

Twisted cubic and point-line incidence matrix in $\text{PG}(3, q)$

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Abstract

We consider the structure of the point-line incidence matrix of the projective space $\text{PG}(3, q)$ connected with orbits of points and lines under the stabilizer group of the twisted cubic. Structures of submatrices with incidences between a union of line orbits and an orbit of points are investigated. For the unions consisting of two or three line orbits, the original submatrices are split into new ones, in which the incidences are also considered. For each submatrix (apart from the ones corresponding to a special type of lines), the numbers of lines through every point and of points lying on every line are obtained. This corresponds to the numbers of ones in columns and rows of the submatrices.

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1 Introduction

Let \mathbb{F}_q be the Galois field with q elements. Let $\text{PG}(N, q)$ be the N -dimensional projective space over \mathbb{F}_q . An n -arc in $\text{PG}(N, q)$, with $n \geq N + 1 \geq 3$, is a set of n points such that no $N + 1$ points belong to the same hyperplane of $\text{PG}(N, q)$, see [1] and the references therein. For an introduction to projective geometry over finite fields see [20, 22, 23].

In $\text{PG}(N, q)$, $2 \leq N \leq q - 2$, a normal rational curve is a $(q + 1)$ -arc projectively equivalent to the arc $\{(t^N, t^{N-1}, \dots, t^2, t, 1) : t \in \mathbb{F}_q\} \cup \{(1, 0, \dots, 0)\}$. In $\text{PG}(3, q)$, the normal rational curve is called a *twisted cubic* [21, 23].

The twisted cubic has many interesting properties and is connected with distinct combinatorial and applied problems, see e.g. [3–5, 7–11, 14–18, 21–23, 26, 28] and the references therein. In particular, using properties of the twisted cubic, spreads in $\text{PG}(3, q)$ are studied [7, 8, 26], optimal multiple covering codes are constructed [3], the weight distributions of cosets and their leaders for the Reed-Solomon codes are obtained [4, 14], the three-level secret sharing schemes are considered [17].

In investigations of the twisted cubic, an important direction is to determine the matrices of the incidences between points, planes, and lines partitioned into orbits under the group G_q fixing the cubic. The orbits of planes and points are known and described in detail [21]. The *point-plane* incidence matrix of $\text{PG}(3, q)$ for all $q \geq 2$ is given in [3] where the numbers of distinct planes through distinct points and, conversely, the numbers of distinct points lying in distinct planes are obtained. (By “distinct planes” we mean “planes from distinct orbits”, and similarly for points and lines.)

For plane-line and point-line incidence matrices a description of line orbits is needed. In [21], the lines in $\text{PG}(3, q)$ are partitioned into classes, each of which is a union of line orbits under G_q ; see Section 2.2. Apart from one class (which is denoted by \mathcal{O}_6), the number and the structure of the orbits forming those unions are independently considered by distinct methods in [16, Sections 3, 8] (for all $q \geq 2$), [4, Section 7] (for all $q \geq 23$), and [18] (for finite fields of characteristic > 3); see also the references therein.

The classification of the line orbits in the class \mathcal{O}_6 is an open problem.

The results on line orbits from [4, 16, 18] are in accordance with each other. The representation and description of the orbits in these papers are distinct; in particular, in [16], the orbits are given in a form which is convenient for the investigations in [15]. More precisely, using the representation of the line orbits in [16], the *plane-line* incidence matrix of $\text{PG}(3, q)$ is given in [15] where, apart from \mathcal{O}_6 , for all $q \geq 2$, the numbers of distinct planes through distinct lines and, vice versa, the numbers of distinct lines lying in distinct planes are obtained. For \mathcal{O}_6 , the corresponding average values are calculated.

In [18], apart from \mathcal{O}_6 , for odd $q \not\equiv 0 \pmod{3}$ the numbers of distinct planes through distinct lines (called “the plane orbit distribution of a line”) and the numbers of distinct points lying on distinct lines (called “the point orbit distribution of a line”) are obtained. For finite fields of characteristic > 3 , the results of [18] on “the plane orbit distribution

of a line” are in accordance with those from [15] on plane-line incidence matrix.

The results of [18] on “the point orbit distribution of a line” are an important step towards the point-line incidence matrix. However, these results are obtained only for odd $q \not\equiv 0 \pmod{3}$ and the computation of the numbers of distinct lines through distinct points has been left open.

In this paper, we obtain the *point-plane* incidence matrix for all $q \geq 2$, leaving open the questions related to \mathcal{O}_6 .

We consider the structure of the point-line incidence matrix with respect to G_q . We use the partitions of planes and lines into orbits and unions of orbits under the group G_q , as described in [16, 21]. We search the structures of the submatrices with incidences between an orbit of points and a union of line orbits. For the unions consisting of two or three line orbits, the original submatrices are split into new ones, in which the incidences are also considered. For each submatrix (apart from the ones related to \mathcal{O}_6), the numbers of distinct points lying on distinct lines and, conversely, the numbers of distinct lines through distinct points are obtained. This corresponds to the numbers of ones in columns and rows of the submatrices.

The results noted are obtained for all $q \geq 2$ including even q and $q \equiv 0 \pmod{3}$. Thus, the gaps of [18] in the point-line incidence matrix are filled.

For \mathcal{O}_6 , some average and cumulative values are calculated.

Many submatrices considered are configurations in the sense of [19], see Definition 2 in Section 2.4. Such configurations are useful in several distinct areas, in particular, to construct bipartite graph codes without the so-called 4-cycles, see e.g. [2, 13, 24] and the references therein.

The paper is organized as follows. Section 2 contains preliminaries. In Section 3, the main results of this paper are summarized. Some useful relations are given in Section 4. The numbers of distinct points lying on distinct lines and, vice versa, the numbers of distinct lines through distinct points are obtained in Sections 5 (for even and odd $q \not\equiv 0 \pmod{3}$) and 6 (for $q \equiv 0 \pmod{3}$). Some general results are given in Section 7.

2 Preliminaries

Throughout the paper, we consider orbits of lines and points under G_q apart from Theorem 6 in Section 3.

2.1 Twisted cubic

In this subsection, including Theorem 1, we summarize some results from [21] useful in this paper.

The space $\text{PG}(N, q)$ contains $\theta_{N,q}$ points and hyperplanes, and $\beta_{N,q}$ lines;

$$\theta_{N,q} = \frac{q^{N+1} - 1}{q - 1}, \quad \beta_{N,q} = \frac{(q^{N+1} - 1)(q^{N+1} - q)}{(q^2 - 1)(q^2 - q)}. \quad (2.1)$$

Let $\pi(c_0, c_1, c_2, c_3)$ be the plane of $\text{PG}(3, q)$ with equation

$$c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3 = 0, \quad c_i \in \mathbb{F}_q. \quad (2.2)$$

We denote $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, $\mathbb{F}_q^+ = \mathbb{F}_q \cup \{\infty\}$. Let $\mathbf{P}(x_0, x_1, x_2, x_3) \in \text{PG}(3, q)$ be a point with homogeneous coordinates $x_i \in \mathbb{F}_q$. Let $P(t)$ be a point with

$$t \in \mathbb{F}_q^+; \quad P(t) = \mathbf{P}(t^3, t^2, t, 1) \text{ if } t \in \mathbb{F}_q; \quad P(\infty) = \mathbf{P}(1, 0, 0, 0). \quad (2.3)$$

Let $\mathcal{C} \subset \text{PG}(3, q)$ be the *twisted cubic* in the canonical form

$$\mathcal{C} = \{P_1, P_2, \dots, P_{q+1}\} = \{P(t) \mid t \in \mathbb{F}_q^+\}. \quad (2.4)$$

where P_1, \dots, P_{q+1} are points no four of which are coplanar.

The *osculating plane* $\pi_{\text{osc}}(t)$ at the point $P(t) \in \mathcal{C}$ has the form

$$\pi_{\text{osc}}(t) = \pi(1, -3t, 3t^2, -t^3) \text{ if } t \in \mathbb{F}_q; \quad \pi_{\text{osc}}(\infty) = \pi(0, 0, 0, 1). \quad (2.5)$$

The $q + 1$ osculating planes form the *osculating developable* Γ to \mathcal{C} . For $q \equiv 0 \pmod{3}$, the osculating developable is a *pencil of planes*.

Definition 1. (i) A *chord* of \mathcal{C} is a line through a pair of real points of \mathcal{C} or a pair of complex conjugate points. If the real points coincide with each other, the chord is a *tangent* to \mathcal{C} ; if they are distinct, we have a *real chord*. For a pair of complex conjugate points, we have an *imaginary chord*.

(ii) An *axis* of Γ is a line of $\text{PG}(3, q)$ which is the intersection of a pair of real planes or complex conjugate planes of Γ . If the real planes coincide with each other, the axis is a *tangent* to \mathcal{C} ; if they are distinct it is a *real axis*. For complex conjugate planes, we have an *imaginary axis*.

The null polarity \mathfrak{A} [20, Sections 2.1.5, 5.3], [21, Theorem 21.1.2] is given by

$$\mathbf{P}(x_0, x_1, x_2, x_3)\mathfrak{A} = \pi(x_3, -3x_2, 3x_1, -x_0), \quad q \not\equiv 0 \pmod{3}. \quad (2.6)$$

Notation 1 We consider $q \equiv \xi \pmod{3}$, $\xi \in \{-1, 0, 1\}$. Values depending of ξ are noted by remarks or by superscripts “ (ξ) ”. The remarks and superscripts “ (ξ) ” are not used if a value is the same for all q or a property holds for all q , or it is not relevant, or it is clear by the context. If a value is the same for $\xi = -1, 1$, then one may use the superscript “ $\neq 0$ ”. Also, in superscripts, instead of “ \bullet ”, one can write “ev” for even q or “od” for odd q . If a value is the same for even and odd q , we may omit “ \bullet ”.

The following notation is used.

G_q the group of projectivities in $\text{PG}(3, q)$ fixing \mathcal{C} ;

\mathbf{Z}_n	cyclic group of order n ;
\mathbf{S}_n	symmetric group of degree n ;
A^{tr}	the transposed matrix of A ;
$\#S$	the cardinality of a set S ;
\overline{AB}	the line through the points A and B ;
\triangleq	the sign “equality by definition”.
	Types π of planes:
Γ -plane	an osculating plane of Γ ;
$d_{\mathcal{C}}$ -plane	a plane containing <i>exactly</i> d distinct points of \mathcal{C} , $d = 0, 2, 3$;
$\overline{1_{\mathcal{C}}}$ -plane	a plane not in Γ containing <i>exactly</i> 1 point of \mathcal{C} ;
\mathfrak{P}	the list of possible types π of planes, $\mathfrak{P} \triangleq \{\Gamma, 2_{\mathcal{C}}, 3_{\mathcal{C}}, \overline{1_{\mathcal{C}}}, 0_{\mathcal{C}}\}$;
π -plane	a plane of type $\pi \in \mathfrak{P}$;
\mathcal{N}_{π}	the orbit of π -planes under G_q , $\pi \in \mathfrak{P}$.
	Types \mathfrak{p} of points with respect to the twisted cubic \mathcal{C}:
\mathcal{C} -point	a point of \mathcal{C} ;
μ_{Γ} -point	a point off \mathcal{C} lying on <i>exactly</i> μ distinct osculating planes;
T-point	a point off \mathcal{C} on a tangent to \mathcal{C} for $\xi \neq 0$;
TO-point	a point off \mathcal{C} on a tangent and one osculating plane for $\xi = 0$;
RC-point	a point off \mathcal{C} on a real chord;
IC-point	a point on an imaginary chord (it always is off \mathcal{C});
$\mathfrak{M}^{(\xi)}$	the list of possible types \mathfrak{p} of points, $\mathfrak{M}^{(\neq 0)} \triangleq \{\mathcal{C}, 0_{\Gamma}, 1_{\Gamma}, 3_{\Gamma}, T, RC, IC\}$, $\mathfrak{M}^{(0)} \triangleq \{\mathcal{C}, (q+1)_{\Gamma}, TO, RC, IC\}$;
$\mathcal{M}_{\mathfrak{p}}^{(\xi)}$	the orbit of \mathfrak{p} -points under G_q , $\mathfrak{p} \in \mathfrak{M}^{(\xi)}$.
	Types λ of lines with respect to the twisted cubic \mathcal{C}:
RC-line	a real chord of \mathcal{C} ;
RA-line	a real axis of Γ for $\xi \neq 0$;
T-line	a tangent to \mathcal{C} ;
IC-line	an imaginary chord of \mathcal{C} ;
IA-line	an imaginary axis of Γ for $\xi \neq 0$;
U Γ	a non-tangent unisecant in a Γ -plane;
Un Γ -line	a unisecant not in a Γ -plane (it is always non-tangent);

EF-line	an external line in a Γ -plane (it cannot be a chord);
En Γ -line	an external line, other than a chord, not in a Γ -plane;
A-line	the axis of the pencil of Γ -planes for $\xi = 0$;
EA-line	an external line meeting the axis of Γ for $\xi = 0$;
$\mathfrak{L}^{(\xi)}$	the list of possible types λ of lines, $\mathfrak{L}^{(\neq 0)} \triangleq \{\text{RC, RA, T, IC, IA, U}\Gamma, \text{Un}\Gamma, \text{E}\Gamma, \text{En}\Gamma\}$ for $\xi \neq 0$, $\mathfrak{L}^{(0)} \triangleq \{\text{RC, T, IC, U}\Gamma, \text{Un}\Gamma, \text{En}\Gamma, \text{A, EA}\}$ for $\xi = 0$;
λ -line	a line of type $\lambda \in \mathfrak{L}^{(\xi)}$;
Orbits of lines. Plane-line incidence matrix. $\pi \in \mathfrak{P}, \lambda \in \mathfrak{L}^{(\xi)}$	
$L_{\lambda\Sigma}^{(\xi)\bullet}$	the total number of orbits of λ -lines;
\mathcal{O}_λ	the union (class) of all $L_{\lambda\Sigma}^{(\xi)\bullet}$ orbits of λ -lines;
\mathcal{O}_{λ_j}	the j -th orbit of the class \mathcal{O}_λ , $j = 1, \dots, L_{\lambda\Sigma}^{(\xi)\bullet}$, $\mathcal{O}_\lambda = \bigcup_{j=1}^{L_{\lambda\Sigma}^{(\xi)\bullet}} \mathcal{O}_{\lambda_j}$;
\mathcal{O}_i	the j -th orbit of the class \mathcal{O}_i ;
λ_j -lines	λ -lines forming the j -th orbit \mathcal{O}_{λ_j} of the class \mathcal{O}_λ ;
$\Lambda_{\lambda_j, \pi}^{(\xi)\bullet}$	the number of lines from an orbit \mathcal{O}_{λ_j} in a π -plane;
$\Lambda_{\lambda, \pi}^{(\xi)\bullet}$	the total number of λ -lines in a π -plane;
$\Pi_{\pi, \lambda_j}^{(\xi)\bullet}$	the exact number of π -planes through a line of an orbit \mathcal{O}_{λ_j} ;
$\Pi_{\pi, \lambda}^{(\xi)\bullet}$	the average number of π -planes through a λ -line over all the λ -lines; if the class \mathcal{O}_λ consists of a <i>single orbit</i> then $\Pi_{\pi, \lambda}^{(\xi)\bullet}$ is <i>the exact number</i> of π -planes through each λ -line;
$\mathcal{I}^{\Pi\Lambda}$	the $\beta_{3,q} \times \theta_{3,q}$ plane-line incidence matrix of $\text{PG}(3, q)$;
$\mathcal{I}_{\pi, \lambda}^{\Pi\Lambda}$	the $\#\mathcal{O}_\lambda \times \#\mathcal{N}_\pi$ submatrix of $\mathcal{I}^{\Pi\Lambda}$ with incidences between π -planes and λ -lines;
$\mathcal{I}_{\pi, \lambda_j}^{\Pi\Lambda}$	the $\#\mathcal{O}_{\lambda_j} \times \#\mathcal{N}_\pi$ submatrix of $\mathcal{I}_{\pi, \lambda}^{\Pi\Lambda}$ with incidences between π -planes and λ_j -lines.

