Invariant measures of torus piecewise isometries

Michael Blank[*][†]

September 2, 2021

Abstract

We study measure-theoretical aspects of torus piecewise isometries. Not much is known about this type of dynamical systems, except for the special case of one-dimensional interval exchange mappings. The last case is fundamentally different from the general situation in the presence of an invariant measure (Lebesgue measure), which helps a lot in the analysis. Due to the absence of good methods of analysis of general systems with discontinuities, even the existence of invariant measures of the torus piecewise isometries was an open question. We establish sufficient conditions for the existence/absence of invariant measures for this class of systems. Technically, our results are based on the approximation of the maps under study by weakly periodic ones.

1 Introduction

By now, we have learned reasonably well how to study hyperbolic (locally expanding/contracting or both) chaotic dynamical systems, thanks to a large extent to the development of the so called operator approach. Contrary to this not much is known about piecewise isometries, except for a special case of one-dimensional interval exchange transformations (IET) and a few similar very exceptional multidimensional systems. It is worth noting that IETs are fundamentally different from the general situation in the clear presence of an invariant measure (Lebesgue measure), which helps a lot in the analysis.

While the IET represent mainly pure theoretical constructions, the general piecewise isometries appear naturally in various applications like contemporary methods of machine learning, some piecewise smooth physical models (in particular Fermi-Ulam models), etc. (see, e.g. [9] for applications in engineering, and further references about other possibilities therein). Indeed, local translations of an IET disrupt its structure completely, while preserve the class of general piecewise isometries. Note also that the maps from this class in general are non-invertible, which adds possibilities for applications for "real life" modeling, but also additional problems for their analysis.

In general from a measure-theoretical point of view one of the first steps in the analysis of a dynamical system is the study of its invariant measures. In some cases one can

[*]Institute for Information Transmission Problems RAS (Kharkevich Institute);

[†]National Research University "Higher School of Economics"; e-mail: blank@iitp.ru

easily find a "good" invariant measure (e.g., for IETs), or to prove its existence if the map, defining the dynamical system, is continuous or satisfies some special properties (e.g., piecewise expanding). For systems with singularities (e.g., discontinuities) even the question about the existence of an invariant measure might become a difficult problem. The class of piecewise isometries represents an example of the latter sort. The point is that the methods (e.g., operator approach or symbolic dynamics) well established for other types of chaotic dynamical systems do not work in this setting and one needs to look for new approaches, one of which will be proposed in the present paper.

Let us start with a simple example of an isometry in $\mathbb{R}^2$ defined by the map $\tilde{T}x := \mathcal{R}x + b$, where $x, b \in \mathbb{R}^2$ are vectors and $\mathcal{R}$ is a rotation matrix. Applying the same map to the 2-dimensional torus X we get $Tx := \tilde{T}x \,(\mathrm{mod}\,1)$, which is no longer a global isometry. Moreover, the map T is not a bijection and has both lines of discontinuities and regions of points having no preimages. On the other hand, one can easily make a partition of X into a finite number of regions, such that the restriction of T to each region is an isometry (see Section 7 for details). This is a basic model for our study. Some nontrivial invariant sets for such a map are demonstrated at Fig. 1. We are interested in asymptotic properties of the dynamics of piecewise isometric torus maps like the one shown in this figure.
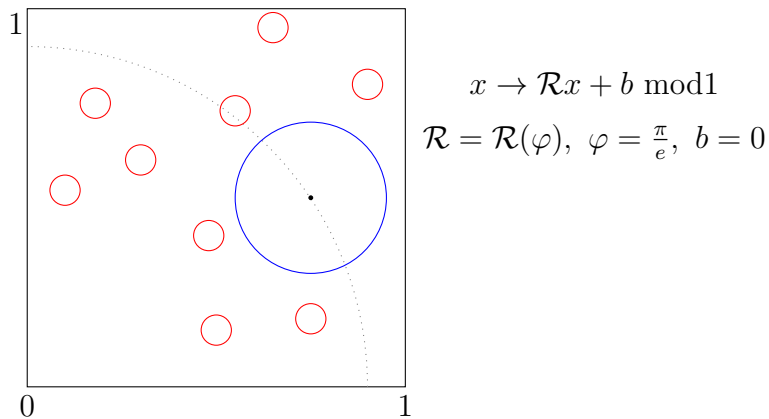

$$x \to \mathcal{R}x + b \;\mathrm{mod}\,1$$
$$\mathcal{R} = \mathcal{R}(\varphi), \;\; \varphi = \tfrac{\pi}{e}, \;\; b = 0$$

Figure 1: Invariant sets of the torus rotation by the angle π/e

Recall that an isometry is a map which preserves distances between all points in the phase space. Thus a piecewise isometry (PWI) is a map T from a space X equipped with a special partition $\{X_i\}$ into itself such that the restriction of T to each element of the partition preserves distances (i.e. is a true isometry). An example of a torus PWI is demonstrated at Fig. 2.

