\mathcal{M}_4, \mathcal{M}_5$, and \mathcal{M}_3 for $\xi = 1, 0$, and -1 , respectively. So, there is $\varphi \in G_q$ such that $P_1\varphi = P_2$. A projectivity maps a line to a line; as all points on IC-lines are placed in “own” orbits (one or two) that do not contain points of other types,

$l_1\varphi$ is an IC-line. Moreover, by Theorem 2.2(v)(a), every point off \mathcal{C} lies on exactly one chord; thus, $l_1\varphi$ is the only imaginary chord containing P_2 , i.e. $l_1\varphi = l_2$. So, $\overline{\mathcal{O}}_1 = \overline{\mathcal{O}}_2$. \square

Corollary 5.5. *Let $q \not\equiv 0 \pmod{3}$. In $\text{PG}(3, q)$, for the osculating developable Γ of the twisted cubic \mathcal{C} (2.2), the imaginary axes (class $\mathcal{O}'_3 = \mathcal{O}_{1A}$) form an orbit under G_q .*

Proof. The assertion follows from Theorems 2.2(iv)(a), 4.3, and 5.4. \square

6 Orbits under G_q of non-tangent unisecants and external lines with respect to the cubic \mathcal{C} (orbits of $U\Gamma$ -, $Un\Gamma$ -, and $E\Gamma$ -lines)

Notation 6.1. In addition to Notation 2.1, the following notation is used.

- P_t the point $P(t)$ of \mathcal{C} with $t \in \mathbb{F}_q^+$, cf. (2.1), (2.2);
- \mathcal{T}_t the tangent line to \mathcal{C} at the point P_t ;
- $G_q^{P_t}$ the subgroup of G_q fixing P_t ;
- \mathcal{O}_{λ_i} the set of lines from \mathcal{O}_λ through P_i , i.e. $\mathcal{O}_{\lambda_i} \triangleq \{\ell \in \mathcal{O}_\lambda | P_i \in \ell\}$.

Lemma 6.2. *The tangent \mathcal{T}_t to \mathcal{C} at the point P_t has the following equation:*

$$\mathcal{T}_\infty \text{ has equation } \begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases} ; \quad \mathcal{T}_1 \text{ has equation } \begin{cases} x_0 = x_1 + x_2 - x_3 \\ x_0 = 3x_2 - 2x_3 \end{cases} ;$$

$$\mathcal{T}_t, t \in \mathbb{F}_q, t \neq 1, \text{ has equation } \begin{cases} x_0 = tx_1 + t^2x_2 - t^3x_3 \\ x_1 = tx_0 + (2t - 3t^3)x_2 + (2t^4 - t^2)x_3 \end{cases} .$$

Proof. The point $P_t = \mathbf{P}(t^3, t^2, t, 1)$, $t \in \mathbb{F}_q$, can be considered as an affine point with respect to the infinite plane $x_3 = 0$. Then the slope of the tangent line to \mathcal{C} at P_t is obtained by deriving the parametric equation of \mathcal{C} and is $(3t^2, 2t, 1)$. It means that \mathcal{T}_t contains the infinite point $Q_t = \mathbf{P}(3t^2, 2t, 1, 0)$. The planes \mathbf{p}_1 of equation $x_0 = tx_1 + t^2x_2 - t^3x_3$ and \mathbf{p}_2 of equation $x_1 = tx_0 + (2t - 3t^3)x_2 + (2t^4 - t^2)x_3$ contain both the points P_t and Q_t .

However, if $t = 1$, $\mathbf{p}_1 = \mathbf{p}_2$, so we consider \mathbf{p}_3 of equation $x_0 = 3x_2 - 2x_3$ as second plane containing both P_t and Q_t . In particular \mathcal{T}_0 has equation $x_0 = 0, x_1 = 0$.

Now consider the projectivity Ψ of equation $x'_0 = x_3, x'_1 = x_2, x'_2 = x_1, x'_3 = x_0$. Then $P_0\Psi = P_\infty, P_\infty\Psi = P_0$, and $P_t\Psi = P_{1/t}$ if $t \neq 0$. It means that $\Psi \in G_q$ and $\mathcal{T}_\infty = \mathcal{T}_0\Psi$ has equation $x_2 = 0, x_3 = 0$. \square

Lemma 6.3. *The general form of the matrix \mathbf{M}^{P_0} corresponding to a projectivity of $G_q^{P_0}$ is as follows:*

$$\mathbf{M}^{P_0} = \begin{bmatrix} 1 & c & c^2 & c^3 \\ 0 & d & 2cd & 3c^2d \\ 0 & 0 & d^2 & 3cd^2 \\ 0 & 0 & 0 & d^3 \end{bmatrix}, \quad c \in \mathbb{F}_q, \quad d \in \mathbb{F}_q^*. \quad (6.1)$$

Proof. Let \mathbf{M} of (2.6) correspond to a projectivity $\Psi \in G_q$. We have $[0, 0, 0, 1] \times \mathbf{M} = [b^3, b^2d, bd^2, d^3]$. Then $\Psi \in G_q^{P_0}$ if and only if $b = 0, d \neq 0$. Also, we should put $a \neq 0$, to provide $ad - bc \neq 0$, see (2.6). One may choose $a = 1$, see (6.1), as we consider points in homogeneous coordinates. \square

Lemma 6.4. *$G_q^{P_i}$ and $G_q^{P_j}$ are conjugate subgroups of $G_q, i, j \in \mathbb{F}_q^+$.*

Proof. As G_q acts transitively on \mathcal{C} , there exists $\Psi \in G_q$ such that $P_i\Psi = P_j$. Then $\Psi^{-1}G_q^{P_i}\Psi = G_q^{P_j}$. In fact, let $\varphi \in G_q^{P_i}$. Then $P_j\Psi^{-1}\varphi\Psi = P_i\varphi\Psi = P_i\Psi = P_j$. On the other hand, let $\gamma \in G_q^{P_j}$. Then $P_i\Psi\gamma\Psi^{-1} = P_j\gamma\Psi^{-1} = P_j\Psi^{-1} = P_i$. It means that $\Psi\gamma\Psi^{-1} \in G_q^{P_i}$, i.e. $\gamma \in \Psi^{-1}G_q^{P_i}\Psi$. \square

Corollary 6.5. *For all $t \in \mathbb{F}_q^+$, we have $\#G_q^{P_t} = q(q-1)$.*

Proof. By (6.1), $\#G_q^{P_0} = q(q-1)$. By Lemma 6.4, there exists $\Psi \in G_q$ such that $\Psi^{-1}G_q^{P_0}\Psi = G_q^{P_t}, t \in \mathbb{F}_q^+$. Then $G_q^{P_0}\Psi = \Psi G_q^{P_t}$. As a finite group and its cosets have the same cardinality, $\#G_q^{P_0} = \#G_q^{P_t} = \#\Psi G_q^{P_t} = \#G_q^{P_t}$. \square

Lemma 6.6. *Let $\lambda \in \{\text{RC}, \text{T}, \text{U}\Gamma, \text{Un}\Gamma\}$. Then $\mathbb{O}_{\lambda_i}G_q^{P_i} = \mathbb{O}_{\lambda_i}$.*

Proof. Let $\ell \in \mathbb{O}_{\lambda_i}, \varphi \in G_q^{P_i}$. As $P_i \in \ell, P_i\varphi = P_i \in \ell\varphi$. For $\lambda \in \{\text{RC}, \text{T}, \text{U}\Gamma, \text{Un}\Gamma\}$, ℓ of type λ implies $\ell\varphi$ of type λ . Therefore, $\ell\varphi \in \mathbb{O}_{\lambda_i}$. On the other hand, if I is the identity element of $G_q^{P_i}$, $\mathbb{O}_{\lambda_i}G_q^{P_i} \supseteq \mathbb{O}_{\lambda_i}I = \mathbb{O}_{\lambda_i}$. \square

Lemma 6.7. *Let ℓ be a line such that $P_i \in \ell$. Let $\mathcal{O}_\ell = \{\ell\varphi | \varphi \in G_q^{P_i}\}$, $\Psi_1, \Psi_2 \in G_q$. If $P_i\Psi_1 = P_i\Psi_2 = P_j$ then $\mathcal{O}_\ell\Psi_1 = \mathcal{O}_\ell\Psi_2$.*

Proof. As $P_i\Psi_1\Psi_2^{-1} = P_j\Psi_2^{-1} = P_i$, we have $\Psi_1\Psi_2^{-1} \in G_q^{P_i}$. Let $\bar{\ell} \in \mathcal{O}_\ell\Psi_1$. Then $\bar{\ell} = \ell\varphi\Psi_1, \varphi \in G_q^{P_i}$. This implies $\bar{\ell}\Psi_2^{-1} = \ell\varphi\Psi_1\Psi_2^{-1} \in \mathcal{O}_\ell$, whence $\bar{\ell} \in \mathcal{O}_\ell\Psi_2$. The proof of the other inclusion is analogous. \square