Theorem 1. [21, Chapter 21] *The following properties of the twisted cubic \mathcal{C} of (2.4) hold:*

(i) *The group G_q acts triply transitively on \mathcal{C} .*

Also, $G_q \cong \text{PGL}(2, q)$ for $q \geq 5$;

$$G_4 \cong \mathbf{S}_5 \cong PGL(2, 4) \cong \mathbf{Z}_2 PGL(2, 4); \quad G_3 \cong \mathbf{S}_4 \mathbf{Z}_2^3; \quad G_2 \cong \mathbf{S}_3 \mathbf{Z}_2^3.$$

The matrix \mathbf{M} corresponding to a projectivity of G_q has the general form

$$\mathbf{M} = \begin{bmatrix} a^3 & a^2c & ac^2 & c^3 \\ 3a^2b & a^2d + 2abc & bc^2 + 2acd & 3c^2d \\ 3ab^2 & b^2c + 2abd & ad^2 + 2bcd & 3cd^2 \\ b^3 & b^2d & bd^2 & d^3 \end{bmatrix}, \quad a, b, c, d \in \mathbb{F}_q, \quad ad - bc \neq 0. \quad (2.7)$$

(ii) (a) Under G_q , $q \geq 5$, there are the following five orbits \mathcal{N}_j of planes:

$$\begin{aligned} \mathcal{N}_1 &= \mathcal{N}_\Gamma = \{\Gamma\text{-planes}\}, & \#\mathcal{N}_1 &= \#\mathcal{N}_\Gamma = q + 1; \\ \mathcal{N}_2 &= \mathcal{N}_{2_\mathcal{C}} = \{2_\mathcal{C}\text{-planes}\}, & \#\mathcal{N}_2 &= \#\mathcal{N}_{2_\mathcal{C}} = q^2 + q; \\ \mathcal{N}_3 &= \mathcal{N}_{3_\mathcal{C}} = \{3_\mathcal{C}\text{-planes}\}, & \#\mathcal{N}_3 &= \#\mathcal{N}_{3_\mathcal{C}} = (q^3 - q)/6; \\ \mathcal{N}_4 &= \mathcal{N}_{\overline{1_\mathcal{C}}} = \{\overline{1_\mathcal{C}}\text{-planes}\}, & \#\mathcal{N}_4 &= \#\mathcal{N}_{\overline{1_\mathcal{C}}} = (q^3 - q)/2; \\ \mathcal{N}_5 &= \mathcal{N}_{0_\mathcal{C}} = \{0_\mathcal{C}\text{-planes}\}, & \#\mathcal{N}_5 &= \#\mathcal{N}_{0_\mathcal{C}} = (q^3 - q)/3. \end{aligned} \quad (2.8)$$

(b) For $q \not\equiv 0 \pmod{3}$, the five orbits $\mathcal{M}_j^{(\neq 0)}$ of points are as follows:

$$\begin{aligned} \mathcal{M}_1^{(\neq 0)} &= \mathcal{M}_\mathcal{C}^{(\neq 0)} = \{\mathcal{C}\text{-points}\}, & \mathcal{M}_2^{(\neq 0)} &= \mathcal{M}_\Gamma^{(\neq 0)} = \{\Gamma\text{-points}\}, \\ \mathcal{M}_3^{(\neq 0)} &= \mathcal{M}_{3_\Gamma}^{(\neq 0)} = \{3_\Gamma\text{-points}\}, & \mathcal{M}_4^{(\neq 0)} &= \mathcal{M}_{1_\Gamma}^{(\neq 0)} = \{1_\Gamma\text{-points}\}, \\ \mathcal{M}_5^{(\neq 0)} &= \mathcal{M}_{0_\Gamma}^{(\neq 0)} = \{0_\Gamma\text{-points}\}; & \#\mathcal{M}_j^{(\neq 0)} &= \#\mathcal{N}_j, \quad j = 1, \dots, 5. \end{aligned} \quad (2.9)$$

$$\text{For } q \equiv 1 \pmod{3}, \quad \mathcal{M}_{3_\Gamma}^{(1)} \cup \mathcal{M}_{0_\Gamma}^{(1)} = \{\text{RC-points}\}, \quad \mathcal{M}_{1_\Gamma}^{(1)} = \{\text{IC-points}\}; \quad (2.10)$$

$$\text{for } q \equiv -1 \pmod{3}, \quad \mathcal{M}_{3_\Gamma}^{(-1)} \cup \mathcal{M}_{0_\Gamma}^{(-1)} = \{\text{IC-points}\}, \quad \mathcal{M}_{1_\Gamma}^{(-1)} = \{\text{RC-points}\}.$$

(c) For $q \equiv 0 \pmod{3}$, the five orbits $\mathcal{M}_j^{(0)}$ of points are as follows:

$$\begin{aligned} \mathcal{M}_1^{(0)} &= \mathcal{M}_\mathcal{C}^{(0)} = \{\mathcal{C}\text{-points}\}, & \mathcal{M}_2^{(0)} &= \mathcal{M}_{(q+1)_\Gamma}^{(0)} = \{(q+1)_\Gamma\text{-points}\}, \\ \#\mathcal{M}_1^{(0)} &= \#\mathcal{M}_\mathcal{C}^{(0)} = \#\mathcal{M}_2^{(0)} = \#\mathcal{M}_{(q+1)_\Gamma}^{(0)} = q + 1; \\ \mathcal{M}_3^{(0)} &= \mathcal{M}_{\text{TO}}^{(0)} = \{\text{TO-points}\}, & \#\mathcal{M}_3^{(0)} &= \#\mathcal{M}_{\text{TO}}^{(0)} = q^2 - 1; \\ \mathcal{M}_4^{(0)} &= \mathcal{M}_{\text{RC}}^{(0)} = \{\text{RC-points}\}, & \mathcal{M}_5^{(0)} &= \mathcal{M}_{\text{IC}}^{(0)} = \{\text{IC-points}\}, \\ \#\mathcal{M}_4^{(0)} &= \#\mathcal{M}_{\text{RC}}^{(0)} = \#\mathcal{M}_5^{(0)} = \#\mathcal{M}_{\text{IC}}^{(0)} = (q^3 - q)/2. \end{aligned} \quad (2.11)$$

(iii) Let $q \not\equiv 0 \pmod{3}$. The null polarity \mathfrak{A} (2.6) interchanges \mathcal{C} and Γ and their corresponding chords and axes. We have

$$\mathcal{M}_j^{(\neq 0)}\mathfrak{A} = \mathcal{N}_j, \quad j = 1, \dots, 5; \quad \mathcal{M}_\mathcal{C}^{(\neq 0)}\mathfrak{A} = \mathcal{N}_\Gamma, \quad \mathcal{M}_\Gamma^{(\neq 0)}\mathfrak{A} = \mathcal{N}_{2_\mathcal{C}}, \quad (2.12)$$

$$\mathcal{M}_{3\Gamma}^{(\neq 0)}\mathfrak{A} = \mathcal{N}_{3\mathcal{C}}, \quad \mathcal{M}_{1\Gamma}^{(\neq 0)}\mathfrak{A} = \mathcal{N}_{1\mathcal{C}}, \quad \mathcal{M}_{0\Gamma}^{(\neq 0)}\mathfrak{A} = \mathcal{N}_{0\mathcal{C}}.$$

(iv) For all q , no two chords of \mathcal{C} meet off \mathcal{C} . Every point off \mathcal{C} lies on exactly one chord of \mathcal{C} .

(v) Let $q \not\equiv 0 \pmod{3}$. No two axes of Γ meet unless they lie in the same plane of Γ . Every plane not in Γ contains exactly one axis of Γ .

2.2 Orbits of lines under the stabilizer group G_q of the twisted cubic

Theorem 2. [16, Section 8] *Let $q \equiv \xi \pmod{3}$, $\xi \in \{1, -1, 0\}$.*

(i) *Let $5 \leq q \leq 37$ and $q = 64$. Then*

(a) *For the total number $L_{\text{En}\Gamma\Sigma}^{(\xi)\bullet}$ of orbits of $\text{En}\Gamma$ -lines we have*

$$L_{\text{En}\Gamma\Sigma}^{(\xi)\text{od}} = 2q - 3 + \xi, \quad L_{\text{En}\Gamma\Sigma}^{(\xi)\text{ev}} = 2q - 2 + \xi. \quad (2.13)$$

(b) *The total number of line orbits in $\text{PG}(3, q)$ is $2q + 7 + \xi$.*

(ii) *Let q be odd, $5 \leq q \leq 37$. Then under G_q , for $\text{En}\Gamma$ -lines, there are*

$(q - \xi)/3$	<i>orbits of length</i>	$q^3 - q,$
$q - 1$	<i>orbits of length</i>	$(q^3 - q)/2,$
$n_q^{(\xi)}$	<i>orbits of length</i>	$(q^3 - q)/4,$

where $n_q^{(1)} = (2q - 11)/3$, $n_q^{(-1)} = (2q - 10)/3$, $n_q^{(0)} = (2q - 6)/3$.

In addition, for $q \in \{7, 13, 19, 25, 31, 37\}$ where $q \equiv 1 \pmod{3}$, there are one orbit of length $(q^3 - q)/12$ and two orbits of length $(q^3 - q)/3$.

(iii) *Let $q = 8, 16, 32, 64$. Then under G_q , for $\text{En}\Gamma$ -lines, there are*

$2 + \xi$ orbits of length $(q^3 - q)/(2 + \xi)$ and $2q - 4$ orbits of length $(q^3 - q)/2$.

Conjecture 1. [16] *The results of Theorem 2 hold for all $q \geq 5$ with the corresponding parity and ξ value.*

For odd $q \not\equiv 0 \pmod{3}$, the conjecture on (2.13) is given also in [18].

The unions (classes) of line orbits are considered in [21, Chapter 21]; they are called \mathcal{O}_i and $\mathcal{O}'_i = \mathcal{O}_i\mathfrak{A}$. In [16] (for all $q \geq 2$), [4] (for all $q \geq 23$), and [18] (for odd $q \not\equiv 0 \pmod{3}$), these classes (apart from \mathcal{O}_6) are investigated; the sizes and the structures of the orbits forming each class are obtained.

Theorem 3 and Table 1 summarize some results from [4, 16, 18, 21] useful in this paper.

Table 1: Unions (classes) \mathcal{O}_i and $\mathcal{O}'_i = \mathcal{O}_i\mathfrak{A}$ of line orbits under G_q in $\text{PG}(3, q)$, $q \equiv \xi \pmod{3}$, $q \geq 5$. $\mathcal{O}_i = \mathcal{O}'_i$, $i = 2, 4, 6$. $L_{\lambda\Sigma}^{(\xi)\bullet}$ is the total number of orbits in the class \mathcal{O}_λ . $\#\mathcal{O}_{\lambda_j}$ is the size of the j -th orbit of a class \mathcal{O}_λ consisting of 2 or 3 orbits

\mathcal{O}_i	\mathcal{O}_λ	content of the class	size of the class	ξ	$L_{\lambda\Sigma}^{(\xi)\text{od}}$	$L_{\lambda\Sigma}^{(\xi)\text{ev}}$	$\#\mathcal{O}_{\lambda_1}$ $\#\mathcal{O}_{\lambda_2}$ $\#\mathcal{O}_{\lambda_3}$
\mathcal{O}_1	\mathcal{O}_{RC}	RC-lines	$(q^2 + q)/2$	any	1	1	
\mathcal{O}'_1	\mathcal{O}_{RA}	RA-lines	$(q^2 + q)/2$	$\neq 0$	1	1	
\mathcal{O}_2	\mathcal{O}_{T}	T-lines	$q + 1$	any	1	1	
\mathcal{O}_3	\mathcal{O}_{IC}	IC-lines	$(q^2 - q)/2$	any	1	1	
\mathcal{O}'_3	\mathcal{O}_{IA}	IA-lines	$(q^2 - q)/2$	$\neq 0$	1	1	
\mathcal{O}_4	\mathcal{O}_{UF}	UF-lines	$q^2 + q$	any	1	2	$q + 1$ $q^2 - 1$
\mathcal{O}_5	$\mathcal{O}_{\text{Un}\Gamma}$	Un Γ -lines	$q^3 - q$	any	2	1	$(q^3 - q)/2$ $(q^3 - q)/2$
\mathcal{O}'_5	$\mathcal{O}_{\text{E}\Gamma}$	E Γ -lines	$q^3 - q$	$\neq 0$	2	1	$(q^3 - q)/2$ $(q^3 - q)/2$
\mathcal{O}_6	$\mathcal{O}_{\text{En}\Gamma}$	En Γ -lines	$(q^2 - q)(q^2 - 1)$	any	$L_{\text{En}\Gamma\Sigma}^{(\xi)\text{od}}$	$L_{\text{En}\Gamma\Sigma}^{(\xi)\text{ev}}$	
\mathcal{O}_7	\mathcal{O}_{A}	A-line	1	0	1	–	
\mathcal{O}_8	\mathcal{O}_{EA}	EA-lines	$(q + 1)(q^2 - 1)$	0	3	–	$q^3 - q$ $(q^2 - 1)/2$ $(q^2 - 1)/2$

In the last column of Table 1, the sizes of the orbits \mathcal{O}_{λ_j} of a class \mathcal{O}_λ consisting of 2 or 3 orbits are given from top to bottom, e.g. for \mathcal{O}_{EA} we have $\mathcal{O}_{\text{EA}_1} = q^3 - q$, $\mathcal{O}_{\text{EA}_2} = (q^2 - 1)/2$, $\mathcal{O}_{\text{EA}_3} = (q^2 - 1)/2$.

Theorem 3. [4,16,18,21] *Let $q \geq 5$. The lines of $\text{PG}(3, q)$ can be partitioned into classes called \mathcal{O}_i and \mathcal{O}'_i , each of which is a union of orbits under G_q . The classification of the unions (classes) of line orbits is given in Table 1.*

If $q \not\equiv 0 \pmod{3}$ we have

$$\begin{aligned} \mathcal{O}'_i &= \mathcal{O}_i\mathfrak{A}, \quad \#\mathcal{O}'_i = \#\mathcal{O}_i, \quad i = 1, \dots, 6; \quad \mathcal{O}_i = \mathcal{O}'_i, \quad i = 2, 4, 6; \\ \mathcal{O}_{\text{RA}} &= \mathcal{O}_{\text{RC}}\mathfrak{A}, \quad \mathcal{O}_{\text{IA}} = \mathcal{O}_{\text{IC}}\mathfrak{A}, \quad \mathcal{O}_{\text{E}\Gamma} = \mathcal{O}_{\text{Un}\Gamma}\mathfrak{A}, \quad \mathcal{O}_\lambda = \mathcal{O}_\lambda\mathfrak{A}, \quad \lambda \in \{\text{T}, \text{UF}, \text{En}\Gamma\}. \end{aligned} \quad (2.14)$$

In [16, Theorem 3.2], the cases when Table 1 holds for $q = 2, 3, 4$ are noted.

Theorem 4. [16, Theorem 4.3] *Let $q \not\equiv 0 \pmod{3}$. Let \mathcal{L} be an orbit of lines under G_q . Then $\mathcal{L}\mathfrak{A}$ also is an orbit of lines under G_q .*

In [18], the line orbits are denoted by \mathcal{L}_i and $\mathcal{L}_i^\perp = \mathcal{L}_i\mathfrak{A}$, $i = 1, \dots, 10$. We give the correspondence between \mathcal{L}_i and the notations of this paper (in [15, 16] the notations are the same as in this paper).

$$\begin{aligned}
\mathcal{L}_1 &= \mathcal{O}'_1 = \mathcal{O}_{\text{RA}}, \quad \mathcal{L}_2 = \mathcal{O}_2 = \mathcal{O}'_2 = \mathcal{O}_{\text{T}}, \quad \mathcal{L}_3 = \mathcal{O}_4 = \mathcal{O}'_1 = \mathcal{O}_{\text{UF}}, \\
\mathcal{L}_4 &= \mathcal{O}'_{5_2} = \mathcal{O}_{\text{E}\Gamma_2}, \quad \mathcal{L}_5 = \mathcal{O}'_{5_1} = \mathcal{O}_{\text{E}\Gamma_1}, \quad \mathcal{L}_6 = \mathcal{L}_1^\perp = \mathcal{O}_1 = \mathcal{O}_{\text{RC}}, \\
\mathcal{L}_7 &= \mathcal{L}_4^\perp = \mathcal{O}_{5_2} = \mathcal{O}_{\text{Un}\Gamma_2}, \quad \mathcal{L}_8 = \mathcal{L}_5^\perp = \mathcal{O}_{5_1} = \mathcal{O}_{\text{Un}\Gamma_1}, \quad \mathcal{L}_9 = \mathcal{O}_3 = \mathcal{O}_{\text{IC}}, \\
\mathcal{L}_{10} &= \mathcal{L}_9^\perp = \mathcal{O}'_3 = \mathcal{O}_{\text{IA}}.
\end{aligned} \tag{2.15}$$

2.3 The plane-line incidence matrix of $\text{PG}(3, q)$

The $\beta_{3,q} \times \theta_{3,q}$ plane-line incidence matrix $\mathcal{I}^{\text{II}\Lambda}$ of $\text{PG}(3, q)$ is considered in [15, 18].

