Shadowing property re(al)visited

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When modeling a time-evolving process, we obtain its approximate realizations. This proximity is due to several reasons. First, we never exactly know the description of the process itself, and second, the presence of various kinds of errors from purely random to rounding errors when implemented on a computer are inevitable. The question of the adequacy of the simulation results is primarily associated with the presence of a real trajectory of the process under study in the vicinity of the obtained realization over the longest possible time interval. This question is especially nontrivial in the case of a chaotic system, since for such systems close trajectories diverge very quickly (often exponentially fast).

At the level of connections between individual trajectories of a hyperbolic system and the corresponding pseudo-trajectories[1], this problem was first posed by D.V. Anosov [1] as a key step of the analysis of structural stability of diffeomorphisms. A similar but much less intuitive approach called "specification" in the same setting was proposed by R. Bowen [2]. Informally, both approaches ensure that errors do not accumulate during the process of modeling: in the systems with shadowing property each approximate trajectory can be uniformly traced by a true trajectory on the arbitrary long period of time. Naturally, this is of great importance in chaotic systems, where even an arbitrary small error in the starting position lead to (exponentially in time) large divergence of trajectories.

Further development demonstrated that for a diffeomorphism the shadowing property implies the uniform hyperbolicity. To some extent, this limits the theory of uniform shadowing to an important but very special class of hyperbolic dynamical systems. The concept of average shadowing introduced in [3] about 30 years ago gave a possibility to extend significantly the range of perturbations under consideration in the theory of shadowing, in particular to be able to deal with perturbations which are small only on average but not uniformly.

The most notorious in the variety of obstacles in the analysis of the shadowing property is that one needs to take into account an infinite number of independent perturbations of the original system. This makes the problem highly nonlocal. It is therefore very desirable to reduce the shadowing problem to the situation with a single perturbation, albeit with tighter control of the approximation accuracy.

To realize this idea in our recent paper [4] we developed a fundamentally new "gluing" construction, consisting in the effective approximation of a pair of consecutive segments of true trajectories.

We restrict ourselves to discrete time dynamical systems, leaving the extension of our approach to continuous time systems (flows) for future research. A discrete time dynamical system is completely defined by a non-necessarily invertible map $T : X \to X$ from a metric space (X, ρ) into itself.

A *trajectory* of the map T starting at a point $x \in X$ is a sequence of points $\vec{x} := \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\} \subset X$, for which $x_0 = x$ and $Tx_i = x_{i+1}$ for all available indices i.

[1] Approximate trajectories of a system under small perturbations.

A *pseudo-trajectory* of the map T is a sequence of points $\vec{y} := \{\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots\} \subset X$, for which the sequence of distances $\{\rho(Ty_i, y_{i+1})\}$ for all available indices i satisfies a certain "smallness" condition.

For a given $\varepsilon > 0$ we say that a pseudo-trajectory $\vec{y}$ is of

(U) *uniform* type, if $\rho(Ty_i, y_{i+1}) \leqslant \varepsilon$ for all available indices i.

(A) *small on average* type, if $\limsup\limits_{n\to\infty} \frac{1}{2n+1} \sum\limits_{i=-n}^{n} \rho(Ty_i, y_{i+1}) \leqslant \varepsilon$.

The idea of *shadowing* in the dynamical systems theory boils down to the question is it possible to approximate pseudo-trajectories of a given dynamical system by true trajectories? Naturally, the answer depends on the type of the approximation.

We say that a true trajectory $\vec{x}$ *shadows* a pseudo-trajectory $\vec{y}$ with accuracy δ (notation δ-shadows):

(U) *uniformly*, if $\rho(x_i, y_i) \leqslant \delta$ for all available indices i.

(A) *on average*, if $\limsup\limits_{n\to\infty} \frac{1}{2n+1} \sum\limits_{i=-n}^{n} \rho(x_i, y_i) \leqslant \delta$.

We say that a DS (T, X, ρ) satisfies the $(\alpha + \beta)$-*shadowing property* (notation $T \in \mathcal{S}(\alpha, \beta)$) with $\alpha \in \{U, A, A', R\}$, $\beta \in \{U, A\}$ if $\forall \delta > 0\ \exists \varepsilon > 0$ such that each ε-pseudo-trajectory of α-type can be shadowed in the β sense with the corresponding accuracy δ.

We say that a trajectory $\vec{z}$ *glues* together semi-trajectories $\vec{x}, \vec{y}$ with accuracy rate $\varphi : \mathbb{Z} \to \mathbb{R}_+$ *strongly* if

$$\rho(x_k, z_k) \leqslant \varphi(k)\rho(x_0, y_0) \quad \forall k < 0, \qquad \rho(y_k, z_k) \leqslant \varphi(k)\rho(x_0, y_0) \quad \forall k \geqslant 0$$

In other words $\vec{z}$ approximates both the backward part of $\vec{x}$ and the forward part of $\vec{y}$ with accuracy controlled by the rate function φ.

We say that the DS (T, X, ρ) satisfies the *gluing property* with the rate-function $\varphi : \mathbb{Z} \to \mathbb{R}$ (notation $T \in G(\varphi)$) if for any pair of trajectories $\vec{x}, \vec{y}$ there is a trajectory $\vec{z}$, which glues them at time $t = 0$ with accuracy φ in the strong/weak sense.

Our main result is the following statement.

Theorem 1. *Let $T : X \to X$ be a map from a metric space (X, ρ) into itself, and let $T \in G(\varphi)$ with $\sum_k \varphi(k) < \infty$. Then $T \in \mathcal{S}(U, U) \cup \mathcal{S}(A, A)$.*

Now we are going to demonstrate our approach for some important classes of dynamical systems (in particular, for non-invertible and discontinuous ones).

Example 1. (Affine mapping) Let $X := \mathbb{R}^d$ with $d \geqslant 1$ with the euclidean metric ρ, A be a $d \times d$ matrix, and $a \in \mathbb{R}^d$. Then $Tx := Ax + a \in \mathcal{S}(U, U) \cup \mathcal{S}(A, A)$ if and only if $E^n = \emptyset$ and either $E^s = \emptyset$ or $E^u = \emptyset$.

Here E^0, E^s, E^u, E^n are linear subspaces of $\mathbb{R}^d$, corresponding to the zero eigenvalue of A, remaining contracting part of A, expanding part of A, and the neutral part (corresponding to eigenvalues of modulys 1).

Example 2. (Anosov diffeomorphism) Let $X := \mathbb{T}^2$ be a unit 2-dimensional torus and let $T : X \to X$ be a uniformly hyperbolic diffeomorphism. Then $T \in \mathcal{S}(U, U) \cup \mathcal{S}(A, A)$.

Example 3. (Nonuniform hyperbolicity) $X := [0,1],\ \alpha,\beta \geqslant 0,\ 0 < c < 1$

$$Tx := \begin{cases} x(1+ax^\alpha) & \text{if } x \leqslant c \\ 1-(1-x)(1+b(1-x)^\beta) & \text{if } x > c \end{cases} .$$

Then $T \in \mathcal{S}(U,U) \cup \mathcal{S}(A,A)$ iff $0 \leqslant \alpha,\beta < 1$ and $c(1+ac^\alpha) = (1-c)(1+b)^\beta = 1$.

References

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