Lemma 6.8. *Let $\lambda \in \{\text{U}\Gamma, \text{Un}\Gamma\}$, $\ell_1, \ell_2 \in \mathcal{O}_{\lambda_i}$, $\mathcal{O}_{\ell_1} = \{\ell_1\varphi | \varphi \in G_q^{P_i}\}$, $\mathcal{O}_{\ell_2} = \{\ell_2\varphi | \varphi \in G_q^{P_i}\}$. If $\mathcal{O}_{\ell_1} \cap \mathcal{O}_{\ell_2} = \emptyset$ then $\mathcal{O}_{\ell_1}G_q \cap \mathcal{O}_{\ell_2}G_q = \emptyset$.*

Proof. Suppose $\bar{\ell} \in \mathcal{O}_{\ell_1}G_q \cap \mathcal{O}_{\ell_2}G_q$. Then $\bar{\ell}$ is a line of the same type λ as ℓ_1 and ℓ_2 , i.e. it is a unisecant of \mathcal{C} , so there exists P_j such that $P_j \in \bar{\ell}$. As $\bar{\ell} \in \mathcal{O}_{\ell_1}G_q$, $\bar{\ell} = \ell_1\varphi_1\Psi_1$, $\varphi_1 \in G_q^{P_i}$, $\Psi_1 \in G_q$. The point P_i belongs to the line $\ell' = \ell_1\varphi_1$. As $P_j \in \bar{\ell} = \ell'\Psi_1$ and $\ell_1, \ell', \bar{\ell}$ are unisecants and $\Psi_1 \in G_q$, $P_i\Psi_1$ is the only point of $\bar{\ell}$ belonging to \mathcal{C} , i.e. $P_i\Psi_1 = P_j$. Analogously, $\bar{\ell} \in \mathcal{O}_{\ell_2}G_q$ implies $\bar{\ell} = \ell_2\varphi_2\Psi_2$, $\varphi_2 \in G_q^{P_i}$, $P_i\Psi_2 = P_j$. Then $P_i\Psi_1\Psi_2^{-1} = P_j\Psi_2^{-1} = P_i$, that implies $\Psi_1\Psi_2^{-1} \in G_q^{P_i}$. Finally, $\ell_1\varphi_1\Psi_1 = \ell_2\varphi_2\Psi_2$ implies $\ell_1\varphi_1\Psi_1\Psi_2^{-1}\varphi_2^{-1} = \ell_2$, whence $\ell_2 \in \mathcal{O}_{\ell_1}$. \square

Lemma 6.9. *Let $\lambda \in \{\text{T}, \text{U}\Gamma, \text{Un}\Gamma\}$, $\ell \in \mathcal{O}_\lambda$, $\mathcal{O}_\ell = \{\ell\varphi | \varphi \in G_q^{P_i}\}$. Then $\#\mathcal{O}_\ell G_q = (q+1) \cdot \#\mathcal{O}_\ell$.*

Proof. Let $G_q^j = \{\varphi \in G_q | P_i\varphi = P_j\}$. The sets $G_q^j, j \in \mathbb{F}_q^+$ form a partition of G_q . In fact, let $\varphi \in G_q$. As G_q is the stabilizer group of \mathcal{C} , $P_i\varphi = P_j \in \mathcal{C}$, so $\varphi \in G_q^j$. On the other hand, if $\varphi \in G_q^j \cap G_q^k$, then $P_j = P_i\varphi = P_k$, so $j = k$. If $\Psi \in G_q^j$, then $\mathcal{O}_\ell\Psi = \mathcal{O}_\ell G_q^j$. In fact, by Lemma 6.7, if $\Psi' \in G_q^j$ then $\mathcal{O}_\ell\Psi' = \mathcal{O}_\ell\Psi$. Finally, consider $\Psi_j \in G_q^j, j \in \mathbb{F}_q^+$. Then $\mathcal{O}_\ell G_q = \bigcup_{j \in \mathbb{F}_q^+} \mathcal{O}_\ell G_q^j = \bigcup_{j \in \mathbb{F}_q^+} \mathcal{O}_\ell\Psi_j$. The sets $\mathcal{O}_\ell\Psi_j, j \in \mathbb{F}_q^+$, are disjoint. In fact, a line $\ell' \in \mathcal{O}_\ell\Psi_m \cap \mathcal{O}_\ell\Psi_n, m \neq n$, would be a line of type λ , i.e. a unisecant of \mathcal{C} , passing through the distinct points $P_m, P_n \in \mathcal{C}$. Moreover, as Ψ_j is a bijection, $\#\mathcal{O}_\ell\Psi_j = \#\mathcal{O}_\ell$. Therefore, $\#\mathcal{O}_\ell G_q = \sum_{j \in \mathbb{F}_q^+} \#\mathcal{O}_\ell\Psi_j = \sum_{j \in \mathbb{F}_q^+} \#\mathcal{O}_\ell = (q+1) \cdot \#\mathcal{O}_\ell$. \square

Lemma 6.10. *Let $\lambda \in \{\text{U}\Gamma, \text{Un}\Gamma\}$, $\ell \in \mathcal{O}_\lambda$. Let P_i be a point of \mathcal{C} . Then there exists a line $\bar{\ell} \in \mathcal{O}_{\lambda_i}$ such that $\ell \in \mathcal{O}_{\bar{\ell}}G_q$, where $\mathcal{O}_{\bar{\ell}} = \{\bar{\ell}\varphi | \varphi \in G_q^{P_i}\}$.*

Proof. The line ℓ is a unisecant, so there exists P_j such that $P_j \in \ell$. As G_q acts transitively on \mathcal{C} , there exists $\Psi \in G_q$ such that $P_j\Psi = P_i$. Let $\bar{\ell} = \ell\Psi$. Then $\bar{\ell}$ is of the same type λ as ℓ , i.e. $\bar{\ell}$ is a unisecant, and $P_j \in \ell$ implies $P_j\Psi = P_i \in \ell\Psi = \bar{\ell}$, i.e. $\bar{\ell} \in \mathcal{O}_{\lambda_i}$. Finally, $\ell = \bar{\ell}\Psi^{-1}$ implies $\ell \in \mathcal{O}_{\bar{\ell}}G_q$. \square

Lemma 6.11. *Let $\lambda \in \{\text{U}\Gamma, \text{Un}\Gamma\}$, let $P_i \in \mathcal{C}$, $\ell^1, \dots, \ell^m \in \mathcal{O}_{\lambda_i}$, $\mathcal{O}_{\ell^j} = \{\ell^j\varphi | \varphi \in G_q^{P_i}\}, j \in 1, \dots, m$. If $\{\mathcal{O}_{\ell^1}, \dots, \mathcal{O}_{\ell^m}\}$ is a partition of \mathcal{O}_{λ_i} , then $\{\mathcal{O}_{\ell^1}G_q, \dots, \mathcal{O}_{\ell^m}G_q\}$ is a partition of \mathcal{O}_λ .*

Proof. Let $\bar{\ell} \in \mathcal{O}_\lambda$. By Lemma 6.10, there exists $\ell' \in \mathbb{O}_{\lambda_i}$ such that $\bar{\ell} \in \mathcal{O}_{\ell'} G_q$, $\mathcal{O}_{\ell'} = \{\ell' \varphi | \varphi \in G_q^{P_i}\}$. By hypothesis, there exists $\bar{\ell}^j, \bar{j} \in \{1, \dots, m\}$, such that $\mathcal{O}_{\ell'} = \mathcal{O}_{\bar{\ell}^j}$. By Lemma 6.8, $\mathcal{O}_{\ell^j} \neq \mathcal{O}_{\ell^k}, j \neq k$, implies $\mathcal{O}_{\ell^j} G_q \neq \mathcal{O}_{\ell^k} G_q$. \square

In the rest of the section we denote by c, d or c_i, d_i the elements of the matrix of the form (6.1) corresponding to a projectivity $\varphi \in G_q^{P_0}$ or $\varphi_i \in G_q^{P_0}$, respectively. Also, given a 3×4 matrix \mathbf{D} , we denote by $\det_i(\mathbf{D})$ the determinant of the 3×3 matrix obtained deleting the i -th column of \mathbf{D} .