In [15], all $q \geq 2$ are considered, including even q and $q \equiv 0 \pmod{3}$, see [15, Section 3], where the results of the paper are summarized. In $\mathcal{I}^{\text{II}\Lambda}$, columns correspond to planes, rows correspond to lines, and there is an entry “1” if the corresponding line lies in the corresponding plane. In [15], $\mathcal{I}^{\text{II}\Lambda}$ is partitioned into $\#\mathcal{O}_\lambda \times \#\mathcal{N}_\pi$ submatrices $\mathcal{I}_{\pi,\lambda}^{\text{II}\Lambda}$, $\lambda \in \mathfrak{L}^{(\xi)}$, $\pi \in \mathfrak{P}$. If $L_{\lambda\Sigma}^{(\xi)\bullet} > 1$, see Table 1, then $\mathcal{I}_{\pi,\lambda}^{\text{II}\Lambda}$ splits into $L_{\lambda\Sigma}^{(\xi)\bullet}$ submatrices $\mathcal{I}_{\pi,\lambda_j}^{\text{II}\Lambda}$, $j = 1, \dots, L_{\lambda\Sigma}^{(\xi)\bullet}$.

The values of $\Pi_{\pi,\lambda}^{(\xi)\bullet}$, $\Lambda_{\lambda,\pi}^{(\xi)\bullet}$ for all $L_{\lambda\Sigma}^{(\xi)\bullet}$ and $\Pi_{\pi,\lambda_j}^{(\xi)\bullet}$, $\Lambda_{\lambda_j,\pi}^{(\xi)\bullet}$ for $L_{\lambda\Sigma}^{(\xi)\bullet} = 2, 3$ are obtained in [15] and collected in [15, Tables 1, 2]. For the class \mathcal{O}_6 , the values of $\Pi_{\pi,\lambda}^{(\xi)\bullet}$ are average over all $\text{En}\Gamma$ -lines.

In [18], for odd $q \not\equiv 0 \pmod{3}$, the ten orbits \mathcal{L}_i , see (2.15), are considered and the corresponding values $\Pi_{\pi,\lambda}^{(\xi)\text{od}}$ are obtained (they are denoted by $OD_2(\ell)$ and are called “the plane orbit distribution of a line ℓ ”). These results of [18] are in accordance with the ones of [15].

2.4 The point-line incidence matrix of $\text{PG}(3, q)$

Notation 2 According to Notation 1, let \mathfrak{p} and λ be the type of a point and of a line and let $\mathfrak{M}^{(\xi)}$ and $\mathfrak{L}^{(\xi)}$ be the lists of the possible types. By default, $\mathfrak{p} \in \mathfrak{M}^{(\xi)}$, $\lambda \in \mathfrak{L}^{(\xi)}$. A \mathfrak{p} -point is a point of the orbit $\mathcal{M}_{\mathfrak{p}}$; a λ -line is a line of the class \mathcal{O}_λ . In addition to Notation 1, the following notation is used:

- $\mathbb{L}_{\lambda_j,\mathfrak{p}}^{(\xi)\bullet}$ the number of lines from an orbit \mathcal{O}_{λ_j} through a \mathfrak{p} -point;
- $\mathbb{L}_{\lambda,\mathfrak{p}}^{(\xi)\bullet}$ the total number of λ -lines through a \mathfrak{p} -point;
- $\mathbb{P}_{\mathfrak{p},\lambda_j}^{(\xi)\bullet}$ the number of \mathfrak{p} -points on a line of an orbit \mathcal{O}_{λ_j} ;
- $\mathbb{P}_{\mathfrak{p},\lambda}^{(\xi)\bullet}$ the average number of \mathfrak{p} -points on a λ -line over all the λ -lines;

	if the class \mathcal{O}_λ consists of a <i>single orbit</i> then $\mathbb{P}_{\mathfrak{p},\lambda}^{(\xi)}$ is <i>the exact number</i> of \mathfrak{p} -points on each λ -line;
\mathcal{I}^{PL}	the $\beta_{3,q} \times \theta_{3,q}$ point-line incidence matrix of $\text{PG}(3, q)$;
$\mathcal{I}_{\mathfrak{p},\lambda}^{\text{PL}}$	the $\#\mathcal{O}_\lambda \times \#\mathcal{M}_{\mathfrak{p}}$ submatrix of \mathcal{I}^{PL} with incidences between \mathfrak{p} -points and λ -lines;
$\mathcal{I}_{\mathfrak{p},\lambda_j}^{\text{PL}}$	the $\#\mathcal{O}_{\lambda_j} \times \#\mathcal{M}_{\mathfrak{p}}$ submatrix of $\mathcal{I}_{\mathfrak{p},\lambda}^{\text{PL}}$ with incidences between \mathfrak{p} -points and λ_j -lines.

In \mathcal{I}^{PL} , columns correspond to points, rows correspond to lines, and there is an entry “1” if the corresponding point lies on the corresponding line. Every column and every row of \mathcal{I}^{PL} contains $\theta_{2,q}$ and $\theta_{1,q}$ ones, respectively, as in $\text{PG}(3, q)$, there are $\theta_{2,q}$ lines through every point and $\theta_{1,q}$ points in every line. Thus, \mathcal{I}^{PL} is a tactical configuration [20, Chapter 2.3], [25, Chapter 7, Section 2]. Moreover, \mathcal{I}^{PL} gives a 2 - $(\theta_{3,q}, \theta_{1,q}, 1)$ design [27] since there is exactly one line through any two points.

Definition 2. [19] A configuration (v_r, b_k) is an incidence structure of v points and b lines such that each line contains k points, each point lies on r lines, and two different points are connected by at most one line. If $v = b$ and, hence, $r = k$, the configuration is symmetric, denoted by v_k .

For an introduction to configurations see [12, 19] and the references therein.

The transposition $(\mathcal{I}^{\text{PL}})^{tr}$ gives the $\theta_{3,q} \times \beta_{3,q}$ line-point incidence matrix. It can be viewed as a (v_r, b_k) configuration with $v = \beta_{3,q}$, $b = \theta_{3,q}$, $r = \theta_{1,q}$, $k = \theta_{2,q}$, as there is at most one point as the intersection of two different lines.

In [18], for odd $q \not\equiv 0 \pmod{3}$, the ten orbits \mathcal{L}_i , see (2.15), are considered and the corresponding values $\mathbb{P}_{\mathfrak{p},\lambda}^{(\xi)\text{od}}$, $\mathbb{P}_{\mathfrak{p},\lambda_j}^{(\xi)\text{od}}$ are obtained (they are denoted by $OD_0(\ell)$ and are called “the point orbit distribution of a line ℓ ”).

These results of [18] are obtained also in this paper by another way. In this process we also obtained the values of $\mathbb{P}_{\mathfrak{p},\lambda}^{(\xi)\text{ev}}$, $\mathbb{P}_{\mathfrak{p},\lambda_j}^{(\xi)\text{ev}}$ for even $q \not\equiv 0 \pmod{3}$ and the values of $\mathbb{L}_{\lambda,\mathfrak{p}}^{(\xi)\bullet}$, $\mathbb{L}_{\lambda_j,\mathfrak{p}}^{(\xi)\bullet}$ for all odd and even $q \not\equiv 0 \pmod{3}$, see the first two tables of Section 3 and Section 5.

Moreover, in this paper we obtained also values $\mathbb{P}_{\mathfrak{p},\lambda}^{(0)\text{od}}$, $\mathbb{P}_{\mathfrak{p},\lambda_j}^{(0)\text{od}}$, $\mathbb{L}_{\lambda,\mathfrak{p}}^{(0)\text{od}}$, and $\mathbb{L}_{\lambda_j,\mathfrak{p}}^{(0)\text{od}}$ for $q \equiv 0 \pmod{3}$, see the last two tables of Section 3 and Section 6.

As we mentioned above, for the class \mathcal{O}_6 , in this paper only average and cumulative results are obtained.

3 The main results

Remark 1. We call $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)\bullet}$ *the average number of \mathbf{p} -points on a λ -line over all the λ -lines.* If the class \mathcal{O}_λ of λ -lines consists of a single orbit, i.e. $L_{\lambda\Sigma}^{(\xi)\bullet} = 1$, then $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)\bullet}$ is *the exact number of \mathbf{p} -points on each λ -line*, see Lemma 1. The situation is always clear by the context. If $L_{\lambda\Sigma}^{(\xi)\bullet} = 1$ then $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)\bullet}$ certainly is an integer. If λ -lines form two or more orbits, i.e. $L_{\lambda\Sigma}^{(\xi)\bullet} \geq 2$, then $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)\bullet}$ may be not integer as well as an integer.

On the other hand, regardless of the number of orbits in \mathcal{O}_λ , for all pairs (\mathbf{p}, λ) , we always have the same total number of λ -lines through each \mathbf{p} -point, i.e. $\mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)\bullet}$ always is an integer, again see Lemma 1.

From now on, we consider $q \geq 5$, apart from Theorem 6. Theorem 6 is obtained by an exhaustive computer search using the computer algebra system Magma [6].

Tables 2–5 and Theorem 5 summarize the results of Sections 4–7 for $q \geq 5$.

For the point-line incidence matrix \mathcal{I}^{PL} of $\text{PG}(3, q)$, $q \equiv \xi \pmod{3}$, Tables 2 (for $q \not\equiv 0 \pmod{3}$) and 4 (for $q \equiv 0 \pmod{3}$) show the values $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)}$ (top entry) and $\mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)}$ (bottom entry) for each pair (\mathbf{p}, λ) , $\mathbf{p} \in \mathfrak{M}^{(\xi)}$, $\lambda \in \mathfrak{L}^{(\xi)}$, where $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)}$ is the exact (if $L_{\lambda\Sigma}^{(\xi)\bullet} = 1$) or average (if $L_{\lambda\Sigma}^{(\xi)\bullet} \geq 2$) number of \mathbf{p} -points on every λ -line, whereas $\mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)}$ always is the exact number of λ -lines through every \mathbf{p} -point. In other words, $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)}$ is the exact or average number of ones in every row of the submatrix $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$ of \mathcal{I}^{PL} , whereas $\mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)}$ always is the exact number of ones in every column of $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$. In Table 2, the superscript (ξ) is $(\neq 0)$ for $\lambda \in \{\text{RA}, \text{T}, \text{IA}, \text{UF}, \text{E}\Gamma\}$ where the values $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)}$, $\mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)}$ are the same for all $q \not\equiv 0 \pmod{3}$.

The total number of orbits of λ -lines is given in Table 1.

In Table 3, the values $\mathbb{P}_{\mathbf{p},\lambda_j}^{(\xi)\bullet}$ and $\mathbb{L}_{\lambda_j,\mathbf{p}}^{(\xi)\bullet}$ are given for the following cases: $q \geq 5$, $\mathbf{p} \in \mathfrak{M}^{(\neq 0)}$; $\lambda = \text{UF}$ with even $q \not\equiv 0 \pmod{3}$ (UF_1 - and UF_2 -lines); $\lambda = \text{Un}\Gamma$ with odd $q \equiv \xi \pmod{3}$, $\xi \in \{1, -1\}$ ($\text{Un}\Gamma_1$ - and $\text{Un}\Gamma_2$ -lines for $\xi = 1$ and $\xi = -1$); $\lambda = \text{E}\Gamma$ with odd $q \not\equiv 0 \pmod{3}$ ($\text{E}\Gamma_1$ - and $\text{E}\Gamma_2$ -lines).

In Table 5, the values $\mathbb{P}_{\mathbf{p},\lambda_j}^{(0)}$ and $\mathbb{L}_{\lambda_j,\mathbf{p}}^{(0)}$ are given for the following cases: $q \equiv 0 \pmod{3}$, $q \geq 9$, $\mathbf{p} \in \mathfrak{M}^{(0)}$, $\lambda \in \{\text{Un}\Gamma, \text{EA}\}$, $j = 1, 2$ if $\lambda = \text{Un}\Gamma$, $j = 1, 2, 3$ if $\lambda = \text{EA}$.

Theorem 5. *Let $q \geq 5$, $q \equiv \xi \pmod{3}$. Let notations be as in Section 2 and Notations 1, 2. The following holds:*

- (i) *In $\text{PG}(3, q)$, for the submatrices $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$ of the point-line incidence matrix \mathcal{I}^{PL} , the values $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)}$ (i.e. the exact or average number of \mathbf{p} -points on a λ -line) and $\mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)}$ (i.e. the exact number of λ -lines through a \mathbf{p} -point) are given in Table 2 (for $\xi \neq 0$) and Table 4 (for $\xi = 0$).*

For the submatrices $\mathcal{I}_{\mathbf{p},\lambda_j}^{\text{PL}}$ corresponding to each of two orbits of the classes $\mathcal{O}_4 = \mathcal{O}_{\text{UF}}$, $\mathcal{O}_5 = \mathcal{O}_{\text{Un}\Gamma}$, and $\mathcal{O}'_5 = \mathcal{O}_{\text{E}\Gamma}$, the values $\mathbb{P}_{\mathbf{p},\lambda_j}^{(\xi)\bullet}$, $\mathbb{L}_{\lambda_j,\mathbf{p}}^{(\xi)\bullet}$ are given in Table 3 (for

Table 2: Values $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)}$ (top entry) and $\mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)}$ (bottom entry) for submatrices $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$ of the point-line incidence matrix of $\text{PG}(3, q)$, $q \equiv \xi \pmod{3}$, $\xi \in \{1, -1\}$, $q \geq 5$, $\mathbf{p} \in \mathfrak{M}^{(\neq 0)}$, $\lambda \in \mathfrak{L}^{(\neq 0)}$. The superscript (ξ) is $(\neq 0)$ if a value is the same for all $q \not\equiv 0 \pmod{3}$