Naturally, the list of available isometries sensitively depends on the choice of a metric. In particular, under the discrete metric each bijection turns out to be an isometry. We will consider a natural family of translational invariant metrics $\tilde{\rho}_p(x, y) := \left(\sum_i |x_i - y_i|^p\right)^{\frac{1}{p}}$, $p \geq 1$, where x_i, y_i are the coordinates of the points $x, y \in X \in \mathbb{R}^d$. The Euclidean metric corresponds to $\tilde{\rho}_2(\cdot, \cdot)$, while the uniform metric to $\tilde{\rho}_\infty(\cdot, \cdot)$. The corresponding family of torus metrics is defined as $\rho_p(x, y) := \min_{z \in \mathbb{Z}^d} \tilde{\rho}_p(x - z, y)$.

The paper is organized as follows. In Section 2 we give a brief historical description of results related to piecewise isometries and motivations of the present research. Section 3 is dedicated to basic definitions and the formulation of the main result – Theorem 3.1. Examples demonstrating the possibility of the absence of invariant measures for PWI and

$$X = \sqcup_{i=1}^{5} X_i$$

$$X_i \to TX_i$$

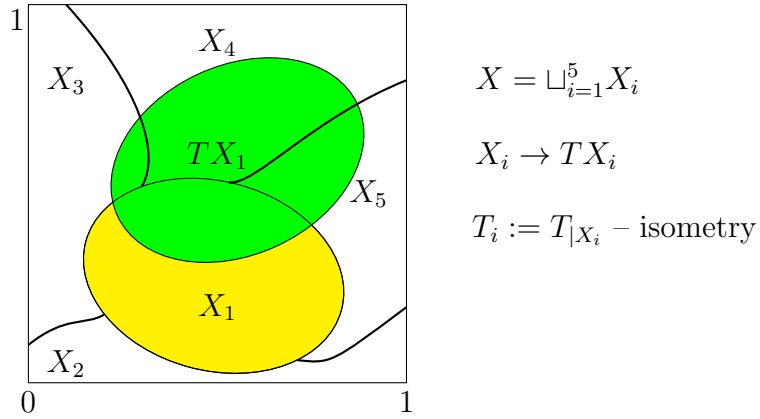$$T_i := T_{|X_i} - \text{isometry}$$

Figure 2: A special partition for a torus isometry. One of the elements of the partition and its image are marked in yellow and green colors correspondingly.

the focusing phenomenon (due to which this happens) will be described in Section 4. In Section 5 we introduce and study weakly periodic maps and semigroups. In Section 6 we apply the approximation by weakly periodic maps to prove Theorem 3.1. Finally in Section 7 we study a seemingly simple example of a PWI: torus restriction of a plane rotation. Note that this situation is excluded from Theorem 3.1 by the extendability assumption.

Acknowledgments. The author is grateful to Gregory Olshansky and Leonid Polterovich for useful discussions.

2 Historical remarks

We start this short review with the discussion of known results related to "bijective" PWI $T : X \to X$, when the union of the images under T of the elements of the special partition $\{X_i\}$ covers the entire phase space X. Since in the multidimensional setting ($d > 1$) this situation is rather exotic, we will discuss only the one-dimensional case, known in the literature as interval exchange transformations (IET). Quotes marks in the word "bijective" emphasize a problem with the boundary points of the elements of the partition $\Gamma := \cup_i \partial X_i$. In fact, the authors either assume that either $TX_i \cap TX_j \in T(\Gamma)$ for $i \neq j$ (may intersect only on boundary points), or instead of the partition consider the covering $\{\text{Clos} X_i\}$. The former case leads to an endomorphism,[1] while the latter to a multi-valued map, because the map is not uniquely defined at Γ. This is especially unpleasant because under the action of a multi-valued map a total measure of the phase space may grow, which cannot happen for a conventional single-valued map. Fortunately, in the case of IET for any version of the action on boundary points the Lebesgue measure is preserved under dynamics. Therefore, since Γ is the set of zero Lebesgue measure, one does not care much about these peculiarities.