Theorem 6.12. *For any $q \geq 5$, in $\text{PG}(3, q)$, for the twisted cubic \mathcal{C} of (2.2), the non-tangent unisecants in a Γ -plane (i.e. $\text{U}\Gamma$ -lines, class $\mathcal{O}_4 = \mathcal{O}_{\text{U}\Gamma}$) form an orbit under G_q if q is odd and two orbits of size $q+1$ and q^2-1 if q is even. Moreover, for q even, the orbit of size $q+1$ consists of the lines in the regulus complementary to that of the tangents. Also, for q even, the $(q+1)$ -orbit and (q^2-1) -orbit can be represented in the form $\{\ell_1 \varphi | \varphi \in G_q\}$ and $\{\ell_2 \varphi | \varphi \in G_q\}$, respectively, where ℓ_j is a line such that $\ell_1 = \overline{P_0 \mathbf{P}(0, 1, 0, 0)}$, $\ell_2 = \overline{P_0 \mathbf{P}(0, 1, 1, 0)}$, $P_0 = \mathbf{P}(0, 0, 0, 1) \in \mathcal{C}$.*

Proof. Let $\mathbb{O}_{\text{U}\Gamma_0} = \{\ell \in \mathcal{O}_{\text{U}\Gamma} | P_0 \in \ell\}$ be the set of $\text{U}\Gamma$ -lines through P_0 . By Lemma 6.6, $\mathbb{O}_{\text{U}\Gamma_0} G_q^{P_0} = \mathbb{O}_{\text{U}\Gamma_0}$, so we can consider the orbits of $\mathbb{O}_{\text{U}\Gamma_0}$ under the group $G_q^{P_0}$. In $\pi_{\text{osc}}(0)$, there are $q+1$ unisecants through P_0 , one of which is a tangent whereas the other q are $\text{U}\Gamma$ -lines; so $\#\mathbb{O}_{\text{U}\Gamma_0} = q$. By (2.3), (2.4), $\pi_{\text{osc}}(0)$ has equation $x_0 = 0$. By Lemma 6.2, the tangent \mathcal{T}_0 to \mathcal{C} at P_0 has equation $x_0 = x_1 = 0$.

Let $P' = \mathbf{P}(0, 1, 0, 0)$, $\ell' = \overline{P' P_0}$. By above, $P' \in \pi_{\text{osc}}(0)$, $P' \notin \mathcal{T}_0$, whence $\ell' \in \mathbb{O}_{\text{U}\Gamma_0}$. Let $\mathcal{O}_{\ell'} = \{\ell' \varphi | \varphi \in G_q^{P_0}\}$. If $\varphi \in G_q^{P_0}$, then, by (6.1), $P' \varphi = \mathbf{P}(0, d, 2cd, 3c^2d) = \mathbf{P}(0, 1, 2c, 3c^2) \notin \mathcal{T}_0$. So, $\ell' \varphi$ is of type $\text{U}\Gamma$ and $P_0 \in \ell'$ implies $P_0 \varphi = P_0 \in \ell' \varphi$, whence $\mathcal{O}_{\ell'} \subseteq \mathbb{O}_{\text{U}\Gamma_0}$.

Now we determine $\#\mathcal{O}_{\ell'}$. Let $\varphi_1, \varphi_2 \in G_q^{P_0}, \varphi_1 \neq \varphi_2$, $Q' = P' \varphi_1$, $R' = P' \varphi_2$. By (6.1) with $d_1, d_2 \neq 0$, we have

$$\begin{aligned} Q' &= \mathbf{P}(0, d_1, 2c_1 d_1, 3c_1^2 d_1) = \mathbf{P}(0, 1, 2c_1, 3c_1^2), \\ R' &= \mathbf{P}(0, d_2, 2c_2 d_2, 3c_2^2 d_2) = \mathbf{P}(0, 1, 2c_2, 3c_2^2). \end{aligned}$$

Obviously, $\ell' \varphi_1 \neq \ell' \varphi_2$ if and only if P_0, Q', R' are not collinear, i.e. the matrix $\mathbf{D}' = [P_0, Q', R']^{tr}$ has the maximum rank. We obtain

$$\det_1(\mathbf{D}') = 2c_2 - 2c_1, \det_2(\mathbf{D}') = \det_3(\mathbf{D}') = \det_4(\mathbf{D}') = 0.$$

If q is odd, fixed $d \neq 0$ in (6.1), and varying $c \in \mathbb{F}_q$, we obtain q different images of ℓ' , i.e. $\mathbb{O}_{\text{Un}\Gamma_0} = \mathcal{O}_{\ell'}$. Then, by Lemma 6.11, $\mathcal{O}_{\ell'}G_q = \mathcal{O}_{\text{Un}\Gamma}$.

Let q be even.

We have $\det_1(\mathbf{D}') = 0$, so $\mathcal{O}_{\ell'} = \{\ell'\}$.

Consider $P'' = \mathbf{P}(0, 1, 1, 0) \notin \mathcal{T}_0$ and $\ell'' = \overline{P''P_0}$. As ℓ' has equation $x_0 = x_2 = 0$, we have $P'' \notin \ell'$; so $\ell'' \neq \ell'$, i.e. $\ell'' \notin \mathcal{O}_{\ell'}$. Let $\mathcal{O}_{\ell''} = \{\ell''\varphi | \varphi \in G_q^{P_0}\}$. If $\varphi \in G_q^{P_0}$, then $\ell''\varphi$ is of type $\text{Un}\Gamma$ and $P_0 \in \ell''$ implies $P_0\varphi = P_0 \in \ell''\varphi$, whence $\mathcal{O}_{\ell''} \subseteq \mathbb{O}_{\text{Un}\Gamma_0}$. Let $\varphi_1, \varphi_2 \in G_q^{P_0}$, $\varphi_1 \neq \varphi_2$, $Q'' = P\varphi_1$, $R'' = P\varphi_2$. By (6.1) with $d_1, d_2 \neq 0$, we have

$$Q'' = \mathbf{P}(0, 1, d_1, c_1^2 + c_1d_1), \quad R'' = \mathbf{P}(0, 1, d_2, c_2^2 + c_2d_2).$$

As above, $\ell''\varphi_1 \neq \ell''\varphi_2$ if and only if P_0, Q'', R'' are not collinear, i.e. the matrix $\mathbf{D}'' = [P_0, Q'', R'']^{\text{tr}}$ has the maximum rank. We obtain

$$\det_1(\mathbf{D}'') = d_2 - d_1, \quad \det_2(\mathbf{D}'') = \det_3(\mathbf{D}'') = \det_4(\mathbf{D}'') = 0.$$

Fixed c and varying $d \in \mathbb{F}^*$, we obtain $q - 1$ different images of ℓ'' , i.e. $\#\mathcal{O}_{\ell''} = q - 1$.

As $\mathcal{O}_{\ell'} \cap \mathcal{O}_{\ell''} = \emptyset$ and $\#\mathbb{O}_{\text{Un}\Gamma_0} = q$, $\{\mathcal{O}_{\ell'}, \mathcal{O}_{\ell''}\}$ is a partition of $\mathbb{O}_{\text{Un}\Gamma_0}$. Then, by Lemma 6.11, $\{\mathcal{O}_{\ell'}G_q, \mathcal{O}_{\ell''}G_q\}$ is a partition of $\mathcal{O}_{\text{Un}\Gamma}$. By Lemma 6.9, $\#\mathcal{O}_{\ell'}G_q = q + 1$, $\#\mathcal{O}_{\ell''}G_q = (q - 1)(q + 1)$.

Finally, on content of the $(q + 1)$ -orbit $\mathcal{O}_{\ell'}G_q$ see Theorem 2.2(iv)(a). \square

Theorem 6.13. *Let $q \geq 5$. In $\text{PG}(3, q)$, for the twisted cubic \mathcal{C} of (2.2), the non-tangent unisecants not in a Γ -plane (i.e. $\text{Un}\Gamma$ -lines, class $\mathcal{O}_5 = \mathcal{O}_{\text{Un}\Gamma}$) form an orbit under G_q if q is even and two orbits of size $\frac{1}{2}(q^3 - q)$ if q is odd. Moreover, for q odd, the two orbits can be represented in the form $\{\ell_j\varphi | \varphi \in G_q\}$, $j = 1, 2$, where ℓ_j is a line such that $\ell_1 = \overline{P_0\mathbf{P}(1, 0, 1, 0)}$, $\ell_2 = \overline{P_0\mathbf{P}(1, 0, \rho, 0)}$, $P_0 = \mathbf{P}(0, 0, 0, 1) \in \mathcal{C}$, ρ is not a square.*

Proof. We act similarly to the proof of Theorem 6.12. Let $\mathbb{O}_{\text{Un}\Gamma_0} = \{\ell \in \mathcal{O}_{\text{Un}\Gamma} | P_0 \in \ell\}$. By Lemma 6.6, $\mathbb{O}_{\text{Un}\Gamma_0}G_q^{P_0} = \mathbb{O}_{\text{Un}\Gamma_0}$, so we can consider the orbits of $\mathbb{O}_{\text{Un}\Gamma_0}$ under $G_q^{P_0}$. In total, through P_0 there are $q^2 + q + 1$ lines, $q + 1$ of which are unisecants in $\pi_{\text{osc}}(0)$, other q are real chords, and the remaining $q^2 - q$ are $\text{Un}\Gamma$ -lines. So, $\#\mathbb{O}_{\text{Un}\Gamma_0} = q^2 - q$. The equation of $\pi_{\text{osc}}(0)$ is $x_0 = 0$. The tangent \mathcal{T}_0 to \mathcal{C} in P_0 has equation $x_0 = x_1 = 0$.