			$\mathcal{M}_1^{(\neq 0)}$	$\mathcal{M}_2^{(\neq 0)}$	$\mathcal{M}_3^{(\neq 0)}$	$\mathcal{M}_4^{(\neq 0)}$	$\mathcal{M}_5^{(\neq 0)}$
			\mathcal{C} -	T-	3Γ -	1Γ -	0Γ -
\mathcal{O}_j	λ -lines	$\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)}$	points	points	points	points	points
\mathcal{O}'_j	$\#\mathcal{O}_\lambda$	$\mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)}$	$q+1$	q^2+q	$\frac{1}{6}(q^3-q)$	$\frac{1}{2}(q^3-q)$	$\frac{1}{3}(q^3-q)$
\mathcal{O}_1	RC-lines	$\mathbb{P}_{\mathbf{p},\text{RC}}^{(1)}$	2	0	$\frac{1}{3}(q-1)$	0	$\frac{2}{3}(q-1)$
	$\frac{1}{2}(q^2+q)$	$\mathbb{L}_{\text{RC},\mathbf{p}}^{(1)}$	q	0	1	0	1
\mathcal{O}_1	RC-lines	$\mathbb{P}_{\mathbf{p},\text{RC}}^{(-1)}$	2	0	0	$q-1$	0
	$\frac{1}{2}(q^2+q)$	$\mathbb{L}_{\text{RC},\mathbf{p}}^{(-1)}$	q	0	0	1	0
\mathcal{O}'_1	RA-lines	$\mathbb{P}_{\mathbf{p},\text{RA}}^{(\neq 0)}$	0	2	$q-1$	0	0
	$\frac{1}{2}(q^2+q)$	$\mathbb{L}_{\text{RA},\mathbf{p}}^{(\neq 0)}$	0	1	3	0	0
\mathcal{O}_2	T-lines	$\mathbb{P}_{\mathbf{p},\text{T}}^{(\neq 0)}$	1	q	0	0	0
\mathcal{O}'_2	$q+1$	$\mathbb{L}_{\text{T},\mathbf{p}}^{(\neq 0)}$	1	1	0	0	0
\mathcal{O}_3	IC-lines	$\mathbb{P}_{\mathbf{p},\text{IC}}^{(1)}$	0	0	0	$q+1$	0
	$\frac{1}{2}(q^2-q)$	$\mathbb{L}_{\text{IC},\mathbf{p}}^{(1)}$	0	0	0	1	0
\mathcal{O}_3	IC-lines	$\mathbb{P}_{\mathbf{p},\text{IC}}^{(-1)}$	0	0	$\frac{1}{3}(q+1)$	0	$\frac{2}{3}(q+1)$
	$\frac{1}{2}(q^2-q)$	$\mathbb{L}_{\text{IC},\mathbf{p}}^{(-1)}$	0	0	1	0	1
\mathcal{O}'_3	IA-lines	$\mathbb{P}_{\mathbf{p},\text{IA}}^{(\neq 0)}$	0	0	0	$q+1$	0
	$\frac{1}{2}(q^2-q)$	$\mathbb{L}_{\text{IA},\mathbf{p}}^{(\neq 0)}$	0	0	0	1	0
\mathcal{O}_4	U Γ -lines	$\mathbb{P}_{\mathbf{p},\text{U}\Gamma}^{(\neq 0)}$	1	1	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	0
\mathcal{O}'_4	q^2+q	$\mathbb{L}_{\text{U}\Gamma,\mathbf{p}}^{(\neq 0)}$	q	1	3	1	0
\mathcal{O}_5	Un Γ -lines	$\mathbb{P}_{\mathbf{p},\text{Un}\Gamma}^{(1)}$	1	1	$\frac{1}{6}(q-4)$	$\frac{1}{2}q$	$\frac{1}{3}(q-1)$
	q^3-q	$\mathbb{L}_{\text{Un}\Gamma,\mathbf{p}}^{(1)}$	q^2-q	$q-1$	$q-4$	q	$q-1$
\mathcal{O}_5	Un Γ -lines	$\mathbb{P}_{\mathbf{p},\text{Un}\Gamma}^{(-1)}$	1	1	$\frac{1}{6}(q-2)$	$\frac{1}{2}(q-2)$	$\frac{1}{3}(q+1)$
	q^3-q	$\mathbb{L}_{\text{Un}\Gamma,\mathbf{p}}^{(-1)}$	q^2-q	$q-1$	$q-2$	$q-2$	$q+1$
\mathcal{O}'_5	E Γ -lines	$\mathbb{P}_{\mathbf{p},\text{E}\Gamma}^{(\neq 0)}$	0	2	$\frac{1}{2}(q-2)$	$\frac{1}{2}q$	0
	q^3-q	$\mathbb{L}_{\text{E}\Gamma,\mathbf{p}}^{(\neq 0)}$	0	$2(q-1)$	$3(q-2)$	q	0
\mathcal{O}_6	En Γ -lines	$\mathbb{P}_{\mathbf{p},\text{En}\Gamma}^{(1)}$	0	1	$\frac{q^2-3q+4}{6(q-1)}$	$\frac{(q+1)(q-2)}{2(q-1)}$	$\frac{q^2+1}{3(q-1)}$
\mathcal{O}'_6	$(q^2-q) \cdot$ (q^2-1)	$\mathbb{L}_{\text{En}\Gamma,\mathbf{p}}^{(1)}$	0	$(q-1)^2$	q^2- $3q+4$	$(q+1) \cdot$ $(q-2)$	q^2+1
\mathcal{O}_6	En Γ -lines	$\mathbb{P}_{\mathbf{p},\text{En}\Gamma}^{(-1)}$	0	1	$\frac{1}{6}(q-2)$	$\frac{1}{2}q$	$\frac{1}{3}(q+1)$
\mathcal{O}'_6	$(q^2-q) \cdot$ (q^2-1)	$\mathbb{L}_{\text{En}\Gamma,\mathbf{p}}^{(-1)}$	0	$(q-1)^2$	$(q-1) \cdot$ $(q-2)$	q^2-q	q^2-1

Table 3: Values $\mathbb{P}_{\mathbf{p},\lambda_j}^{(\xi)\bullet}$ (top entry) and $\mathbb{L}_{\lambda_j,\mathbf{p}}^{(\xi)\bullet}$ (bottom entry) for submatrices $\mathcal{I}_{\mathbf{p},\lambda_j}^{\mathbb{P}\mathbb{L}}$ of the point-line incidence matrix of $\text{PG}(3, q)$, $q \geq 5$, $\mathbf{p} \in \mathfrak{M}^{(\neq 0)}$; $j = 1, 2$; $\lambda = \text{U}\Gamma$ with even $q \not\equiv 0 \pmod{3}$ ($\text{U}\Gamma_1$ - and $\text{U}\Gamma_2$ -lines); $\lambda = \text{Un}\Gamma$ with odd $q \equiv \xi \pmod{3}$, $\xi \in \{1, -1\}$ ($\text{Un}\Gamma_1$ - and $\text{Un}\Gamma_2$ -lines for $\xi = 1$ and $\xi = -1$); $\lambda = \text{E}\Gamma$ with odd $q \not\equiv 0 \pmod{3}$ ($\text{E}\Gamma_1$ - and $\text{E}\Gamma_2$ -lines)

			$\mathcal{M}_1^{(\neq 0)}$	$\mathcal{M}_2^{(\neq 0)}$	$\mathcal{M}_3^{(\neq 0)}$	$\mathcal{M}_4^{(\neq 0)}$	$\mathcal{M}_5^{(\neq 0)}$
			\mathcal{C} -	T-	3Γ -	1Γ -	0Γ -
\mathcal{O}_{i_j}	λ_j -lines	$\mathbb{P}_{\mathbf{p},\lambda_j}^{(\xi)}$	points	points	points	points	points
\mathcal{O}'_{i_j}	$\#\mathcal{O}_{\lambda_j}$	$\mathbb{L}_{\lambda_j,\mathbf{p}}^{(\xi)}$	$q + 1$	$q^2 + q$	$\frac{q^3 - q}{6}$	$\frac{q^3 - q}{2}$	$\frac{q^3 - q}{3}$
\mathcal{O}_{4_1}	$\text{U}\Gamma_1$ -lines	$\mathbb{P}_{\mathbf{p},\text{U}\Gamma_1}^{(\neq 0)\text{ev}}$	1	q	0	0	0
	$q + 1$	$\mathbb{L}_{\text{U}\Gamma_1,\mathbf{p}}^{(\neq 0)\text{ev}}$	1	1	0	0	0
\mathcal{O}_{4_2}	$\text{U}\Gamma_2$ -lines	$\mathbb{P}_{\mathbf{p},\text{U}\Gamma_2}^{(\neq 0)\text{ev}}$	1	0	$\frac{1}{2}q$	$\frac{1}{2}q$	0
	$q^2 - 1$	$\mathbb{L}_{\text{U}\Gamma_2,\mathbf{p}}^{(\neq 0)\text{ev}}$	$q - 1$	0	3	1	0
\mathcal{O}_{5_1}	$\text{Un}\Gamma_1$ -	$\mathbb{P}_{\mathbf{p},\text{Un}\Gamma_1}^{(1)\text{od}}$	1	0	$\frac{1}{6}(q - 1)$	$\frac{1}{2}(q + 1)$	$\frac{1}{3}(q - 1)$
	lines						
1	$\frac{1}{2}(q^3 - q)$	$\mathbb{L}_{\text{Un}\Gamma_1,\mathbf{p}}^{(1)\text{od}}$	$\frac{1}{2}(q^2 - q)$	0	$\frac{1}{2}(q - 1)$	$\frac{1}{2}(q + 1)$	$\frac{1}{2}(q - 1)$
\mathcal{O}_{5_2}	$\text{Un}\Gamma_2$ -	$\mathbb{P}_{\mathbf{p},\text{Un}\Gamma_2}^{(1)\text{od}}$	1	2	$\frac{1}{6}(q - 7)$	$\frac{1}{2}(q - 1)$	$\frac{1}{3}(q - 1)$
	lines						
1	$\frac{1}{2}(q^3 - q)$	$\mathbb{L}_{\text{Un}\Gamma_2,\mathbf{p}}^{(1)\text{od}}$	$\frac{1}{2}(q^2 - q)$	$q - 1$	$\frac{1}{2}(q - 7)$	$\frac{1}{2}(q - 1)$	$\frac{1}{2}(q - 1)$
\mathcal{O}_{5_1}	$\text{Un}\Gamma_1$ -	$\mathbb{P}_{\mathbf{p},\text{Un}\Gamma_1}^{(-1)\text{od}}$	1	0	$\frac{1}{6}(q + 1)$	$\frac{1}{2}(q - 1)$	$\frac{1}{3}(q + 1)$
	lines						
-1	$\frac{1}{2}(q^3 - q)$	$\mathbb{L}_{\text{Un}\Gamma_1,\mathbf{p}}^{(-1)\text{od}}$	$\frac{1}{2}(q^2 - q)$	0	$\frac{1}{2}(q + 1)$	$\frac{1}{2}(q - 1)$	$\frac{1}{2}(q + 1)$
\mathcal{O}_{5_2}	$\text{Un}\Gamma_2$ -	$\mathbb{P}_{\mathbf{p},\text{Un}\Gamma_2}^{(-1)\text{od}}$	1	2	$\frac{1}{6}(q - 5)$	$\frac{1}{2}(q - 3)$	$\frac{1}{3}(q + 1)$
	lines						
-1	$\frac{1}{2}(q^3 - q)$	$\mathbb{L}_{\text{Un}\Gamma_2,\mathbf{p}}^{(-1)\text{od}}$	$\frac{1}{2}(q^2 - q)$	$q - 1$	$\frac{1}{2}(q - 5)$	$\frac{1}{2}(q - 3)$	$\frac{1}{2}(q + 1)$
\mathcal{O}'_{5_1}	$\text{E}\Gamma_1$ -lines	$\mathbb{P}_{\mathbf{p},\text{E}\Gamma_1}^{(\neq 0)\text{od}}$	0	1	$\frac{1}{2}(q - 1)$	$\frac{1}{2}(q + 1)$	0
	$\frac{1}{2}(q^3 - q)$	$\mathbb{L}_{\text{E}\Gamma_1,\mathbf{p}}^{(\neq 0)\text{od}}$	0	$\frac{1}{2}(q - 1)$	$\frac{3}{2}(q - 1)$	$\frac{1}{2}(q + 1)$	0
\mathcal{O}'_{5_2}	$\text{E}\Gamma_2$ -lines	$\mathbb{P}_{\mathbf{p},\text{E}\Gamma_2}^{(\neq 0)\text{od}}$	0	3	$\frac{1}{2}(q - 3)$	$\frac{1}{2}(q - 1)$	0
	$\frac{1}{2}(q^3 - q)$	$\mathbb{L}_{\text{E}\Gamma_2,\mathbf{p}}^{(\neq 0)\text{od}}$	0	$\frac{3}{2}(q - 1)$	$\frac{3}{2}(q - 3)$	$\frac{1}{2}(q - 1)$	0

Table 4: Values $\mathbb{P}_{\mathbf{p},\lambda}^{(0)}$ (top entry) and $\mathbb{L}_{\lambda,\mathbf{p}}^{(0)}$ (bottom entry) for submatrices $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$ of the point-line incidence matrix \mathcal{I}^{PL} of $\text{PG}(3, q)$, $q \equiv 0 \pmod{3}$, $q \geq 9$, $\lambda \in \mathfrak{L}^{(0)}$, $\mathbf{p} \in \mathfrak{M}^{(0)}$

			$\mathcal{M}_1^{(0)}$	$\mathcal{M}_2^{(0)}$	$\mathcal{M}_3^{(0)}$	$\mathcal{M}_4^{(0)}$	$\mathcal{M}_5^{(0)}$
			\mathcal{C} -	$(q+1)_{\Gamma}$ -	TO-	RC-	IC-
	λ -lines	$\mathbb{P}_{\mathbf{p},\lambda}^{(0)}$	points	points	points	points	points
\mathcal{O}_j	$\#\mathcal{O}_\lambda$	$\mathbb{L}_{\lambda,\mathbf{p}}^{(0)}$	$q+1$	$q+1$	q^2-1	$\frac{1}{2}(q^3-q)$	$\frac{1}{2}(q^3-q)$
\mathcal{O}_1	RC-lines	$\mathbb{P}_{\mathbf{p},\text{RC}}^{(0)}$	2	0	0	$q-1$	0
	$\frac{1}{2}(q^2+q)$	$\mathbb{L}_{\text{RC},\mathbf{p}}^{(0)}$	q	0	0	1	0
\mathcal{O}_2	T-lines	$\mathbb{P}_{\mathbf{p},\text{T}}^{(0)}$	1	1	$q-1$	0	0
	$q+1$	$\mathbb{L}_{\text{T},\mathbf{p}}^{(0)}$	1	1	1	0	0
\mathcal{O}_3	IC-lines	$\mathbb{P}_{\mathbf{p},\text{IC}}^{(0)}$	0	0	0	0	$q+1$
	$\frac{1}{2}(q^2-q)$	$\mathbb{L}_{\text{IC},\mathbf{p}}^{(0)}$	0	0	0	0	1
\mathcal{O}_4	U Γ -lines	$\mathbb{P}_{\mathbf{p},\text{U}\Gamma}^{(0)}$	1	1	0	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$
	q^2+q	$\mathbb{L}_{\text{U}\Gamma,\mathbf{p}}^{(0)}$	q	q	0	1	1
\mathcal{O}_5	Un Γ -lines	$\mathbb{P}_{\mathbf{p},\text{Un}\Gamma}^{(0)}$	1	0	1	$\frac{1}{2}(q-2)$	$\frac{1}{2}q$
	q^3-q	$\mathbb{L}_{\text{Un}\Gamma,\mathbf{p}}^{(0)}$	q^2-q	0	q	$q-2$	q
\mathcal{O}_6	En Γ -lines	$\mathbb{P}_{\mathbf{p},\text{En}\Gamma}^{(0)}$	0	0	1	$\frac{q^2-q+1}{2(q-1)}$	$\frac{q^2-q-1}{2(q-1)}$
	$(q^2-q) \cdot (q^2-1)$	$\mathbb{L}_{\text{En}\Gamma,\mathbf{p}}^{(0)}$	0	0	q^2-q	q^2-q+1	q^2-q-1
\mathcal{O}_7	A-lines	$\mathbb{P}_{\mathbf{p},\text{A}}^{(0)}$	0	$q+1$	0	0	0
	1	$\mathbb{L}_{\text{A},\mathbf{p}}^{(0)}$	0	1	0	0	0
\mathcal{O}_8	EA-lines	$\mathbb{P}_{\mathbf{p},\text{EA}}^{(0)}$	0	1	$\frac{q}{q+1}$	$\frac{q^2}{2(q+1)}$	$\frac{q^2}{2(q+1)}$
	$(q+1) \cdot (q^2-1)$	$\mathbb{L}_{\text{EA},\mathbf{p}}^{(0)}$	0	q^2-1	q	q	q

Table 5: Values $\mathbb{P}_{\mathbf{p},\lambda_j}^{(0)}$ (top entry) and $\mathbb{L}_{\lambda_j,\mathbf{p}}^{(0)}$ (bottom entry) for submatrices $\mathcal{I}_{\mathbf{p},\lambda_j}^{\text{PL}}$ of the point-line incidence matrix \mathcal{I}^{PL} of $\text{PG}(3,q)$, $q \equiv 0 \pmod{3}$, $q \geq 9$, $\mathbf{p} \in \mathfrak{M}^{(0)}$, $\lambda \in \{\text{Un}\Gamma, \text{EA}\}$, $j = 1, 2$ if $\lambda = \text{Un}\Gamma$, $j = 1, 2, 3$ if $\lambda = \text{EA}$