Let $P' = \mathbf{P}(1, 0, 1, 0)$ and $\ell' = \overline{P'P_0} \notin \pi_{\text{osc}}(0)$. Also, ℓ' is not a real chord, as ℓ' has equation $x_0 = x_2, x_1 = 0$ and $\mathcal{C} \cap \ell' = P_0$. Thus, ℓ' is a $\text{Un}\Gamma$ -line.

Let $\mathcal{O}_{\ell'} = \{\ell'\varphi | \varphi \in G_q^{P_0}\}$. We have $\mathcal{O}_{\ell'} \subseteq \mathbb{O}_{\text{Un}\Gamma_0}$, as $\ell'\varphi$ is a $\text{Un}\Gamma$ -line and $P_0 \in \ell'$ implies $P_0\varphi = P_0 \in \ell'\varphi$.

We find $\#\mathcal{O}_{\ell'}$. Let $\varphi_1, \varphi_2 \in G_q^{P_0}, \varphi_1 \neq \varphi_2, Q' = P'\varphi_1, R' = P'\varphi_2$. By (6.1),

$$Q' = \mathbf{P}(1, c_1, c_1^2 + d_1^2, c_1^3 + 3c_1d_1^2), R' = \mathbf{P}(1, c_2, c_2^2 + d_2^2, c_2^3 + 3c_2d_2^2).$$

Obviously, $\ell'\varphi_1 \neq \ell'\varphi_2$ if and only if P_0, Q', R' are not collinear, i.e. the matrix $\mathbf{D}' = [P_0, Q', R']^{tr}$ has the maximum rank. We obtain

$$\begin{aligned} \det_1(\mathbf{D}') &= c_1(c_2^2 + d_2^2) - c_2(c_1^2 + d_1^2), \quad \det_2(\mathbf{D}') = c_2^2 + d_2^2 - (c_1^2 + d_1^2), \\ \det_3(\mathbf{D}') &= c_2 - c_1, \quad \det_4(\mathbf{D}') = 0. \end{aligned}$$

If $c_2 \neq c_1$, then $\det_3(\mathbf{D}') \neq 0$.

If q is even and $c_2 = c_1$, then $\det_2(\mathbf{D}') = d_2^2 - d_1^2 = (d_2 - d_1)^2$, so $\det_2(\mathbf{D}') = 0$ if and only if $d_2 = d_1$. Therefore, $\varphi_1 \neq \varphi_2$ implies $\ell'\varphi_1 \neq \ell'\varphi_2$. It means that $\mathcal{O}_{\ell'} = \mathbb{O}_{\text{Un}\Gamma_0}$ and $\#\mathcal{O}_{\ell'} = \#G_q^{P_0} = q(q-1)$, see Corollary 6.5. Then, by Lemma 6.11, $\mathcal{O}_{\ell'}G_q = \mathcal{O}_{\text{UR}}$ and by Lemma 6.9, $\#\mathcal{O}_{\ell'}G_q = q(q-1)(q+1)$.

Let q be odd.

If $c_2 = c_1$, then $\det_2(\mathbf{D}') = (d_2 - d_1)(d_2 + d_1)$, so $\det_2(\mathbf{D}') = 0$ if $d_1 = -d_2$. In this case also $\det_1(\mathbf{D}') = 0$. Therefore, given $\varphi_1 \in G_q^{P_0}$, if and only if we take $\varphi_2 \in G_q^{P_0}$ with $c_2 = c_1, d_2 = -d_1$, then $\varphi_1 \neq \varphi_2$ and $\ell'\varphi_1 = \ell'\varphi_2$. It means that $\#\mathcal{O}_{\ell'} = \frac{1}{2}q(q-1)$, see (6.1).

Consider $P'' = \mathbf{P}(1, 0, \rho, 0)$, ρ is not a square, and $\ell'' = \overline{P''P_0} \notin \pi_{\text{osc}}(0)$. Also, ℓ'' is not a real chord, as ℓ'' has equation $\rho x_0 = x_2, x_1 = 0$ and $\mathcal{C} \cap \ell'' = P_0$. Thus, ℓ'' is a $\text{Un}\Gamma$ -line. Also $\ell'' \notin \mathcal{O}_{\ell'}$. In fact, if $\ell'' \in \mathcal{O}_{\ell'}$, then $\varphi \in G_q^{P_0}$ such that $P_0, P'\varphi, P''$ are collinear would exist. It means that the matrix $\mathbf{D}_{\varphi} = [P_0, P'\varphi, P'']^{tr}$ should have rank 2. As $P'\varphi = \mathbf{P}(0, c, c^2 + d^2, c^3 + 3cd^2)$, we have

$$\det_1(\mathbf{D}_{\varphi}) = -\rho c, \quad \det_2(\mathbf{D}_{\varphi}) = c^2 + d^2 - \rho, \quad \det_3(\mathbf{D}_{\varphi}) = c, \quad \det_4(\mathbf{D}_{\varphi}) = 0.$$

Thus, $\det_3(\mathbf{D}_{\varphi}) = 0$ implies $c = 0$. Then $\det_2(\mathbf{D}_{\varphi}) = d^2 - \rho$, that cannot be equal to 0 as ρ is not a square; contradiction.

Let $\mathcal{O}_{\ell''} = \{\ell''\varphi | \varphi \in G_q^{P_0}\}$. Let $\varphi_1, \varphi_2 \in G_q^{P_0}, \varphi_1 \neq \varphi_2, Q'' = P''\varphi_1, R'' = P''\varphi_2$. By (6.1),

$$Q'' = \mathbf{P}(1, c_1, c_1^2 + \rho d_1^2, c_1^3 + 3\rho c_1 d_1^2), R'' = \mathbf{P}(1, c_2, c_2^2 + \rho d_2^2, c_2^3 + 3\rho c_2 d_2^2).$$

Obviously, $\ell''\varphi_1 \neq \ell''\varphi_2$ if and only if P_0, Q'', R'' are not collinear, i.e. the matrix $\mathbf{D}'' = [P_0, Q'', R'']^{tr}$ has the maximum rank. We have

$$\det_1(\mathbf{D}'') = c_1(c_2^2 + \rho d_2^2) - c_2(c_1^2 + \rho d_1^2), \quad \det_2(\mathbf{D}'') = c_2^2 + \rho d_2^2 - (c_1^2 + \rho d_1^2),$$

$$\det_3(\mathbf{D}'') = c_2 - c_1, \quad \det_4(\mathbf{D}'') = 0.$$

If $c_2 \neq c_1$, then $\det_3(\mathbf{D}'') \neq 0$. If $c_2 = c_1$ then $\det_2(\mathbf{D}'') = \rho(d_2 - d_1)(d_2 + d_1)$, so $\det_2(\mathbf{D}'') = 0$ if $d_1 = -d_2$. In this case also $\det_1(\mathbf{D}'') = 0$. Therefore, given $\varphi_1 \in G_q^{P_0}$ if and only if we take $\varphi_2 \in G_q^{P_0}$ with $c_2 = c_1, d_2 = -d_1$ then we obtain $\varphi_1 \neq \varphi_2$ and $\ell''\varphi_1 = \ell'\varphi_2$. It means that $\#\mathcal{O}_{\ell''} = \frac{1}{2}q(q-1)$, see (6.1).

As $\mathcal{O}_{\ell'} \cap \mathcal{O}_{\ell''} = \emptyset$ and $\#\mathcal{O}_{\text{Un}\Gamma_0} = q(q-1)$, $\{\mathcal{O}_{\ell'}, \mathcal{O}_{\ell''}\}$ is a partition of $\mathcal{O}_{\text{Un}\Gamma_0}$. Then, by Lemma 6.11, $\{\mathcal{O}_{\ell'}G_q, \mathcal{O}_{\ell''}G_q\}$ is a partition of $\mathcal{O}_{\text{Un}\Gamma}$. By Lemma 6.9, $\#\mathcal{O}_{\ell'}G_q = \#\mathcal{O}_{\ell''}G_q = \frac{1}{2}q(q-1)(q+1)$. \square

Corollary 6.14. *Let $q \not\equiv 0 \pmod{3}$. In $\text{PG}(3, q)$, for the twisted cubic \mathcal{C} of (2.2), the external lines in a Γ -plane (class $\mathcal{O}'_5 = \mathcal{O}_{\text{E}\Gamma}$) form an orbit under G_q if q is even and two orbits of size $(q^3 - q)/2$ if q is odd. Moreover, for q odd, the two orbits can be represented in the form $\{\ell_j\varphi \mid \varphi \in G_q\}$, $j = 1, 2$, where $\ell_j = \mathfrak{p}_0 \cap \mathfrak{p}_j$ is the intersection line of planes \mathfrak{p}_0 and \mathfrak{p}_j such that $\mathfrak{p}_0 = \pi(1, 0, 0, 0) = \pi_{\text{osc}}(0)$, $\mathfrak{p}_1 = \pi(0, -3, 0, -1)$, $\mathfrak{p}_2 = \pi(0, -3\rho, 0, -1)$, ρ is not a square, cf. (2.3), (2.4).*