			$\mathcal{M}_1^{(0)}$	$\mathcal{M}_2^{(0)}$	$\mathcal{M}_3^{(0)}$	$\mathcal{M}_4^{(0)}$	$\mathcal{M}_5^{(0)}$
			\mathcal{C} -	$(q+1)_{\Gamma}$ -	TO-	RC-	IC-
	λ_j -lines	$\mathbb{P}_{\mathbf{p},\lambda}^{(0)}$	points	points	points	points	points
\mathcal{O}_{i_j}	$\#\mathcal{O}_{\lambda_j}$	$\mathbb{L}_{\lambda,\mathbf{p}}^{(0)}$	$q+1$	$q+1$	q^2-1	$\frac{q^3-q}{2}$	$\frac{q^3-q}{2}$
\mathcal{O}_{5_1}	Un Γ_1 -	$\mathbb{P}_{\mathbf{p},\text{Un}\Gamma_1}^{(0)}$	1	0	0	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q+1)$
$\xi =$	lines						
0	$\frac{1}{2}(q^3-q)$	$\mathbb{L}_{\text{Un}\Gamma_1,\mathbf{p}}^{(0)}$	$\frac{1}{2}(q^2-q)$	0	0	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q+1)$
\mathcal{O}_{5_2}	Un Γ_2 -	$\mathbb{P}_{\mathbf{p},\text{Un}\Gamma_2}^{(0)}$	1	0	2	$\frac{1}{2}(q-3)$	$\frac{1}{2}(q-1)$
$\xi =$	lines						
0	$\frac{1}{2}(q^3-q)$	$\mathbb{L}_{\text{Un}\Gamma_2,\mathbf{p}}^{(0)}$	$\frac{1}{2}(q^2-q)$	0	q	$\frac{1}{2}(q-3)$	$\frac{1}{2}(q-1)$
\mathcal{O}_{8_1}	EA $_1$ -lines	$\mathbb{P}_{\mathbf{p},\text{EA}_1}^{(0)}$	0	1	1	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$
	q^3-q	$\mathbb{L}_{\text{EA}_1,\mathbf{p}}^{(0)}$	0	q^2-q	q	$q-1$	$q-1$
\mathcal{O}_{8_2}	EA $_2$ -lines	$\mathbb{P}_{\mathbf{p},\text{EA}_2}^{(0)}$	0	1	0	q	0
	$\frac{1}{2}(q^2-1)$	$\mathbb{L}_{\text{EA}_2,\mathbf{p}}^{(0)}$	0	$\frac{1}{2}(q-1)$	0	1	0
\mathcal{O}_{8_3}	EA $_3$ -lines	$\mathbb{P}_{\mathbf{p},\text{EA}_3}^{(0)}$	0	1	0	0	q
	$\frac{1}{2}(q^2-1)$	$\mathbb{L}_{\text{EA}_3,\mathbf{p}}^{(0)}$	0	$\frac{1}{2}(q-1)$	0	0	1

$\xi \neq 0$). For the submatrices $\mathcal{I}_{\mathbf{p},\lambda_j}^{\text{PL}}$ corresponding to each of two orbits of the class $\mathcal{O}_5 = \mathcal{O}_{\text{Un}\Gamma}$ and to each of three orbits of the class $\mathcal{O}_8 = \mathcal{O}_{\text{EA}}$, the values $\mathbb{P}_{\mathbf{p},\lambda_j}^{(0)}$, $\mathbb{L}_{\lambda_j,\mathbf{p}}^{(0)}$ are given in Table 5 (for $\xi = 0$).

- (ii) Let a class \mathcal{O}_λ consist of a single orbit according to Table 1. Then, in Tables 2 and 4, the values of $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)}$, $\mathbf{p} \in \mathfrak{M}^{(\xi)}$, are the exact numbers of \mathbf{p} -points on every λ -line.
- (iii) Let $q \equiv 1 \pmod{3}$. Let $V^{(1)} = \{\mathcal{O}_1 = \mathcal{O}_{\text{RC}}, \mathcal{O}_2 = \mathcal{O}_{\text{T}}, \mathcal{O}'_3 = \mathcal{O}_{\text{IA}}\}$. Then, cf. Theorem 1(iv), no two lines of $V^{(1)}$ meet off \mathcal{C} . Every point off \mathcal{C} lies on exactly one line of $V^{(1)}$.
- (iv) Let $q \equiv 0 \pmod{3}$. Let $\mathcal{W}^{(0)} = \{\mathcal{O}_2 = \mathcal{O}_{\text{T}}, \mathcal{O}_4 = \mathcal{O}_{\text{U}\Gamma}\}$. Let $\mathbb{M} = \mathcal{C} \cup \text{A-line}$ be the union of the twisted cubic and the A-line. Then no two lines of $\mathcal{W}^{(0)}$ meet off \mathbb{M} . Every point off \mathbb{M} lies on exactly one line of $\mathcal{W}^{(0)}$, cf. Theorems 1(iv) and 5(iii).
- (v) Let $\mathbf{p} \in \mathfrak{M}^{(\xi)}$. Let a class \mathcal{O}_λ consist of a single orbit. Then the submatrix $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$ of \mathcal{I}^{PL} is a (v_r, b_k) configuration of Definition 2 with $v = \#\mathcal{M}_{\mathbf{p}}$, $b = \#\mathcal{O}_\lambda$, $r = \mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)}$,

$k = \mathbb{P}_{\mathbf{p},\lambda}^{(\xi)}$. Also, up to rearrangement of rows and columns, the submatrices $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$ with $\mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)} = 1$ can be viewed as a concatenation of $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)}$ identity matrices of order $\#\mathcal{O}_\lambda$. The same holds for the submatrices $\mathcal{I}_{\mathbf{p},\lambda_j}^{\text{PL}}$.

- (vi) Let $(\lambda, \mathbf{p}) \in \{(\text{U}\Gamma, \mathcal{C}), (\text{Un}\Gamma, \mathcal{C})\}$ if $\xi \neq 0$, and $(\lambda, \mathbf{p}) \in \{(\text{Un}\Gamma, \mathcal{C}), (\text{EA}, (q+1)_\Gamma)\}$ if $\xi = 0$. Then, independently of the number of orbits in the class \mathcal{O}_λ , we have exactly one \mathbf{p} -point on every λ -line. Up to rearrangement of rows and columns, the submatrices $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$ can be viewed as a vertical concatenation of $\mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)}$ identity matrices of order $\#\mathcal{M}_\mathbf{p}$.

Theorem 6. Let the types of lines and points be as in Tables 1 – 5.

- (i) Let $q = 2$. The group $G_2 \cong \mathbf{S}_3\mathbf{Z}_2^3$ contains 8 subgroups isomorphic to $\text{PGL}(2, 2)$ divided into two conjugacy classes. For one of these subgroups, the matrices corresponding to the projectivities of the subgroup assume the form described by (2.7). For line and point orbits under this subgroup (and only under it) the point-line incidence matrix has the form of Tables 2 and 3 for even $q \equiv -1 \pmod{3}$ and also Table 1 holds.
- (ii) Let $q = 3$. The group $G_3 \cong \mathbf{S}_4\mathbf{Z}_2^3$ contains 24 subgroups isomorphic to $\text{PGL}(2, 3)$ divided into four conjugacy classes. For one of these subgroups, the matrices corresponding to the projectivities of the subgroup assume the form described by (2.7). For line and point orbits under this subgroup (and only under it) the point-line incidence matrix has the form of Tables 4 and 5 for $q \equiv 0 \pmod{3}$ and also Table 1 holds.
- (iii) Let $q = 4$. The group $G_4 \cong \mathbf{S}_5 \cong \text{P}\Gamma\text{L}(2, 4)$ contains one subgroup isomorphic to $\text{PGL}(2, 4)$. The matrices corresponding to the projectivities of this subgroup assume the form described by (2.7) and for line and point orbits under this subgroup the point-line incidence matrix has the form of Tables 2 and 3 for even $q \equiv 1 \pmod{3}$ and also Table 1 holds.
- (iv) For line orbits under the subgroups of G_q noted in the points (i)–(iii) of this theorem, Theorem 2 holds also if $q = 2, 3, 4$.

4 Some useful relations

In this section, we omit the superscripts “ (ξ) ”, “od”, and “ev” as they are the same for all terms in a formula; in particular, we use \mathfrak{L} and $L_{\lambda\Sigma}$ instead of $\mathfrak{L}^{(\xi)}$ and $L_{\lambda\Sigma}^{(\xi)\text{od}}, L_{\lambda\Sigma}^{(\xi)\text{ev}}$. In the rest of the paper, when relations of this section are applied, we add the superscripts if they are necessary by the context.

Lemma 1. *The following holds:*

- (i) *The number $\mathbb{L}_{\lambda_j, \mathbf{p}}$ of lines from an orbit \mathcal{O}_{λ_j} through a point of an orbit $\mathcal{M}_{\mathbf{p}}$ is the same for all points of $\mathcal{M}_{\mathbf{p}}$.*
- (ii) *The total number $\mathbb{L}_{\lambda, \mathbf{p}}$ of lines from an orbit union \mathcal{O}_{λ} through a point of an orbit $\mathcal{M}_{\mathbf{p}}$ is the same for all points of $\mathcal{M}_{\mathbf{p}}$. We have*

$$\mathbb{L}_{\lambda, \mathbf{p}} = \sum_{j=1}^{L_{\lambda\Sigma}} \mathbb{L}_{\lambda_j, \mathbf{p}}. \quad (4.1)$$

- (iii) *The number $\mathbb{P}_{\mathbf{p}, \lambda_j}$ of points from an orbit $\mathcal{M}_{\mathbf{p}}$ on a line of an orbit \mathcal{O}_{λ_j} is the same for all lines of \mathcal{O}_{λ_j} .*
- (iv) *The average number $\mathbb{P}_{\mathbf{p}, \lambda}$ of points from an orbit $\mathcal{M}_{\mathbf{p}}$ on a line of a union \mathcal{O}_{λ} over all lines of \mathcal{O}_{λ} satisfies the following relations:*

$$\mathbb{L}_{\lambda, \mathbf{p}} \cdot \#\mathcal{M}_{\mathbf{p}} = \mathbb{P}_{\mathbf{p}, \lambda} \cdot \#\mathcal{O}_{\lambda}; \quad (4.2)$$

$$\mathbb{P}_{\mathbf{p}, \lambda} = \frac{1}{\#\mathcal{O}_{\lambda}} \sum_{j=1}^{L_{\lambda\Sigma}} (\mathbb{P}_{\mathbf{p}, \lambda_j} \cdot \#\mathcal{O}_{\lambda_j}). \quad (4.3)$$

- (v) *If $L_{\lambda\Sigma} = 1$, then \mathcal{O}_{λ} is an orbit and the number of points from $\mathcal{M}_{\mathbf{p}}$ on a line of \mathcal{O}_{λ} is the same for all the lines of \mathcal{O}_{λ} . In this case, $\mathbb{P}_{\mathbf{p}, \lambda}$ is certainly an integer. If $\mathbb{P}_{\mathbf{p}, \lambda}$ is not an integer then the class \mathcal{O}_{λ} contains more than one orbit, i.e. $L_{\lambda\Sigma} \geq 2$.*

Proof. (i) Consider points \mathbf{p}_1 and \mathbf{p}_2 of $\mathcal{M}_{\mathbf{p}}$. Denote by ℓ a line of \mathcal{O}_{λ_j} . Let $S(\mathbf{p}_1)$ and $S(\mathbf{p}_2)$ be the subsets of \mathcal{O}_{λ_j} such that $S(\mathbf{p}_1) = \{\ell \in \mathcal{O}_{\lambda_j} | \mathbf{p}_1 \in \ell\}$, $S(\mathbf{p}_2) = \{\ell \in \mathcal{O}_{\lambda_j} | \mathbf{p}_2 \in \ell\}$. There exists $\varphi \in G_q$ such that $\mathbf{p}_2 = \mathbf{p}_1\varphi$. Clearly, φ embeds $S(\mathbf{p}_1)$ in $S(\mathbf{p}_2)$, i.e. $S(\mathbf{p}_1)\varphi \subseteq S(\mathbf{p}_2)$ and $\#S(\mathbf{p}_1) \leq \#S(\mathbf{p}_2)$. In the same way, φ^{-1} embeds $S(\mathbf{p}_2)$ in $S(\mathbf{p}_1)$, i.e. $\#S(\mathbf{p}_2) \leq \#S(\mathbf{p}_1)$. Thus, $\#S(\mathbf{p}_2) = \#S(\mathbf{p}_1)$.

- (ii) For a fixed λ , orbits \mathcal{O}_{λ_j} do not intersect each other.
- (iii) The assertion can be proved similarly to the case (i).
- (iv) The cardinality C_1 of the multiset consisting of the lines of \mathcal{O}_{λ} through all the points of $\mathcal{M}_{\mathbf{p}}$ is equal to $\mathbb{L}_{\lambda, \mathbf{p}} \cdot \#\mathcal{M}_{\mathbf{p}}$. The cardinality C_2 of the multiset consisting of the points of $\mathcal{M}_{\mathbf{p}}$ on all the lines of \mathcal{O}_{λ} is $\mathbb{P}_{\mathbf{p}, \lambda} \cdot \#\mathcal{O}_{\lambda}$. Every C_i is the number of ones in the incidence submatrix $\mathcal{I}_{\mathbf{p}, \lambda}^{\text{PL}}$ of \mathcal{I}^{PL} . Thus, $C_1 = C_2$.

The assertion (4.3) holds as \mathcal{O}_{λ} is *partitioned* into $L_{\lambda\Sigma}$ orbits \mathcal{O}_{λ_j} .

(v) The assertion follows from the case (iii). \square

Corollary 1. *If $\mathbb{P}_{\mathbf{p},\lambda} = 0$ then $\mathbb{L}_{\lambda,\mathbf{p}} = 0$ and vice versa.*

Proof. The assertions follow from (4.2). \square

Theorem 7. *Let the lines of $\text{PG}(3, q)$ be partitioned under G_q into $\#\mathfrak{L}$ classes \mathcal{O}_λ where every class is a union of orbits of λ -lines, $\lambda \in \mathfrak{L}$. Also, let $\text{PG}(3, q)$ be partitioned under G_q by $\#\mathfrak{M}$ orbits $\mathcal{M}_{\mathbf{p}}$ of \mathbf{p} -points, $\mathbf{p} \in \mathfrak{M}$. The following holds:*

$$\sum_{\mathbf{p} \in \mathfrak{M}} \mathbb{P}_{\mathbf{p},\lambda} = q + 1, \quad \lambda \text{ is fixed}; \quad (4.4)$$

$$\sum_{\lambda \in \mathfrak{L}} \mathbb{L}_{\lambda,\mathbf{p}} = \beta_{2,q} = q^2 + q + 1, \quad \mathbf{p} \text{ is fixed}. \quad (4.5)$$

Proof. Relations (4.4) and (4.5) hold as $\text{PG}(3, q)$ is *partitioned* under G_q by unions of line orbits and by orbits of points. In total, in $\text{PG}(3, q)$, there are $q + 1$ points on every line and $\beta_{2,q}$ lines through every point. \square

Corollary 2. *The following holds:*

$$\mathbb{P}_{\mathbf{p},\lambda} = \frac{\mathbb{L}_{\lambda,\mathbf{p}} \cdot \#\mathcal{M}_{\mathbf{p}}}{\#\mathcal{O}_\lambda}, \quad \mathbb{P}_{\mathbf{p},\lambda_j} = \frac{\mathbb{L}_{\lambda_j,\mathbf{p}} \cdot \#\mathcal{M}_{\mathbf{p}}}{\#\mathcal{O}_{\lambda_j}}; \quad (4.6)$$

$$\mathbb{L}_{\lambda,\mathbf{p}} = \frac{\mathbb{P}_{\mathbf{p},\lambda} \cdot \#\mathcal{O}_\lambda}{\#\mathcal{M}_{\mathbf{p}}}, \quad \mathbb{L}_{\lambda_j,\mathbf{p}} = \frac{\mathbb{P}_{\mathbf{p},\lambda_j} \cdot \#\mathcal{O}_{\lambda_j}}{\#\mathcal{M}_{\mathbf{p}}}; \quad (4.7)$$

$$\mathbb{P}_{\mathbf{p}^*,\lambda} = q + 1 - \sum_{\mathbf{p} \in \mathfrak{M} \setminus \{\mathbf{p}^*\}} \mathbb{P}_{\mathbf{p},\lambda}, \quad \lambda \text{ is fixed}, \quad \mathbf{p}^* \in \mathfrak{M}; \quad (4.8)$$

$$\mathbb{P}_{\mathbf{p}^*,\lambda_j} = q + 1 - \sum_{\mathbf{p} \in \mathfrak{M} \setminus \{\mathbf{p}^*\}} \mathbb{P}_{\mathbf{p},\lambda_j}, \quad \lambda_j \text{ is fixed}, \quad \mathbf{p}^* \in \mathfrak{M}; \quad (4.9)$$

$$\mathbb{L}_{\lambda^*,\mathbf{p}} = q^2 + q + 1 - \sum_{\lambda \in \mathfrak{L} \setminus \{\lambda^*\}} \mathbb{L}_{\lambda,\mathbf{p}}, \quad \mathbf{p} \text{ is fixed}, \quad \lambda^* \in \mathfrak{L}. \quad (4.10)$$

Proof. The assertions directly follow from (4.2), (4.4), (4.5). \square

Remark 2. Let $q \equiv 0 \pmod{3}$. By Section 2, Γ -planes form a pencil with the A-line as the axis. Only lines lying in a Γ -plane can intersect the axis.