Proof. The assertion follows from Theorems 2.2(iv)(a), 4.3, and 6.13. The null polarity \mathfrak{A} (2.5) maps the points $P_0 = \mathbf{P}(0, 0, 0, 1)$, $P' = \mathbf{P}(1, 0, 1, 0)$, and $P'' = \mathbf{P}(1, 0, \rho, 0)$ of Proof of Theorem 6.13 to the planes $\mathfrak{p}_0 = \pi(1, 0, 0, 0)$, $\mathfrak{p}_1 = \pi(0, -3, 0, -1)$, and $\mathfrak{p}_2 = \pi(0, -3\rho, 0, -1)$, respectively. The $\text{Un}\Gamma$ -lines $\ell' = \overline{P_0P'}$ and $\ell'' = \overline{P_0P''}$ are mapped to EA-lines so that $\ell'\mathfrak{A} = \mathfrak{p}_0 \cap \mathfrak{p}_1 \triangleq \ell_1$ and $\ell''\mathfrak{A} = \mathfrak{p}_0 \cap \mathfrak{p}_2 \triangleq \ell_2$. \square

7 Orbits under G_q of external lines with respect to the cubic \mathcal{C} meeting the axis of the pencil of osculating planes, $q \equiv 0 \pmod{3}$ (orbits of EA-lines)

In the following we consider $q \equiv 0 \pmod{3}$, $q \geq 9$, and denote by ℓ_A the axis of Γ and by P_A the point $\mathbf{P}(0, 1, 0, 0)$. The line ℓ_A is the intersection of the osculating planes, so has equation $x_0 = x_3 = 0$, and $P_A \in \ell_A$. Recall that by Theorem 2.2(iv)(b), ℓ_A is fixed by G_q .

Notation 7.1. In addition to Notations 2.1 and 6.1, the following notation is used.

$$P_t^A \quad \text{the point } \mathbf{P}(0, 1, t, 0) \text{ of } \ell_A \text{ with } t \in \mathbb{F}_q;$$

P_∞^A	the point $\mathbf{P}(0, 0, 1, 0)$ of ℓ_A ;
$G_q^{P_t^A}$	the subgroup of G_q fixing P_t^A with $t \in \mathbb{F}_q^+$;
\mathcal{O}_{EA_i}	the set of lines from \mathcal{O}_{EA} through P_i^A ,
	i.e. $\mathcal{O}_{EA_i} \triangleq \{\ell \in \mathcal{O}_{EA} \mid P_i^A \in \ell\}$.

Lemma 7.2. *Let $q \equiv 0 \pmod{3}$, $q \geq 9$. The group G_q acts transitively on ℓ_A .*

Proof. If we take $\varphi \in G_q$ whose matrix in the form (2.6) has $a = 0, b = c = d = 1$, then $P_0^A \varphi = \mathbf{P}(0, 0, 1, 0) = P_\infty^A$. If we take $\varphi \in G_q$ whose matrix in the form (2.6) has $a = d = 1, b = 0, c = -n$, then $P_0^A \varphi = \mathbf{P}(0, 1, n, 0) = P_n^A$. \square

Lemma 7.3. *The general form of the matrix \mathbf{M} corresponding to a projectivity of $G_q^{P_0^A}$ is as follows:*

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & -bd & d^2 & 0 \\ b^3 & b^2d & bd^2 & d^3 \end{bmatrix}, \quad b \in \mathbb{F}_q, d \in \mathbb{F}_q^*. \quad (7.1)$$

Proof. Let \mathbf{M} be the matrix corresponding to a projectivity $\Psi \in G_q$; by (2.6), $[0, 1, 0, 0] \times \mathbf{M} = [0, a^2d - abc, bc^2 - acd, 0]$. Then $\Psi \in G_q^{P_0^A}$ if and only if

$$bc^2 - acd = 0, \quad a^2d - abc \neq 0. \quad (7.2)$$

If $a = 0$, then $bc^2 = 0$ that implies $\det(\mathbf{M}) = 0$, contradiction, so we can fix $a = 1$. Then the 1-st equality of (7.2) becomes $c(bc - d) = 0$. If $a = 1$ and $bc - d = 0$, also $a^2d - abc = 0$. Therefore, $c = 0, d \neq 0$. \square

Lemma 7.2 allows to prove the following lemmas and corollary in analogous way to Lemmas 6.4, 6.6–6.11, and Corollary 6.5.

Lemma 7.4. *$G_q^{P_i^A}$ and $G_q^{P_j^A}$ are conjugate subgroups of G_q .*

Proof. By Lemma 7.2, G_q acts transitively on ℓ_A , so there exists $\Psi \in G_q$ such that $P_i^A \Psi = P_j^A$. Then $\Psi^{-1} G_q^{P_i^A} \Psi = G_q^{P_j^A}$. In fact, let $\varphi \in G_q^{P_i^A}$. Then $P_j^A \Psi^{-1} \varphi \Psi = P_i^A \varphi \Psi = P_i^A \Psi = P_j^A$. On the other hand, let $\gamma \in G_q^{P_j^A}$. Then $P_i^A \Psi \gamma \Psi^{-1} = P_j^A \gamma \Psi^{-1} = P_j^A \Psi^{-1} = P_i^A$. It means that $\Psi \gamma \Psi^{-1} \in G_q^{P_i^A}$, i.e. $\gamma \in \Psi^{-1} G_q^{P_i^A} \Psi$. \square

Corollary 7.5. For all $t \in \mathbb{F}_q^+$, we have $\#G_q^{P_t^A} = q(q-1)$.

Proof. By (7.1), $\#G_q^{P_0^A} = q(q-1)$. By Lemma 7.4, there exists $\Psi \in G_q$ such that $\Psi^{-1}G_q^{P_0^A}\Psi = G_q^{P_t^A}$, $t \in \mathbb{F}_q^+$. Then $G_q^{P_0^A}\Psi = \Psi G_q^{P_t^A}$. As a finite group and its cosets have the same cardinality, $\#G_q^{P_0^A} = \#G_q^{P_0^A}\Psi = \#\Psi G_q^{P_t^A} = \#G_q^{P_t^A}$. \square

Lemma 7.6. We have $\mathbb{O}_{\text{EA}_i}G_q^{P_i^A} = \mathbb{O}_{\text{EA}_i}$.

Proof. Let $\ell \in \mathbb{O}_{\text{EA}_i}$, $\varphi \in G_q^{P_i^A}$. Then $P_i^A \in \ell$, so $P_i^A\varphi = P_i^A \in \ell\varphi$. As ℓ of type EA implies $\ell\varphi$ of type EA, $\ell\varphi \in \mathbb{O}_{\text{EA}_i}$. On the other hand, if I is the identity element of $G_q^{P_i^A}$, $\mathbb{O}_{\lambda_i}G_q^{P_i^A} \supseteq \mathbb{O}_{\lambda_i}I = \mathbb{O}_{\lambda_i}$. \square

Lemma 7.7. Let $\ell \in \mathbb{O}_{\text{EA}_i}$, $\mathcal{O}_\ell = \{\ell\varphi \mid \varphi \in G_q^{P_i^A}\}$, $\Psi_1, \Psi_2 \in G_q$. If $P_i^A\Psi_1 = P_j^A\Psi_2 = P_j^A$ then $\mathcal{O}_\ell\Psi_1 = \mathcal{O}_\ell\Psi_2$.

Proof. As $P_i^A\Psi_1\Psi_2^{-1} = P_j^A\Psi_2^{-1} = P_i^A$, $\Psi_1\Psi_2^{-1} \in G_q^{P_i^A}$. Let $\bar{\ell} \in \mathcal{O}_\ell\Psi_1$. Then $\bar{\ell} = \ell\varphi\Psi_1$, $\varphi \in G_q^{P_i^A}$. This implies $\bar{\ell}\Psi_2^{-1} = \ell\varphi\Psi_1\Psi_2^{-1} \in \mathcal{O}_\ell$, whence $\bar{\ell} \in \mathcal{O}_\ell\Psi_2$. The proof of the other inclusion is analogous. \square

Lemma 7.8. Let $\ell_1, \ell_2 \in \mathbb{O}_{\text{EA}_i}$, $\mathcal{O}_{\ell_1} = \{\ell_1\varphi \mid \varphi \in G_q^{P_i^A}\}$, $\mathcal{O}_{\ell_2} = \{\ell_2\varphi \mid \varphi \in G_q^{P_i^A}\}$. If $\mathcal{O}_{\ell_1} \cap \mathcal{O}_{\ell_2} = \emptyset$ then $\mathcal{O}_{\ell_1}G_q \cap \mathcal{O}_{\ell_2}G_q = \emptyset$.