By definition, an EA-line necessarily intersects the A-line; therefore an EA-line always lies in a Γ -plane and intersects all the other lines belonging to this plane including the only tangent. Also, in [15, Tables 1, 2, Theorem 3.3(iv), Corollary 7.2] it is proved that we have exactly one Γ -plane through every EA-line and, in every Γ -plane, there are $q^2 - 1$ EA-lines such that $q^2 - q$ from them belong to the orbit $\mathcal{O}_{\text{EA}_1}$ while the remaining $q - 1$

ones are equally divided into the orbits $\mathcal{O}_{\text{EA}_2}$, $\mathcal{O}_{\text{EA}_3}$. In total, in every Γ -plane, we have $q^2 - 1$ intersections of EA-lines and the A-line.

In addition, by definition, every Γ -plane contains a tangent and q U Γ -lines intersecting the A-line in distinct points. Thus, in every Γ -plane, through a $(q+1)_\Gamma$ -point (i.e. a point of the A-line) we have a unisecant, the A-line, and $q - 1$ EA-lines.

Remark 3. By [15, Table 1, Theorem 3.3(vi)], all $q+1$ planes through an imaginary chord are $\overline{\text{I}}_{\mathcal{E}}$ -planes forming a pencil. The $\binom{q}{2}(q+1)$ -orbit of all $\overline{\text{I}}_{\mathcal{E}}$ -planes can be partitioned into $\binom{q}{2}$ pencils of planes having an imaginary chord as the axis. Only lines lying in a $\overline{\text{I}}_{\mathcal{E}}$ -plane can intersect an IC-line. If, in average, there are $\Pi_{\overline{\text{I}}_{\mathcal{E}},\lambda}$ $\overline{\text{I}}_{\mathcal{E}}$ -planes through a λ -line ($\lambda \neq \text{IC}$) then every λ -line intersects, in average, $\Pi_{\overline{\text{I}}_{\mathcal{E}},\lambda}$ IC-lines and contains, in average, $\Pi_{\overline{\text{I}}_{\mathcal{E}},\lambda}$ IC-points. So,

$$\mathbb{P}_{\text{IC},\lambda} = \Pi_{\overline{\text{I}}_{\mathcal{E}},\lambda}, \quad \lambda \in \mathfrak{L}. \quad (4.11)$$

whence, by (4.7), Theorem 1(ii)(a)(c), and [15, equation (4.9)], we have

$$\mathbb{L}_{\lambda,\text{IC}} = \frac{\mathbb{P}_{\text{IC},\lambda} \cdot \#\mathcal{O}_\lambda}{\#\mathcal{M}_{\text{IC}}} = \frac{\Pi_{\overline{\text{I}}_{\mathcal{E}},\lambda} \cdot \#\mathcal{O}_\lambda}{\#\mathcal{N}_{\overline{\text{I}}_{\mathcal{E}}}} = \Lambda_{\lambda,\overline{\text{I}}_{\mathcal{E}}}, \quad \lambda \in \mathfrak{L}. \quad (4.12)$$

Similarly, for the j -th orbit \mathcal{O}_{λ_j} we have

$$\mathbb{P}_{\text{IC},\lambda_j} = \Pi_{\overline{\text{I}}_{\mathcal{E}},\lambda_j}, \quad \mathbb{L}_{\lambda_j,\text{IC}} = \Lambda_{\lambda_j,\overline{\text{I}}_{\mathcal{E}}}, \quad j = 1, \dots, L_{\lambda\Sigma}, \quad \lambda \in \mathfrak{L}. \quad (4.13)$$

The values of $\Pi_{\overline{\text{I}}_{\mathcal{E}},\lambda}$, $\Pi_{\overline{\text{I}}_{\mathcal{E}},\lambda_j}$, $\Lambda_{\lambda,\overline{\text{I}}_{\mathcal{E}}}$, and $\Lambda_{\lambda_j,\overline{\text{I}}_{\mathcal{E}}}$ can be taken from [15, Tables 1, 2].

Theorem 8. Let $q \not\equiv 0 \pmod{3}$. Let $\pi \in \mathfrak{B}$, $\mathfrak{p} \in \mathfrak{M}^{(\neq 0)}$, and $\{\lambda_a, \lambda_b\} \subset \mathfrak{L}^{(\neq 0)}$ be such that $\mathcal{M}_{\mathfrak{p}}^{(\neq 0)}\mathfrak{A} = \mathcal{N}_\pi$ and $\mathcal{O}_{\lambda_a} = \mathcal{O}_{\lambda_b}\mathfrak{A}$. Then

$$\mathcal{M}_{\mathfrak{p}}^{(\neq 0)} = \mathcal{N}_\pi\mathfrak{A}, \quad \mathcal{O}_{\lambda_a}\mathfrak{A} = \mathcal{O}_{\lambda_b}. \quad (4.14)$$

Proof. By definition, see [20, Sections 2.1.5, 5.3], a polarity is involutory, i.e. $\mathfrak{A}^2 = \mathfrak{J}$, where \mathfrak{J} is the identity. Therefore, $\mathfrak{A}^{-1} = \mathfrak{A}$. \square

Corollary 3. Let $q \not\equiv 0 \pmod{3}$. The following holds:

$$\mathcal{M}_j^{(\neq 0)} = \mathcal{N}_j\mathfrak{A}, \quad \#\mathcal{M}_j^{(\neq 0)} = \#\mathcal{N}_j, \quad j = 1, \dots, 5; \quad \mathcal{M}_{\mathcal{E}}^{(\neq 0)} = \mathcal{N}_\Gamma\mathfrak{A}, \quad (4.15)$$

$$\begin{aligned} \mathcal{M}_\Gamma^{(\neq 0)} &= \mathcal{N}_{2_{\mathcal{E}}}\mathfrak{A}, \quad \mathcal{M}_{3_\Gamma}^{(\neq 0)} = \mathcal{N}_{3_{\mathcal{E}}}\mathfrak{A}, \quad \mathcal{M}_{1_\Gamma}^{(\neq 0)} = \mathcal{N}_{\overline{\text{I}}_{\mathcal{E}}}\mathfrak{A}, \quad \mathcal{M}_{0_\Gamma}^{(\neq 0)} = \mathcal{N}_{0_{\mathcal{E}}}\mathfrak{A}; \\ \mathcal{O}'_i\mathfrak{A} &= \mathcal{O}_i, \quad \#\mathcal{O}'_i = \#\mathcal{O}_i, \quad i = 1, \dots, 6, \quad q \not\equiv 0 \pmod{3}; \\ \mathcal{O}_{\text{RA}}\mathfrak{A} &= \mathcal{O}_{\text{RC}}, \quad \mathcal{O}_{\text{IA}}\mathfrak{A} = \mathcal{O}_{\text{IC}}, \quad \mathcal{O}_{\text{E}\Gamma}\mathfrak{A} = \mathcal{O}_{\text{Un}\Gamma}; \quad \mathcal{O}_\lambda\mathfrak{A} = \mathcal{O}_\lambda, \quad \lambda \in \{\text{T}, \text{U}\Gamma, \text{En}\Gamma\}. \end{aligned} \quad (4.16)$$

Proof. We use (2.12), Table 1, and Theorems 1(ii)(iii), 3, 8. \square

5 The numbers of λ -lines through \mathfrak{p} -points and of \mathfrak{p} -points on λ -lines, $q \not\equiv 0 \pmod{3}$

Remind that we consider $q \geq 5$; also $q \equiv \xi \pmod{3}$.

Notation 3 In addition to Notations 1 and 2 we denote the following:

$$\begin{aligned} \pi(\mathfrak{p}) \in \mathfrak{P} & \quad \text{the plane type such that } \mathcal{M}_{\mathfrak{p}}^{(\neq 0)} \mathfrak{A} = \mathcal{N}_{\pi(\mathfrak{p})}, \mathfrak{p} \in \mathfrak{M}^{(\neq 0)}, \xi \neq 0; \\ \lambda(\tilde{\lambda}) \in \mathfrak{L}^{(\neq 0)} & \quad \text{the line type such that } \mathcal{O}_{\lambda(\tilde{\lambda})} = \mathcal{O}_{\tilde{\lambda}} \mathfrak{A}, \tilde{\lambda} \in \mathfrak{L}^{(\neq 0)}, \xi \neq 0; \\ \lambda_j(\tilde{\lambda}_j) & \quad \text{the line type of the } j\text{-th orbit of the class } \mathcal{O}_{\lambda(\tilde{\lambda})} \text{ corresponding} \\ & \quad \text{to the } j\text{-th orbit of the class } \mathcal{O}_{\tilde{\lambda}} \text{ so that } \mathcal{O}_{\lambda_j(\tilde{\lambda}_j)} = \mathcal{O}_{\tilde{\lambda}_j} \mathfrak{A}. \end{aligned}$$

Theorem 9. *Let $q \not\equiv 0 \pmod{3}$. The following holds:*

$$\begin{aligned} \pi(\mathcal{C}) = \Gamma, \quad \pi(\mathbb{T}) = 2_{\mathcal{C}}, \quad \pi(3_{\Gamma}) = 3_{\mathcal{C}}, \quad \pi(1_{\Gamma}) = \overline{1_{\mathcal{C}}}, \quad \pi(0_{\Gamma}) = 0_{\mathcal{C}}; & \quad (5.1) \\ \lambda(\text{RC}) = \text{RA}, \quad \lambda(\text{RA}) = \text{RC}, \quad \lambda(\mathbb{T}) = \mathbb{T}, \quad \lambda(\text{IC}) = \text{IA}, \quad \lambda(\text{IA}) = \text{IC}, \\ \lambda(\text{U}\Gamma) = \text{U}\Gamma, \quad \lambda(\text{Un}\Gamma) = \text{E}\Gamma, \quad \lambda(\text{E}\Gamma) = \text{Un}\Gamma, \quad \lambda(\text{En}\Gamma) = \text{En}\Gamma; \\ \lambda_j(\text{U}\Gamma_j) = \text{U}\Gamma_j, \quad \lambda_j(\text{Un}\Gamma_j) = \text{E}\Gamma_j, \quad \lambda_j(\text{E}\Gamma_j) = \text{Un}\Gamma_j, \quad j = 1, 2. \end{aligned}$$

Proof. The assertions directly follow from (2.12), (2.14), (4.14)–(4.16), Theorems 1(iii), 3, 8, and Corollary 3. \square

Theorem 10. *Let $q \not\equiv 0 \pmod{3}$. Let $\mathfrak{p} \in \mathfrak{M}^{(\neq 0)}$, $\tilde{\lambda} \in \mathfrak{L}^{(\neq 0)}$. Then*

$$\mathbb{P}_{\mathfrak{p}, \tilde{\lambda}}^{(\xi)} = \Pi_{\pi(\mathfrak{p}), \lambda(\tilde{\lambda})}^{(\xi)}, \quad \mathbb{L}_{\tilde{\lambda}, \mathfrak{p}}^{(\xi)} = \Lambda_{\lambda(\tilde{\lambda}), \pi(\mathfrak{p})}^{(\xi)}; \quad \mathbb{P}_{\mathfrak{p}, \tilde{\lambda}_j}^{(\xi)} = \Pi_{\pi(\mathfrak{p}), \lambda_j(\tilde{\lambda}_j)}^{(\xi)}, \quad \mathbb{L}_{\tilde{\lambda}_j, \mathfrak{p}}^{(\xi)} = \Lambda_{\lambda_j(\tilde{\lambda}_j), \pi(\mathfrak{p})}^{(\xi)}.$$

Proof. We have $\mathcal{M}_{\mathfrak{p}}^{(\neq 0)} \mathfrak{A} = \mathcal{N}_{\pi(\mathfrak{p})}$, $\mathcal{O}_{\lambda(\tilde{\lambda})} = \mathcal{O}_{\tilde{\lambda}} \mathfrak{A}$. By Theorem 8, $\mathcal{M}_{\mathfrak{p}}^{(\neq 0)} = \mathcal{N}_{\pi(\mathfrak{p})} \mathfrak{A}$, $\mathcal{O}_{\lambda(\tilde{\lambda})} \mathfrak{A} = \mathcal{O}_{\tilde{\lambda}}$. The incidences between $\pi(\mathfrak{p})$ -planes and $\lambda(\tilde{\lambda})$ -lines are saved for \mathfrak{p} -points and $\tilde{\lambda}$ -lines. The same holds for orbits \mathcal{O}_{λ_j} . \square

Corollary 4. *Let $q \not\equiv 0 \pmod{3}$. Let $\mathfrak{p} \in \mathfrak{M}^{(\neq 0)}$. Let $\pi(\mathfrak{p}) \in \mathfrak{P}$ be as in (5.1). For $\xi = 1, -1$, the following holds:*

$$\begin{aligned} \mathbb{P}_{\mathfrak{p}, \text{RC}}^{(\xi)} &= \Pi_{\pi(\mathfrak{p}), \text{RA}}^{(\xi)}, \quad \mathbb{L}_{\text{RC}, \mathfrak{p}}^{(\xi)} = \Lambda_{\text{RA}, \pi(\mathfrak{p})}^{(\xi)}, \quad \mathbb{P}_{\mathfrak{p}, \text{RA}}^{(\neq 0)} = \Pi_{\pi(\mathfrak{p}), \text{RC}}, \quad \mathbb{L}_{\text{RA}, \mathfrak{p}}^{(\neq 0)} = \Lambda_{\text{RC}, \pi(\mathfrak{p})}; \\ \mathbb{P}_{\mathfrak{p}, \text{IC}}^{(\xi)} &= \Pi_{\pi(\mathfrak{p}), \text{IA}}^{(\xi)}, \quad \mathbb{L}_{\text{IC}, \mathfrak{p}}^{(\xi)} = \Lambda_{\text{IA}, \pi(\mathfrak{p})}^{(\xi)}, \quad \mathbb{P}_{\mathfrak{p}, \text{IA}}^{(\neq 0)} = \Pi_{\pi(\mathfrak{p}), \text{IC}}, \quad \mathbb{L}_{\text{IA}, \mathfrak{p}}^{(\neq 0)} = \Lambda_{\text{IC}, \pi(\mathfrak{p})}; \\ \mathbb{P}_{\mathfrak{p}, \text{Un}\Gamma}^{(\xi)} &= \Pi_{\pi(\mathfrak{p}), \text{E}\Gamma}^{(\xi)}, \quad \mathbb{L}_{\text{Un}\Gamma, \mathfrak{p}}^{(\xi)} = \Lambda_{\text{E}\Gamma, \pi(\mathfrak{p})}^{(\xi)}, \quad \mathbb{P}_{\mathfrak{p}, \text{E}\Gamma}^{(\neq 0)} = \Pi_{\pi(\mathfrak{p}), \text{Un}\Gamma}, \quad \mathbb{L}_{\text{E}\Gamma, \mathfrak{p}}^{(\neq 0)} = \Lambda_{\text{Un}\Gamma, \pi(\mathfrak{p})}; \\ \mathbb{P}_{\mathfrak{p}, \lambda}^{(\xi)} &= \Pi_{\pi(\mathfrak{p}), \lambda}^{(\xi)}, \quad \mathbb{L}_{\lambda, \mathfrak{p}}^{(\xi)} = \Lambda_{\lambda, \pi(\mathfrak{p})}^{(\xi)}, \quad \lambda \in \{\mathbb{T}, \text{U}\Gamma, \text{En}\Gamma\}. \\ \mathbb{P}_{\mathfrak{p}, \text{U}\Gamma_j}^{(\neq 0)} &= \Pi_{\pi(\mathfrak{p}), \text{U}\Gamma_j}^{(\neq 0)}, \quad \mathbb{L}_{\text{U}\Gamma_j, \mathfrak{p}}^{(\neq 0)} = \Lambda_{\text{U}\Gamma_j, \pi(\mathfrak{p})}^{(\neq 0)}, \quad \mathbb{P}_{\mathfrak{p}, \text{Un}\Gamma_j}^{(\xi)} = \Pi_{\pi(\mathfrak{p}), \text{E}\Gamma_j}^{(\xi)}, \end{aligned}$$

$$\mathbb{L}_{\text{Un}\Gamma_j, \mathfrak{p}}^{(\xi)} = \Lambda_{\text{E}\Gamma_j, \pi(\mathfrak{p})}^{(\xi)}, \quad \mathbb{P}_{\mathfrak{p}, \text{E}\Gamma_j}^{(\neq 0)} = \Pi_{\pi(\mathfrak{p}), \text{Un}\Gamma_j}, \quad \mathbb{L}_{\text{E}\Gamma_j, \mathfrak{p}}^{(\neq 0)} = \Lambda_{\text{Un}\Gamma_j, \pi(\mathfrak{p})}, \quad j = 1, 2.$$

Proof. We use Theorems 9 and 10. □

Now we are able to form Tables 2 and 3 using Corollary 4, and the values of $\Pi_{\pi, \lambda}^{(\xi)}$ and $\Lambda_{\lambda, \pi}^{(\xi)}$ from [15, Tables 1, 2].

6 The numbers of λ -lines through \mathfrak{p} -points and of \mathfrak{p} -points on λ -lines, $q \equiv 0 \pmod{3}$

In this section, we consider $q \equiv 0 \pmod{3}$. The values of $\#\mathcal{M}_{\mathfrak{p}}$, $\#\mathcal{O}_{\lambda}$, $\#\mathcal{O}_{\lambda_j}$, needed for (4.6), (4.7), are taken from (2.9)–(2.11) and Table 1. When we use (4.8)–(4.10), the values $\mathbb{P}_{\mathfrak{p}, \lambda}$, $\mathbb{P}_{\mathfrak{p}, \lambda_j}$, and $\mathbb{L}_{\lambda, \mathfrak{p}}$, obtained above, are summed up.

Note that if some of the relations (4.1)–(4.13) are not used directly in proofs then they can be used to check the results.

Theorem 11. *For RC-lines the following holds:*

$$\begin{aligned} \mathbb{P}_{\mathcal{C}, \text{RC}}^{(0)} &= 2, \quad \mathbb{P}_{\text{RC}, \text{RC}}^{(0)} = q - 1, \quad \mathbb{L}_{\text{RC}, \mathcal{C}}^{(0)} = q, \quad \mathbb{L}_{\text{RC}, \text{RC}}^{(0)} = 1, \\ \mathbb{P}_{\mathfrak{p}, \text{RC}}^{(0)} &= \mathbb{L}_{\text{RC}, \mathfrak{p}}^{(0)} = 0, \quad \mathfrak{p} \in \{(q+1)_{\Gamma}, \text{TO}, \text{IC}\}. \end{aligned}$$

Proof. By definition, a real chord contains exactly two \mathcal{C} -points; the other $q - 1$ points of the chord are RC-points. So, $\mathbb{P}_{\mathcal{C}, \text{RC}}^{(0)} = 2$, $\mathbb{P}_{\text{RC}, \text{RC}}^{(0)} = q - 1$. Also, by definition, there are q RC-lines through every \mathcal{C} -point, i.e. $\mathbb{L}_{\text{RC}, \mathcal{C}}^{(0)} = q$.

By Theorem 1(iv), two chords do not intersect each other off \mathcal{C} ; therefore, $\mathbb{L}_{\text{RC}, \text{RC}}^{(0)} = 1$, $\mathbb{P}_{\mathfrak{p}, \text{RC}}^{(0)} = \mathbb{L}_{\text{RC}, \mathfrak{p}}^{(0)} = 0$, $\mathfrak{p} \in \{\text{TO}, \text{IC}\}$.

Finally, by Remark 2, only lines lying in a Γ -plane can intersect the A-line. As an RC-line contains two \mathcal{C} -points, it cannot lie in a Γ -plane, whence $\mathbb{P}_{(q+1)_{\Gamma}, \text{RC}}^{(0)} = \mathbb{L}_{\text{RC}, (q+1)_{\Gamma}}^{(0)} = 0$. □

Theorem 12. *For T-lines the following holds:*

$$\begin{aligned} \mathbb{P}_{\mathcal{C}, \text{T}}^{(0)} &= \mathbb{P}_{(q+1)_{\Gamma}, \text{T}}^{(0)} = \mathbb{L}_{\text{T}, \mathcal{C}}^{(0)} = \mathbb{L}_{\text{T}, (q+1)_{\Gamma}}^{(0)} = \mathbb{L}_{\text{T}, \text{TO}}^{(0)} = 1, \quad \mathbb{P}_{\text{TO}, \text{T}}^{(0)} = q - 1, \\ \mathbb{P}_{\mathfrak{p}, \text{T}}^{(0)} &= \mathbb{L}_{\text{T}, \mathfrak{p}}^{(0)} = 0, \quad \mathfrak{p} \in \{\text{RC}, \text{IC}\}. \end{aligned}$$

Proof. By definition, a tangent contains exactly one \mathcal{C} -point and there is one tangent through every \mathcal{C} -point. Thus, $\mathbb{P}_{\mathcal{C}, \text{T}}^{(0)} = \mathbb{L}_{\text{T}, \mathcal{C}}^{(0)} = 1$. Also, a tangent lies in a Γ -plane and, hence, intersects the A-line. This implies $\mathbb{P}_{(q+1)_{\Gamma}, \text{T}}^{(0)} = 1$ whence, by (4.7), $\mathbb{L}_{\text{T}, (q+1)_{\Gamma}}^{(0)} = 1$. The remaining $q - 1$ points of the tangent are TO-points, i.e. $\mathbb{P}_{\text{TO}, \text{T}}^{(0)} = q - 1$.

By Theorem 1(iv), two chords do not intersect each other off \mathcal{C} ; therefore, $\mathbb{L}_{T,TO}^{(0)} = 1$, $\mathbb{P}_{RC,T}^{(0)} = \mathbb{P}_{IC,T}^{(0)} = \mathbb{L}_{T,RC}^{(0)} = \mathbb{L}_{T,IC}^{(0)} = 0$. \square

Theorem 13. *For IC-lines the following holds:*

$$\mathbb{P}_{IC,IC}^{(0)} = q + 1, \mathbb{L}_{IC,IC}^{(0)} = 1, \mathbb{P}_{p,IC}^{(0)} = \mathbb{L}_{IC,p}^{(0)} = 0, \mathbf{p} \in \{\mathcal{C}, (q+1)_\Gamma, TO, RC\}.$$

Proof. By definition, all points of an IC-line are IC-points; this implies $\mathbb{P}_{IC,IC}^{(0)} = q + 1$, $\mathbb{P}_{p,IC}^{(0)} = \mathbb{L}_{IC,p}^{(0)} = 0$, $\mathbf{p} \in \{\mathcal{C}, (q+1)_\Gamma, TO, RC\}$. By Theorem 1(iv), two IC-lines do not intersect each other; therefore, $\mathbb{L}_{IC,IC}^{(0)} = 1$. \square

Theorem 14. *For U Γ -lines the following holds:*

$$\begin{aligned} \mathbb{P}_{\mathcal{C},U\Gamma}^{(0)} = \mathbb{P}_{(q+1)_\Gamma,U\Gamma}^{(0)} = 1, \mathbb{L}_{U\Gamma,\mathcal{C}}^{(0)} = \mathbb{L}_{U\Gamma,(q+1)_\Gamma}^{(0)} = q, \mathbb{P}_{TO,U\Gamma}^{(0)} = \mathbb{L}_{U\Gamma,TO}^{(0)} = 0, \\ \mathbb{L}_{U\Gamma,RC}^{(0)} = \mathbb{L}_{U\Gamma,IC}^{(0)} = 1, \mathbb{P}_{RC,U\Gamma}^{(0)} = \mathbb{P}_{IC,U\Gamma}^{(0)} = \frac{1}{2}(q-1). \end{aligned}$$

Proof. By definition, a U Γ -line contains exactly one \mathcal{C} -point and there are q U Γ -lines through every \mathcal{C} -point, i.e. $\mathbb{P}_{\mathcal{C},U\Gamma}^{(0)} = 1$, $\mathbb{L}_{U\Gamma,\mathcal{C}}^{(0)} = q$. A U Γ -line lies in a Γ -plane and, hence, intersects the A-line. This implies $\mathbb{P}_{(q+1)_\Gamma,U\Gamma}^{(0)} = 1$ whence, by (4.7), $\mathbb{L}_{U\Gamma,(q+1)_\Gamma}^{(0)} = q$.

By definition, U Γ -lines and T-lines lie in Γ -planes. If a U Γ -line and a T-line belong to the same Γ -plane, then their common point is a \mathcal{C} -point. Otherwise they are skew. So, a U Γ -line cannot intersect a T-line off \mathcal{C} . As all TO-points are off \mathcal{C} , we have $\mathbb{P}_{TO,U\Gamma}^{(0)} = \mathbb{L}_{U\Gamma,TO}^{(0)} = 0$.

By [3, Table 2, Theorem 5.13(ii)], there is exactly one Γ -plane, say π_P , through an RC-point P . Let $Q \in \pi_P$ be the contact point of \mathcal{C} and π_P . By Theorem 1(iv), the line \overline{PQ} cannot be either a real chord or a tangent, hence \overline{PQ} is a U Γ -line. Thus, $\mathbb{L}_{U\Gamma,RC}^{(0)} = 1$. Similarly, it can be shown that $\mathbb{L}_{U\Gamma,IC}^{(0)} = 1$. Now, by (4.6), we obtain $\mathbb{P}_{RC,U\Gamma}^{(0)} = \mathbb{P}_{IC,U\Gamma}^{(0)} = (q-1)/2$. \square

Theorem 15. *For the A-line the following holds:*

$$\mathbb{P}_{(q+1)_\Gamma,A}^{(0)} = q + 1, \mathbb{L}_{A,(q+1)_\Gamma}^{(0)} = 1, \mathbb{P}_{p,A}^{(0)} = \mathbb{L}_{A,p}^{(0)} = 0, \mathbf{p} \in \{\mathcal{C}, TO, RC, IC\}.$$

Proof. By definition, all points of the A-line are $(q+1)_\Gamma$ -points and there is one A-line through every $(q+1)_\Gamma$ -point; this implies $\mathbb{P}_{(q+1)_\Gamma,A}^{(0)} = q + 1$, $\mathbb{L}_{A,(q+1)_\Gamma}^{(0)} = 1$, and $\mathbb{P}_{p,A}^{(0)} = \mathbb{L}_{A,p}^{(0)} = 0$, $\mathbf{p} \neq (q+1)_\Gamma$. \square

Theorem 16. *The following holds:*

$$\mathbb{P}_{\mathcal{C},Un\Gamma}^{(0)} = \mathbb{P}_{\mathcal{C},Un\Gamma_v}^{(0)} = 1, \mathbb{L}_{Un\Gamma,\mathcal{C}}^{(0)} = q^2 - q, \mathbb{L}_{Un\Gamma_v,\mathcal{C}}^{(0)} = \frac{1}{2}(q^2 - q), \quad v = 1, 2;$$

$$\mathbb{P}_{\mathcal{C},\lambda}^{(0)} = \mathbb{L}_{\lambda,\mathcal{C}}^{(0)} = 0, \quad \lambda \in \{\text{En}\Gamma, \text{EA}, \text{EA}_j\}, \quad j = 1, 2, 3.$$

Proof. By definition, a $\text{Un}\Gamma$ -line contains exactly one \mathcal{C} -point, i.e. $\mathbb{P}_{\mathcal{C},\text{Un}\Gamma}^{(0)} = \mathbb{P}_{\mathcal{C},\text{Un}\Gamma_v}^{(0)} = 1$, whence, by (4.7), $\mathbb{L}_{\text{Un}\Gamma,\mathcal{C}}^{(0)} = q^2 - q$, $\mathbb{L}_{\text{Un}\Gamma_v,\mathcal{C}}^{(0)} = (q^2 - q)/2$. Also, by definition, $\mathbb{P}_{\mathcal{C},\lambda}^{(0)} = 0$, $\lambda \in \{\text{En}\Gamma, \text{EA}, \text{EA}_j\}$, whence, by Corollary 1, $\mathbb{L}_{\lambda,\mathcal{C}}^{(0)} = 0$. \square

Theorem 17. *A $\text{Un}\Gamma$ - and an $\text{En}\Gamma$ -line cannot intersect the A-line whereas an EA-line necessary intersects it. The following holds:*

$$\begin{aligned} \mathbb{P}_{(q+1)\Gamma,\lambda}^{(0)} &= \mathbb{L}_{\lambda,(q+1)\Gamma}^{(0)} = 0, \quad \lambda \in \{\text{Un}\Gamma, \text{Un}\Gamma_v, \text{En}\Gamma\}, \quad v = 1, 2; \\ \mathbb{P}_{(q+1)\Gamma,\lambda}^{(0)} &= 1, \quad \lambda \in \{\text{EA}, \text{EA}_j\}, \quad j = 1, 2, 3; \\ \mathbb{L}_{\text{EA},(q+1)\Gamma}^{(0)} &= q^2 - 1, \quad \mathbb{L}_{\text{EA}_1,(q+1)\Gamma}^{(0)} = q^2 - q, \quad \mathbb{L}_{\text{EA}_j,(q+1)\Gamma}^{(0)} = \frac{1}{2}(q - 1), \quad j = 2, 3; \end{aligned}$$

where $\mathbb{P}_{(q+1)\Gamma,\text{EA}}^{(0)} = 1$ is the exact number of $(q+1)\Gamma$ -points on an EA-line.

Proof. By definition, $\text{Un}\Gamma$ - and $\text{En}\Gamma$ -lines do not lie in any Γ -plane; it implies, due to Remark 2, $\mathbb{P}_{(q+1)\Gamma,\lambda}^{(0)} = \mathbb{L}_{\lambda,(q+1)\Gamma}^{(0)} = 0$, $\lambda \in \{\text{Un}\Gamma, \text{Un}\Gamma_v, \text{En}\Gamma\}$. Also, by definition, an EA-line necessary intersects the A-line that gives $\mathbb{P}_{(q+1)\Gamma,\lambda}^{(0)} = 1$, $\lambda \in \{\text{EA}, \text{EA}_j\}$, as the exact value. Now, by (4.7), we obtain $\mathbb{L}_{\text{EA},(q+1)\Gamma}^{(0)}$ and $\mathbb{L}_{\text{EA}_j,(q+1)\Gamma}^{(0)}$. \square

Theorem 18. *For $\text{Un}\Gamma$ -lines the following holds:*

$$\begin{aligned} \mathbb{L}_{\text{Un}\Gamma,\text{TO}}^{(0)} &= \mathbb{L}_{\text{Un}\Gamma,\text{IC}}^{(0)} = q, \quad \mathbb{P}_{\text{TO},\text{Un}\Gamma}^{(0)} = 1, \quad \mathbb{P}_{\text{IC},\text{Un}\Gamma}^{(0)} = \frac{1}{2}q, \\ \mathbb{L}_{\text{Un}\Gamma,\text{RC}}^{(0)} &= q - 2, \quad \mathbb{P}_{\text{RC},\text{Un}\Gamma}^{(0)} = \frac{1}{2}(q - 2). \end{aligned}$$

Proof. Let \mathcal{T} be a tangent to \mathcal{C} at a point P . Let $B \in \mathcal{T}$ be a TO-point. Let ℓ be a line through B and one of the q points of $\mathcal{C} \setminus \{P\}$. By Theorem 1(iv), ℓ can be neither a real chord nor a tangent, hence it is a non-tangent unisecant. By [3, Table 2, Theorem 5.13(ii)], there is exactly one Γ -plane, say π_B , through the TO-point B . Obviously, $\mathcal{T} \in \pi_B$ and P is the contact point of \mathcal{C} and π_B . Thus, ℓ does not lie in a Γ -plane, i.e. ℓ is a $\text{Un}\Gamma$ -line and we have $\#\mathcal{C} \setminus \{P\}$ $\text{Un}\Gamma$ -lines through every TO-point. So, $\mathbb{L}_{\text{Un}\Gamma,\text{TO}}^{(0)} = q$, whence, by (4.6), $\mathbb{P}_{\text{TO},\text{Un}\Gamma}^{(0)} = 1$.

Let \overline{PQ} be a real chord through \mathcal{C} -points P and Q . Let $B \in \overline{PQ}$ be an RC-point. By [3, Table 2, Theorem 5.13(ii)], there is exactly one Γ -plane, say π_B , through the RC-point B . Let $R \in \pi_B$ be the contact point of \mathcal{C} and π_B . Let ℓ be a line through P and one of the $q - 2$ points of $\mathcal{C} \setminus \{P, Q, R\}$. By Theorem 1(iv), ℓ can be neither a real chord nor a tangent; also, $\ell \notin \pi_B$. Thus, ℓ is a $\text{Un}\Gamma$ -line and we have $\#\mathcal{C} \setminus \{P, Q, R\}$ $\text{Un}\Gamma$ -lines through every RC-point. So, $\mathbb{L}_{\text{Un}\Gamma,\text{RC}}^{(0)} = q - 2$, whence, by (4.6), $\mathbb{P}_{\text{RC},\text{Un}\Gamma}^{(0)} = (q - 2)/2$.