Proof. Suppose $\bar{\ell} \in \mathcal{O}_{\ell_1}G_q \cap \mathcal{O}_{\ell_2}G_q$. Then also $\bar{\ell}$ is a line of type EA; let $P_j^A = \ell_A \cap \bar{\ell}$. As $\bar{\ell} \in \mathcal{O}_{\ell_1}G_q$, $\bar{\ell} = \ell_1\varphi_1\Psi_1$, $\varphi_1 \in G_q^{P_i^A}$, $\Psi_1 \in G_q$. Let $\ell' = \ell_1\varphi_1$. As $P_i^A \in \ell_1$, $P_i^A\varphi_1 = P_i^A \in \ell_1\varphi_1 = \ell'$. As $P_j^A \in \bar{\ell} = \ell'\Psi_1$ and $\ell_1, \ell', \bar{\ell}$ are of type EA and $\Psi_1 \in G_q$, $P_i^A\Psi_1$ is the only point of $\bar{\ell}$ belonging to ℓ_A , i.e. $P_i^A\Psi_1 = P_j^A$. Analogously, $\bar{\ell} \in \mathcal{O}_{\ell_2}G_q$ implies $\bar{\ell} = \ell_2\varphi_2\Psi_2$, $\varphi_2 \in G_q^{P_i^A}$, $P_i^A\Psi_2 = P_j^A$. Then $P_i^A\Psi_1\Psi_2^{-1} = P_j^A\Psi_2^{-1} = P_i^A$ that implies $\Psi_1\Psi_2^{-1} \in G_q^{P_i^A}$. Finally, $\ell_1\varphi_1\Psi_1 = \ell_2\varphi_2\Psi_2$ implies $\ell_1\varphi_1\Psi_1\Psi_2^{-1}\varphi_2^{-1} = \ell_2$, whence $\ell_2 \in \mathcal{O}_{\ell_1}$. \square

Lemma 7.9. Let $\ell \in \mathbb{O}_{\text{EA}_i}$, $\mathcal{O}_\ell = \{\ell\varphi \mid \varphi \in G_q^{P_i^A}\}$. Then $\#\mathcal{O}_\ell G_q = (q+1) \cdot \#\mathcal{O}_\ell$.

Proof. Let $G_i^j = \{\varphi \in G_q \mid P_i^A \varphi = P_j^A\}$. The sets $G_i^j, j \in \mathbb{F}_q^+$ form a partition of G_q . In fact, let $\varphi \in G_q$. By Theorem 2.2(iv)(b), the line ℓ_A is fixed by G_q , so $P_i^A \varphi = P_j^A \in \ell_A$: it means that $\varphi \in G_i^j$. On the other hand, if $\varphi \in G_i^j \cap G_i^k$, then $P_j^A = P_i^A \varphi = P_k^A$, so $j = k$. If $\Psi \in G_i^j$, then $\mathcal{O}_\ell \Psi = \mathcal{O}_\ell G_i^j$. In fact, by Lemma 7.7, if $\Psi' \in G_i^j$ then $\mathcal{O}_\ell \Psi' = \mathcal{O}_\ell \Psi$. Finally, consider $\Psi_j \in G_i^j, j \in \mathbb{F}_q^+$. Then $\mathcal{O}_\ell G_q = \bigcup_{j \in \mathbb{F}_q^+} \mathcal{O}_\ell G_i^j = \bigcup_{j \in \mathbb{F}_q^+} \mathcal{O}_\ell \Psi_j$. The sets $\mathcal{O}_\ell \Psi_j, j \in \mathbb{F}_q^+$, are disjoint. In fact, a line $\ell \in \mathcal{O}_\ell \Psi_m \cap \mathcal{O}_\ell \Psi_n, m \neq n$, would be a line of type EA passing through the distinct points $P_m^A, P_n^A \in \ell_A$. Moreover, as Ψ_j is a bijection, $\#\mathcal{O}_\ell \Psi_j = \#\mathcal{O}_\ell$. Therefore, $\#\mathcal{O}_\ell G_q = \sum_{j \in \mathbb{F}_q^+} \#\mathcal{O}_\ell \Psi_j = \sum_{j \in \mathbb{F}_q^+} \#\mathcal{O}_\ell = (q+1) \cdot \#\mathcal{O}_\ell$. \square

Lemma 7.10. *Let $\ell \in \mathbb{O}_{EA}$. Let P_i^A be a point of ℓ_A . Then there exists a line $\bar{\ell} \in \mathbb{O}_{EA_i}$ such that $\ell \in \mathcal{O}_{\bar{\ell}} G_q$, where $\mathcal{O}_{\bar{\ell}} = \{\bar{\ell} \varphi \mid \varphi \in G_q^{P_i^A}\}$.*

Proof. As $\ell \in \mathbb{O}_{EA}$, there exists $P_j^A \in \ell_A$, such that $P_j^A \in \ell$. By Lemma 7.2, G_q acts transitively on ℓ_A , so there exists $\Psi \in G_q$ such that $P_j^A \Psi = P_i^A$. Let $\bar{\ell} = \ell \Psi$. Then $\bar{\ell}$ is of type EA and $P_j^A \in \ell$ implies $P_j^A \Psi = P_i^A \in \ell \Psi = \bar{\ell}$, i.e. $\bar{\ell} \in \mathbb{O}_{EA_i}$. Finally, $\ell = \bar{\ell} \Psi^{-1}$ implies $\ell \in \mathcal{O}_{\bar{\ell}} G_q$. \square

Lemma 7.11. *Let $P_i^A \in \ell_A, \ell^1, \dots, \ell^m, \in \mathbb{O}_{EA_i}, \mathcal{O}_{\ell^j} = \{\ell^j \varphi \mid \varphi \in G_q^{P_i^A}\}, j \in 1, \dots, m$. If $\{\mathcal{O}_{\ell^1}, \dots, \mathcal{O}_{\ell^m}\}$ is a partition of \mathbb{O}_{EA_i} , then $\{\mathcal{O}_{\ell^1} G_q, \dots, \mathcal{O}_{\ell^m} G_q\}$ is a partition of \mathbb{O}_{EA} .*

Proof. Let $\bar{\ell} \in \mathbb{O}_{EA}$. By Lemma 7.10, there exists $\ell' \in \mathbb{O}_{EA_i}$ such that $\bar{\ell} \in \mathcal{O}_{\ell'} G_q, \mathcal{O}_{\ell'} = \{\ell' \varphi \mid \varphi \in G_q^{P_i^A}\}$. By hypothesis, there exists $\ell^{\bar{j}}, \bar{j} \in \{1, \dots, m\}$, such that $\mathcal{O}_{\ell'} = \mathcal{O}_{\ell^{\bar{j}}}$. By Lemma 7.8, $\mathcal{O}_{\ell^j} \neq \mathcal{O}_{\ell^k}, j \neq k$, implies $\mathcal{O}_{\ell^j} G_q \neq \mathcal{O}_{\ell^k} G_q$. \square

Lemma 7.12. *We have $\#\mathbb{O}_{EA_i} = q^2 - 1$.*

Proof. No real cord contains the point P_0^A . In fact, the line $\overline{P_0^A P_\infty}$ has equation $x_2 = x_3 = 0$ and contains no point $P_t, t \in \mathbb{F}$. The points P_0^A, P_{t_1}, P_{t_2} , with $t_1, t_2 \in \mathbb{F}, t_1 \neq t_2$ are collinear if and only if the matrix $\mathbf{M}_{P_0^A, P_{t_1}, P_{t_2}} = [P_0^A, P_{t_1}, P_{t_2}]^{tr}$ has rank 2, but $\det_1(\mathbf{M}_{P_0^A, P_{t_1}, P_{t_2}}) = t_1 - t_2 \neq 0$.

In total $q^2 + q + 1$ lines pass through the point P_0^A . One is ℓ_A , other $q + 1$ are unisecants to \mathcal{C} . Therefore, the remaining $q^2 - 1$ lines are of type EA.

The same holds for every point of ℓ_A . In fact, let $\bar{\ell}$ be a RC-line through a point P_i^A . Then by Lemma 7.2, as G_q acts transitively on ℓ_A , there exists $\Psi \in G_q$ such that $P_i^A \Psi = P_0^A$ and $\bar{\ell} \Psi$ would be an RC-line through P_0^A , contradiction. \square

Theorem 7.13. *For any $q \equiv 0 \pmod{3}$, $q \geq 9$, in $\text{PG}(3, q)$, for the twisted cubic \mathcal{C} of (2.2), the external lines meeting the axis of Γ (i.e. EA-lines, class $\mathcal{O}_8 = \mathcal{O}_{\text{EA}}$) form three orbits under G_q of size $q^3 - q$, $(q^2 - 1)/2$, $(q^2 - 1)/2$. Moreover, the $(q^3 - q)$ -orbit and the two $(q^2 - 1)/2$ -orbits can be represented in the form $\{\ell_1 \varphi | \varphi \in G_q\}$ and $\{\ell_j \varphi | \varphi \in G_q\}$, $j = 2, 3$, respectively, where ℓ_j are lines such that $\ell_1 = \overline{P_0^A \mathbf{P}(0, 0, 1, 1)}$, $\ell_2 = \overline{P_0^A \mathbf{P}(1, 0, 1, 0)}$, $\ell_3 = \overline{P_0^A \mathbf{P}(1, 0, \rho, 0)}$, $P_0^A = \mathbf{P}(0, 1, 0, 0)$, ρ is not a square.*

Proof. Let $\mathcal{O}_{\text{EA}_0} = \{\ell \in \text{EA} | P_0^A \in \ell\}$. By Lemma 7.6, $\mathcal{O}_{\text{EA}_0} G_q^{P_0^A} = \mathcal{O}_{\text{EA}_0}$, so we can consider the orbits of $\mathcal{O}_{\text{EA}_0}$ under the group $G_q^{P_0^A}$. Let $P' = \mathbf{P}(0, 0, 1, 1)$ and $\ell' = \overline{P' P_0^A}$. The line ℓ' has equation $x_0 = 0, x_2 = x_3$, so $\ell' \cap \mathcal{C} = \emptyset$. Let $\mathcal{O}_{\ell'} = \{\ell' \varphi | \varphi \in G_q^{P_0^A}\}$. We find $\#\mathcal{O}_{\ell'}$. Let $\varphi_1, \varphi_2 \in G_q^{P_0^A}$, $\varphi_1 \neq \varphi_2$, $Q' = P' \varphi_1$, $R' = P' \varphi_2$. By (7.1) with $d_1, d_2 \neq 0$,

$$\begin{aligned} Q' &= \mathbf{P}(b_1^3, -b_1 d_1 + b_1^2 d_1, d_1^2 + b_1 d_1^2, d_1^3), \\ R' &= \mathbf{P}(b_2^3, -b_2 d_2 + b_2^2 d_2, d_2^2 + b_2 d_2^2, d_2^3). \end{aligned}$$

Obviously, $\ell' \varphi_1 \neq \ell' \varphi_2$ if and only if P_0^A, Q', R' are not collinear, i.e. if and only if the matrix $\mathbf{D}' = [P_0^A, Q', R']^{tr}$ has maximum rank. Then

$$\begin{aligned} \det_1(\mathbf{D}') &= (d_1^2 + b_1 d_1^2) d_2^3 - (d_2^2 + b_2 d_2^2) d_1^3, \quad \det_2(\mathbf{D}') = 0 \\ \det_3(\mathbf{D}') &= d_1^3 b_2^3 - d_2^3 b_1^3 = (d_1 b_2 - d_2 b_1)^3, \\ \det_4(\mathbf{D}') &= (d_1^2 + b_1 d_1^2) b_2^3 - (d_2^2 + b_2 d_2^2) b_1^3. \end{aligned}$$

If $d_1 b_2 - d_2 b_1 \neq 0$, then $\det_3(\mathbf{D}') \neq 0$. If $b_2 = d_2 b_1 / d_1$, then $\det_1(\mathbf{D}') = d_1^2 d_2^2 (d_1 - d_2)$. Therefore, $\det_1(\mathbf{D}') = 0$ if and only if $d_1 = d_2$ that implies $b_1 = b_2$, i.e. $\varphi_1 = \varphi_2$. Therefore, $\#\mathcal{O}_{\ell'} = \#G_q^{P_0^A} = q(q - 1)$.

Now, let $P'' = \mathbf{P}(1, 0, 1, 0)$, $\ell'' = \overline{P_0^A P''}$, $P''' = \mathbf{P}(1, 0, \rho, 0)$, ρ not a square in \mathbb{F}_q , $\ell''' = \overline{P_0^A P'''}$. As ℓ'' has equation $x_3 = 0, x_0 = x_2$, and ℓ''' has equation $x_3 = 0, \rho x_0 = x_2$, no point of \mathcal{C} belongs to ℓ'', ℓ''' and $\ell'', \ell''' \in \text{EA}$. Moreover, $\ell'', \ell''' \notin \mathcal{O}_{\ell'}$. In fact, let $P = \mathbf{P}(1, 0, s, 0)$, $s \neq 0$, $\ell = \overline{P_0^A P}$; if $\ell \in \mathcal{O}_{\ell'}$, $\varphi \in G_q^{P_0^A}$ such that $P_0^A, P' \varphi, P$ are collinear would exist. It means

that the matrix $\mathbf{D}'_\varphi = [P_0^A, P'\varphi, P]^{tr}$ should have rank 2, but as $P'\varphi = \mathbf{P}(b^3, -bd + b^2d, d^2 + bd^2, d^3)$ with $d \neq 0$, $\det_1(\mathbf{D}'_\varphi) = -sd^3 \neq 0$.

Let $\mathcal{O}_{\ell''} = \{\ell''\varphi | \varphi \in G_q^{P_0^A}\}$. We find $\#\mathcal{O}_{\ell''}$. Let $\varphi_1, \varphi_2 \in G_q^{P_0^A}$, $\varphi_1 \neq \varphi_2$, $Q'' = P''\varphi_1$, $R'' = P''\varphi_2$. By (7.1) with $d_1, d_2 \neq 0$,

$$Q'' = \mathbf{P}(1, -b_1d_1, d_1^2, 0), \quad R'' = \mathbf{P}(1, -b_2d_2, d_2^2, 0).$$

Obviously, $\ell''\varphi_1 \neq \ell''\varphi_2$ if and only if P_0^A, Q'', R'' are not collinear, i.e. if and only if the matrix $\mathbf{D}'' = [P_0^A, Q'', R'']^{tr}$ has maximum rank. We have

$$\det_1(\mathbf{D}'') = \det_2(\mathbf{D}'') = \det_3(\mathbf{D}'') = 0, \quad \det_4(\mathbf{D}'') = (d_1 + d_2)(d_1 - d_2).$$

If $d_1 = d_2$, $\det_4(\mathbf{D}'') = 0 \forall b$; if $d_1 = -d_2$, $\det_4(\mathbf{D}'') = 0 \forall b$. It means that $\#\mathcal{O}_{\ell''} = \frac{1}{2}(q-1)$. It holds $\ell''' \notin \mathcal{O}_{\ell''}$. In fact, if $\ell''' \in \mathcal{O}_{\ell''}$, $\varphi \in G_q^{P_0^A}$ such that $P_0^A, P''\varphi, P'''$ are collinear would exist. It means that the matrix $\mathbf{D}''_\varphi = [P_0^A, P''\varphi, P''']^{tr}$ should have rank 2, but as $P''\varphi = \mathbf{P}(1, -bd, d^2, 0)$, we have $\det_4(\mathbf{D}''_\varphi) = d^2 - \rho \neq 0$ as ρ is not a square.

Let $\mathcal{O}_{\ell'''} = \{\ell'''\varphi | \varphi \in G_q^{P_0^A}\}$. We find $\#\mathcal{O}_{\ell'''}$. Let $\varphi_1, \varphi_2 \in G_q^{P_0^A}$, $\varphi_1 \neq \varphi_2$, $Q''' = P'''\varphi_1$, $R''' = P'''\varphi_2$. By (7.1) with $d_1, d_2 \neq 0$, we have

$$Q''' = \mathbf{P}(1, -\rho b_1d_1, \rho d_1^2, 0), \quad R''' = \mathbf{P}(1, -\rho b_2d_2, \rho d_2^2, 0).$$

Obviously, $\ell'''\varphi_1 \neq \ell'''\varphi_2$ if and only if P_0^A, Q''', R''' are not collinear, i.e. if and only if the matrix $\mathbf{D}''' = [P_0^A, Q''', R''']^{tr}$ has maximum rank. We have

$$\det_1(\mathbf{D}''') = \det_2(\mathbf{D}''') = \det_3(\mathbf{D}''') = 0, \quad \det_4(\mathbf{D}''') = \rho(d_1 + d_2)(d_1 - d_2).$$

If $d_1 = d_2$, $\det_4(\mathbf{D}''') = 0 \forall b$; if $d_1 = -d_2$, $\det_4(\mathbf{D}''') = 0 \forall b$. It means that $\#\mathcal{O}_{\ell'''} = \frac{1}{2}(q-1)$. As $\mathcal{O}_{\ell'}$, $\mathcal{O}_{\ell''}$, $\mathcal{O}_{\ell'''}$ are pairwise disjoint and by Lemma 7.12 $\#\mathbb{O}_{\text{EA}_0} = q(q-1)$, $\{\mathcal{O}_{\ell'}, \mathcal{O}_{\ell''}, \mathcal{O}_{\ell'''}\}$ is a partition of \mathbb{O}_{EA_0} . Then, by Lemma 7.11, $\{\mathcal{O}_{\ell'}G_q, \mathcal{O}_{\ell''}G_q, \mathcal{O}_{\ell'''}G_q\}$ is a partition of \mathcal{O}_{EA} . By Lemma 7.9, $\#\mathcal{O}_{\ell'}G_q = q(q-1)(q+1)$, $\#\mathcal{O}_{\ell''}G_q = \#\mathcal{O}_{\ell'''}G_q = \frac{1}{2}(q-1)(q+1)$. \square

8 Open problems for $\text{En}\Gamma$ -lines and their solutions for $5 \leq q \leq 37$ and $q = 64$

We introduce sets $Q_\bullet^{(\epsilon)}$ of q values with the natural subscripts ‘‘od’’ and ‘‘ev’’.

$$Q_{\text{od}}^{(0)} = \{9, 27\}, \quad Q_{\text{od}}^{(1)} = \{7, 13, 19, 25, 31, 37\}, \quad Q_{\text{od}}^{(-1)} = \{5, 11, 17, 23, 29\};$$

$$Q_{\text{ev}} = \{8, 16, 32, 64\}.$$

Theorem 8.1 has been proved by an exhaustive computer search using the symbol calculation system Magma [4].