Finally, let \mathcal{IC} be an imaginary chord and $B \in \mathcal{IC}$ be an IC-point. By [3, Table 2], there is exactly one Γ -plane, say π_B , through B . Let $R \in \pi_B$ be the contact point of \mathcal{C} and π_B . Similarly to above, all the q lines through B and a point of $\mathcal{C} \setminus \{R\}$ are Un Γ -lines; this gives $\mathbb{L}_{\text{Un}\Gamma, \text{IC}}^{(0)} = q$, whence, by (4.6), $\mathbb{P}_{\text{IC}, \text{Un}\Gamma}^{(0)} = q/2$. \square

Theorem 19. *For TO-points the following holds:*

$$\begin{aligned} \mathbb{P}_{\text{TO}, \text{EA}}^{(0)} &= \frac{q}{q+1}, \quad \mathbb{L}_{\text{EA}, \text{TO}}^{(0)} = \mathbb{L}_{\text{EA}_1, \text{TO}}^{(0)} = q, \quad \mathbb{L}_{\text{En}\Gamma, \text{TO}}^{(0)} = q^2 - q, \\ \mathbb{P}_{\text{TO}, \text{EA}_1}^{(0)} &= \mathbb{P}_{\text{TO}, \text{En}\Gamma}^{(0)} = 1, \quad \mathbb{P}_{\text{TO}, \text{EA}_j}^{(0)} = \mathbb{L}_{\text{EA}_j, \text{TO}}^{(0)} = 0, \quad j = 2, 3. \end{aligned}$$

Proof. By Remark 2, in every Γ -plane, $q - 1$ EA-lines intersect the only tangent at its common point with the A-line while the remaining $q^2 - q$ ones intersect the tangent in TO-points. Thus, in total, there are $(q^2 - q) \cdot \#\mathcal{N}_\Gamma = (q^2 - q)(q + 1)$ TO-points on all $(q + 1)(q^2 - 1)$ EA-lines. The average number is $\mathbb{P}_{\text{TO}, \text{EA}}^{(0)} = q/(q + 1)$, whence, by (4.7), $\mathbb{L}_{\text{EA}, \text{TO}}^{(0)} = q$.

An EA-line intersects exactly one tangent either in its common point with the A-line or in a TO-point. Therefore, $\mathbb{P}_{\text{TO}, \text{EA}_j}^{(0)} \in \{0, 1\}$. If for $j = 2, 3$, we put $\mathbb{P}_{\text{TO}, \text{EA}_j}^{(0)} = 1$ then, by (4.7), we obtain $\mathbb{L}_{\text{EA}_j, \text{TO}}^{(0)} = 1/2$ that is not an integer, contradiction. So, $\mathbb{P}_{\text{TO}, \text{EA}_j}^{(0)} = \mathbb{L}_{\text{EA}_j, \text{TO}}^{(0)} = 0$, $j = 2, 3$, whence, by (4.3), $\mathbb{P}_{\text{TO}, \text{EA}_1}^{(0)} = 1$ and, by (4.7), $\mathbb{L}_{\text{EA}_1, \text{TO}}^{(0)} = q$.

Finally, by (4.10), $\mathbb{L}_{\text{En}\Gamma, \text{TO}}^{(0)} = q^2 - q$, whence, by (4.6), $\mathbb{P}_{\text{TO}, \text{En}\Gamma}^{(0)} = 1$. \square

Remark 4. By Remark 2 and Theorem 19, it can be seen that in every Γ -plane, the $q - 1$ EA-lines from the orbits $\mathcal{O}_{\text{EA}_2}$ and $\mathcal{O}_{\text{EA}_3}$ intersect the only tangent of this plane at its common point with the A-line (it is not a TO-point). At the same time, the $q^2 - q$ EA-lines from the orbit $\mathcal{O}_{\text{EA}_1}$ intersect the tangent in TO-points.

Theorem 20. *For IC-points the following holds:*

$$\begin{aligned} \mathbb{P}_{\text{IC}, \text{En}\Gamma}^{(0)} &= \frac{q^2 - q - 1}{2(q - 1)}, \quad \mathbb{L}_{\text{En}\Gamma, \text{IC}}^{(0)} = q^2 - q - 1; \quad \mathbb{P}_{\text{IC}, \text{EA}}^{(0)} = \frac{q^2}{2(q + 1)}, \quad \mathbb{L}_{\text{EA}, \text{IC}}^{(0)} = q; \\ \mathbb{P}_{\text{IC}, \text{EA}_1}^{(0)} &= \frac{1}{2}(q - 1), \quad \mathbb{L}_{\text{EA}_1, \text{IC}}^{(0)} = q - 1, \quad \mathbb{P}_{\text{IC}, \text{EA}_2}^{(0)} = \mathbb{L}_{\text{EA}_2, \text{IC}}^{(0)} = 0, \\ \mathbb{P}_{\text{IC}, \text{EA}_3}^{(0)} &= q, \quad \mathbb{L}_{\text{EA}_3, \text{IC}}^{(0)} = 1. \end{aligned}$$

Proof. The assertions follow from Remark 3 with (4.11)–(4.13) and [15, Tables 1, 2]. \square

Theorem 21. *For RC-points the following holds:*

$$\mathbb{P}_{\text{RC}, \text{En}\Gamma}^{(0)} = \frac{q^2 - q + 1}{2(q - 1)}, \quad \mathbb{L}_{\text{En}\Gamma, \text{RC}}^{(0)} = q^2 - q + 1;$$

$$\begin{aligned}\mathbb{P}_{\text{RC,EA}}^{(0)} &= \frac{q^2}{2(q+1)}, \quad \mathbb{L}_{\text{EA,RC}}^{(0)} = q; \quad \mathbb{P}_{\text{RC,EA}_1}^{(0)} = \frac{1}{2}(q-1), \quad \mathbb{L}_{\text{EA}_1,\text{RC}}^{(0)} = q-1, \\ \mathbb{P}_{\text{RC,EA}_2}^{(0)} &= q, \quad \mathbb{L}_{\text{EA}_2,\text{RC}}^{(0)} = 1, \quad \mathbb{P}_{\text{RC,EA}_3}^{(0)} = \mathbb{L}_{\text{EA}_3,\text{IC}}^{(0)} = 0.\end{aligned}$$

Proof. The values of $\mathbb{P}_{\text{RC},\lambda}^{(0)}$, $\mathbb{P}_{\text{RC},\lambda_j}^{(0)}$ are obtained by (4.8), (4.9). Then we obtain $\mathbb{L}_{\lambda,\text{RC}}^{(0)}$, $\mathbb{L}_{\lambda_j,\text{RC}}^{(0)}$ by (4.7). \square

Theorem 22. *For $\text{Un}\Gamma_v$ -lines, $v = 1, 2$, the following holds:*

$$\begin{aligned}\mathbb{P}_{\text{TO,Un}\Gamma_1}^{(0)} &= \mathbb{L}_{\text{Un}\Gamma_1,\text{TO}}^{(0)} = 0, \quad \mathbb{P}_{\text{TO,Un}\Gamma_2}^{(0)} = 2, \quad \mathbb{L}_{\text{Un}\Gamma_2,\text{TO}}^{(0)} = q, \\ \mathbb{P}_{\text{IC,Un}\Gamma_1}^{(0)} &= \mathbb{L}_{\text{Un}\Gamma_1,\text{IC}}^{(0)} = \frac{1}{2}(q+1), \quad \mathbb{P}_{\text{IC,Un}\Gamma_2}^{(0)} = \mathbb{L}_{\text{Un}\Gamma_2,\text{IC}}^{(0)} = \frac{1}{2}(q-1), \\ \mathbb{P}_{\text{RC,Un}\Gamma_1}^{(0)} &= \mathbb{L}_{\text{Un}\Gamma_1,\text{RC}}^{(0)} = \frac{1}{2}(q-1), \quad \mathbb{P}_{\text{RC,Un}\Gamma_2}^{(0)} = \mathbb{L}_{\text{Un}\Gamma_2,\text{RC}}^{(0)} = \frac{1}{2}(q-3).\end{aligned}$$

Proof. By Theorem 19, $\mathbb{P}_{\text{TO,En}\Gamma}^{(0)} = 1$ whence $\mathbb{P}_{\text{TO,Un}\Gamma_1}^{(0)} + \mathbb{P}_{\text{TO,Un}\Gamma_2}^{(0)} = 2$, by (4.3). If for $v = 1, 2$, we put $\mathbb{P}_{\text{TO,Un}\Gamma_v}^{(0)} = 1$, then, by (4.7), we obtain $\mathbb{L}_{\text{Un}\Gamma_v,\text{TO}}^{(0)} = q/2$ that is not an integer, contradiction. So, $\mathbb{P}_{\text{TO,Un}\Gamma_v}^{(0)} \in \{0, 2\}$.

By [15, Table 1], through a T-line we have one Γ -plane and q $2_{\mathcal{C}}$ -planes; also, every $2_{\mathcal{C}}$ -plane contains one T-line. Therefore, through a T-line and a $\text{Un}\Gamma$ -line meeting the T-line in a TO-point we have a $2_{\mathcal{C}}$ -plane. By [15, Table 2], there are one and three $2_{\mathcal{C}}$ -planes through a $\text{Un}\Gamma_1$ - and a $\text{Un}\Gamma_2$ -line, respectively. Therefore, a $\text{Un}\Gamma_1$ - and a $\text{Un}\Gamma_2$ -line intersect one and three T-lines, respectively. This means that $\mathbb{P}_{\text{TO,Un}\Gamma_1}^{(0)} = 0$, $\mathbb{P}_{\text{TO,Un}\Gamma_2}^{(0)} = 2$, and also every $\text{Un}\Gamma_v$ -line intersects one T-line at a \mathcal{C} -point. Now, by (4.7), we obtain $\mathbb{L}_{\text{Un}\Gamma_1,\text{TO}}^{(0)} = 0$, $\mathbb{L}_{\text{Un}\Gamma_2,\text{TO}}^{(0)} = q$.

By [15, Table 2], $\Pi_{\overline{1_{\mathcal{C}}},\text{Un}\Gamma_1} = \Lambda_{\text{Un}\Gamma_1,\overline{1_{\mathcal{C}}}} = (q+1)/2$, $\Pi_{\overline{1_{\mathcal{C}}},\text{Un}\Gamma_2} = \Lambda_{\text{Un}\Gamma_2,\overline{1_{\mathcal{C}}}} = (q-1)/2$ whence, by (4.13), $\mathbb{P}_{\text{IC,Un}\Gamma_1}^{(0)} = \mathbb{L}_{\text{Un}\Gamma_1,\text{IC}}^{(0)} = (q+1)/2$, $\mathbb{P}_{\text{IC,Un}\Gamma_2}^{(0)} = \mathbb{L}_{\text{Un}\Gamma_2,\text{IC}}^{(0)} = (q-1)/2$.

Finally, RC-points lie only on RC-lines. By [15, Table 1], through an RC-line there are two $2_{\mathcal{C}}$ -planes and $q-1$ $3_{\mathcal{C}}$ -planes. A $\text{Un}\Gamma$ -line lying in a $2_{\mathcal{C}}$ -plane intersects the RC-line of this plane at a \mathcal{C} -point. A $\text{Un}\Gamma$ -line lying in a $3_{\mathcal{C}}$ -plane intersects two RC-lines of this plane at a \mathcal{C} -point and one RC-line at an RC-point. Therefore, $\mathbb{P}_{\text{TO,Un}\Gamma_v}^{(0)} = \Pi_{3_{\mathcal{C}},\text{Un}\Gamma_v}^{(0)}$. By [15, Table 2], $\Pi_{3_{\mathcal{C}},\text{Un}\Gamma_1}^{(0)} = (q-1)/2$, $\Pi_{3_{\mathcal{C}},\text{Un}\Gamma_2}^{(0)} = (q-3)/2$ whence together with (4.7) we obtain the remaining assertions. \square

Now we form Tables 4 and 5 using the results of this section.

7 Some general results

Theorem 23. *Let $\mathfrak{p} \in \mathfrak{M}^{(\xi)}$. Let a class \mathcal{O}_λ consist of a single orbit. Then the submatrix $\mathcal{I}_{\mathfrak{p},\lambda}^{\text{PL}}$ of \mathcal{I}^{PL} is a (v_r, b_k) configuration of Definition 2 with $v = \#\mathcal{M}_{\mathfrak{p}}$, $b = \#\mathcal{O}_\lambda$, $r = \mathbb{L}_{\lambda,\mathfrak{p}}^{(\xi)}$,*

$k = \mathbb{P}_{\mathbf{p},\lambda}^{(\xi)}$. Also, up to rearrangement of rows and columns, the submatrices $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$ with $\mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)} = 1$ can be viewed as a concatenation of $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)}$ identity matrices of order $\#\mathcal{O}_\lambda$. The same holds for the submatrices $\mathcal{I}_{\mathbf{p},\lambda_j}^{\text{PL}}$.

Proof. As the class \mathcal{O}_λ is an orbit, $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$ contains $\mathbb{P}_{\mathbf{p},\lambda}$ (resp. $\mathbb{L}_{\lambda,\mathbf{p}}$) ones in every row (resp. column), see Lemma 1. In $\text{PG}(3, q)$, two lines are either skew or intersect at a point. Therefore, two points of $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$ are connected by at most one line. If $\mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)} = 1$, $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$ contains $\mathbb{P}_{\mathbf{p},\lambda}^{(\xi)}$ (resp. 1) ones in every row (resp. column). \square

Theorem 24. *Let $(\lambda, \mathbf{p}) \in \{(\text{U}\Gamma, \mathcal{C}), (\text{Un}\Gamma, \mathcal{C})\}$ if $q \not\equiv 0 \pmod{3}$, and $(\lambda, \mathbf{p}) \in \{(\text{Un}\Gamma, \mathcal{C}), (\text{EA}, (q+1)_\Gamma)\}$ if $q \equiv 0 \pmod{3}$. Then, independently of the number of orbits in the class \mathcal{O}_λ , we have exactly one \mathbf{p} -point on every λ -line. Up to rearrangement of rows and columns, the submatrices $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$ can be viewed as a vertical concatenation of $\mathbb{L}_{\lambda,\mathbf{p}}^{(\xi)}$ identity matrices of order $\#\mathcal{M}_\mathbf{p}$.*

Proof. By Tables 2–5, in the considered submatrices $\mathcal{I}_{\mathbf{p},\lambda}^{\text{PL}}$ there is exactly one unit in every row. \square

Theorem 25. *Let $q \equiv 1 \pmod{3}$. Let $V^{(1)} = \{\mathcal{O}_1 = \mathcal{O}_{\text{RC}}, \mathcal{O}_2 = \mathcal{O}_{\text{T}}, \mathcal{O}'_3 = \mathcal{O}_{\text{IA}}\}$. Then, cf. Theorem 1(iv), no two lines of $V^{(1)}$ meet off \mathcal{C} . Every point off \mathcal{C} lies on exactly one line of $V^{(1)}$.*

Proof. The assertions follow from Table 2. Also, they can be proved using Theorem 1(ii)(iv) with (2.10) and [26, Theorem 1]. \square

Theorem 26. *Let $q \equiv 0 \pmod{3}$. Let $\mathcal{W}^{(0)} = \{\mathcal{O}_2 = \mathcal{O}_{\text{T}}, \mathcal{O}_4 = \mathcal{O}_{\text{U}\Gamma}\}$. Let $\mathbb{M} = \mathcal{C} \cup \text{A-line}$ be the union of the twisted cubic and the A-line. Then no two lines of $\mathcal{W}^{(0)}$ meet off \mathbb{M} . Every point off \mathbb{M} lies on exactly one line of $\mathcal{W}^{(0)}$, cf. Theorems 1(iv) and 25.*

Proof. The assertions follow from Table 4. Note also that for $q \equiv 0 \pmod{3}$, Γ -planes form a pencil with the axis A-line and, in turn, in each plane $\pi_{\text{osc}}(t)$, q $\text{U}\Gamma$ -lines and the tangent form the pencil of lines through the point $P(t)$. \square

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