Theorem 8.1. *For $q \in Q_{\text{od}}^{(1)} \cup Q_{\text{od}}^{(-1)} \cup Q_{\text{od}}^{(0)}$ and $q \in Q_{\text{ev}}$, all the results of Sections 3–7 are confirmed by computer search. In addition, the following holds, see Notation 2.1.*

(i) *Let $q \equiv \xi \pmod{3}$, $\xi \in \{1, -1, 0\}$. Let $q \in Q_{\text{od}}^{(1)} \cup Q_{\text{od}}^{(-1)} \cup Q_{\text{od}}^{(0)}$ be odd. Then we have the following:*

The total number of $\text{En}\Gamma$ -line orbits is $L_{\text{En}\Gamma\Sigma}^{(\xi)\text{od}} = 2q - 3 + \xi$.

The total number of line orbits in $\text{PG}(3, q)$ is $L_{\Sigma}^{(\xi)} = 2q + 7 + \xi$.

Under G_q , for $\text{En}\Gamma$ -lines with $\xi \in \{1, -1, 0\}$, there are

$(2q - 6 - 4.5\xi^2 - 0.5\xi)/3$	<i>orbits of length</i>	$(q^3 - q)/4$,
$q - 1$	<i>orbits of length</i>	$(q^3 - q)/2$,
$(q - \xi)/3$	<i>orbits of length</i>	$q^3 - q$.

In addition, for $q \in Q_{\text{od}}^{(1)}$, there are

- 1 *orbit of length* $(q^3 - q)/12$,
- 2 *orbits of length* $(q^3 - q)/3$.

(ii) *Let $q \equiv \xi \pmod{3}$, $\xi \in \{1, -1\}$. Let $q \in Q_{\text{ev}}$ be even. Then we have the following:*

The total number of $\text{En}\Gamma$ -line orbits is $L_{\text{En}\Gamma\Sigma}^{(\xi)\text{ev}} = 2q - 2 + \xi$.

The total number of line orbits in $\text{PG}(3, q)$ is $L_{\Sigma}^{(\xi)} = 2q + 7 + \xi$.

Under G_q , for $\text{En}\Gamma$ -lines, there are

- $2 + \xi$ *orbits of length* $(q^3 - q)/(2 + \xi)$;
- $2q - 4$ *orbits of length* $(q^3 - q)/2$.

Conjecture 8.2. *The results of Theorem 8.1 hold for all $q \geq 5$ with the corresponding parity and ξ value.*

Open problems. Find the number, sizes and the structures of orbits of the class $\mathcal{O}_6 = \mathcal{O}_{\text{En}\Gamma}$ (i.e. external lines, other than chord, not in a Γ -plane). Prove the corresponding results of Theorem 8.1 for all $q \geq 5$.

Acknowledgments

The research of S. Marcugini, and F. Pambianco was supported in part by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INDAM) and by University of Perugia (Project: Curve, codici e configurazioni di punti, Base Research Fund 2018).

References

- [1] S. Ball, M. Lavrauw, Arcs in finite projective spaces, EMS Surv. Math. Sci. 6(1/2) (2019) 133–172, [https : //dx.doi.org/10.4171/EMSS/33](https://dx.doi.org/10.4171/EMSS/33).
- [2] D. Bartoli, A.A. Davydov, S. Marcugini, F. Pambianco, On planes through points off the twisted cubic in $\text{PG}(3,q)$ and multiple covering codes, Finite Fields Appl. 67, Oct. 2020, paper 101710, [https : //doi.org/10.1016/j.ffa.2020.101710](https://doi.org/10.1016/j.ffa.2020.101710).
- [3] G. Bonoli, O. Polverino, The twisted cubic in $\text{PG}(3, q)$ and translation spreads in $H(q)$, Discrete Math. 296 (2005) 129–142, [https : //doi.org/10.1016/j.disc.2005.03.010](https://doi.org/10.1016/j.disc.2005.03.010).
- [4] W. Bosma, J. Cannon, C. Playoust, The Magma Algebra System. I. The User Language, J. Symbolic Comput. 24 (1997) 235–265, [https : //doi.org/10.1006/jsc.1996.0125](https://doi.org/10.1006/jsc.1996.0125).
- [5] A.A. Bruen and J.W.P. Hirschfeld, Applications of line geometry over finite fields I: The twisted cubic, Geom. Dedicata 6 (1977) 495–509, [https : //doi.org/10.1007/BF00147786](https://doi.org/10.1007/BF00147786).
- [6] I. Cardinali, G. Lunardon, O. Polverino, R. Trombetti, Spreads in $H(q)$ and 1-systems of $Q(6, q)$, European J. Combin. 23 (2002) 367–376, [https : //dx.doi.org/10.1006/eujc.2001.0578](https://dx.doi.org/10.1006/eujc.2001.0578).
- [7] L.R.A. Casse, Projective Geometry: An Introduction, Oxford Univ. Press, New-York, 2006.
- [8] L.R.A. Casse and D.G. Glynn, The solution to Beniamino Segre’s problem $I_{r,q}$, $r = 3$, $q = 2^h$, Geom. Dedicata 13 (1982) 157–163, [https : //doi.org/10.1007/BF00147659](https://doi.org/10.1007/BF00147659).

- [9] L.R.A. Casse, D.G. Glynn, On the uniqueness of $(q+1)_4$ -arcs of $\text{PG}(4, q)$, $q = 2^h$, $h \geq 3$, *Discrete Math.* **48**(2-3) (1984) 173–186,
[https://doi.org/10.1016/0012-365X\(84\)90180-8](https://doi.org/10.1016/0012-365X(84)90180-8).
- [10] A. Cossidente, J.W.P. Hirschfeld, L. Storme, Applications of line geometry, III: The quadric Veronesean and the chords of a twisted cubic, *Austral. J. Combin.* **16** (1997) 99–111,
<https://ajc.maths.uq.edu.au/pdf/16/ocr-ajc-v16-p99.pdf>.
- [11] A.A. Davydov, S. Marcugini, F. Pambianco, Twisted cubic and plane-line incidence matrix in $\text{PG}(3, q)$, arXiv:2103.11248 [math.CO] (2021),
<https://arxiv.org/abs/2103.11248>.
- [12] M. Giulietti, R. Vincenti, Three-level secret sharing schemes from the twisted cubic, *Discrete Math.* **310** (2010) 3236–3240,
<https://dx.doi.org/10.1016/j.disc.2009.11.040>.
- [13] J.W.P. Hirschfeld, *Projective Geometries over Finite Fields*, 2nd edition, Oxford University Press, Oxford, 1999.
- [14] J.W.P. Hirschfeld, *Finite Projective Spaces of Three Dimensions*, Oxford University Press, Oxford, 1985.
- [15] J.W.P. Hirschfeld, L. Storme, The packing problem in statistics, coding Theory and finite projective spaces: Update 2001, in: A. Blokhuis, J.W.P. Hirschfeld, D. Jungnickel, J.A. Thas (Eds.), *Finite Geometries (Proc. 4th Isle of Thorns Conf., July 16-21, 2000)*, *Dev. Math.*, vol. 3, Dordrecht: Kluwer, 2001, pp. 201–246,
https://dx.doi.org/10.1007/978-1-4613-0283-4_13.
- [16] J.W.P. Hirschfeld, J.A. Thas, Open problems in finite projective spaces, *Finite Fields Appl.* **32** (2015) 44–81,
<https://dx.doi.org/10.1016/j.ffa.2014.10.006>.
- [17] G. Lunardon, O. Polverino, On the Twisted Cubic of $\text{PG}(3, q)$, *J. Algebr. Combin.* **18** (2003) 255–262,
<https://dx.doi.org/10.1023/B:JACO.0000011940.77655.b4>.
- [18] Maple 16. Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario. <https://www.maplesoft.com/products/maple/>.

- [19] M. Zannetti, F. Zuanni, Note on three-character $(q+1)$ -sets in $\text{PG}(3, q)$,
Austral. J. Combin. 47 (2010) 37–40,
[https : //ajc.maths.uq.edu.au/pdf/47/ajc_v47_p037.pdf](https://ajc.maths.uq.edu.au/pdf/47/ajc_v47_p037.pdf).