Analysis of IETs is a very popular field of research and the list of publications related to them is very long and still actively growing. Therefore we mention only some basic publications in a historical order and refer the reader to look for further reference therein.

---

[1] Except for orientation preserving PWI with a very special structure of the partition, consisting of semi-closed intervals of type $X_i := [a_i, a_{i+1})$.

The first results and important constructions related to orientation preserving IET were described in V.I. Oseledets (1966) [19], M. Keane (1975) [13], A. Katok (1980) [12], W. Veech (1982) [23], H. Masur (1982) [16], M. Viana (2006) [24], A.I. Bufetov(2006) [5]. Starting already from the work of W. Veech the main object of interest here was shifted from a pure dynamical point of view to connections with geometry, in particular with Teichmuller flows. Since the latter goes out of the scope of the present paper, we will not discuss this.

In the present paper we are interested mainly in measure-theoretic points of view to dynamics, which is relatively simple and well understood in the case of IETs.

The situation is becoming rather different if a PWI is not "bijective" even in the above mentioned weak sense. This means that the images of the elements of the special partition might have arbitrary intersections. The main body of results in this direction are again about the one-dimensional orientation preserving case, known in the literature as interval translation maps (ITM). The most interesting among them in a historical order belongs to M. Boshernitzan & I. Kornfeld (1995) [3], J. Buzzi & P. Hubert (2004) [6], H. Bruin (2007,2012) [4], S. Marmi, P. Moussa, J-C. Yoccoz (2012) [17], D. Volk (2014) [25]. A further generalization when a general cover is used instead of a special partition (which inevitably leads to a multi-valued map) was considered in A. Skripchenko & S. Troubetzkoy (2015) [21]. In any case the problem with the boundary points already mentioned in the case of IET is becoming a serious obstacle here. Nevertheless from a topological point of view or in the discussion of attractors one can go around it in order to reduce the analysis of ITM to a better undrestood IET. A popular approach introduced in [3] for the case of the circle orientation preserving PWI is to classify them into finite and infinite types with respect to the number of iterations under which the images of the phase space are stabilized. Namely, the finite type is characterized by the property $T^{n+1}X = T^n X$ for some $n < \infty$, and otherwise we refer to an infinite type. Indeed, for the finite type case in the one-dimensional orientation preserving case one obtains an ITM in a finite number of iterations.

A number of attempts to follow this idea in the multidimensional setting have been tried (see, e.g. [25, 22] and further references therein). Unfortunately, in distinction to the one-dimensional case, the multidimensional dynamics is much more complicated and cannot be reduced to some version of ITM, which has been demonstrated by A. Goetz in [11]. In his example X is an isosceles triangle divided by the special partition into two smaller isosceles triangles X_1 and X_2. Under the action of the piecewise isometry T the smaller triangles are rotated one clockwise and another anticlockwise in such a way that their images intersect only along boundaries and their closures cover X completely. Similarly to the discussion above, there is a problem how to define the action of T on the boundary of the special partition, but on the level of numerics this is not so important. The aim of the author was to study numerically the structure of limit sets under dynamics, which turns out to be extremely complicated. Moreover the simulation indicates that this structure is self similar up to a very fine scales. This conjecture is still not proven on the mathematical level.

From the most interesting for us measure-theoretic point of view, there are no results at all for $d > 1$ and only a few in the one-dimensional setting, which we will discuss now. The main problem here is that there is no obvious candidate for an invariant measure, while the images of a good enough reference measure in general might concentrate at boundary points, which complicates their analysis. In [6] it has been shown that in the absence of periodic points the number of ergodic invariant measures of an orientation preserving

circle PWI cannot be larger than the number of elements of the special partition. It is somewhat strange that having this answer the authors even did not consider a question whether at least one invariant measure is present. Observe that PWI maps are necessarily discontinuous and the classical Krylov-Bogolyubov argument cannot be applied. The first step to overcome this difficulty was done by B. Pires (2016) [20], where the existence was proven under some (uncheckable) technical assumptions.[2] Let us mention also two other papers [10, 22], where the analysis again was based on a number of (uncheckable) technical assumptions. The first and the only real result about the existence of invariant measures of an orientation preserving circle PWI was obtained by S. Kryzhevich (2020) [14] by means of rational approximations. We will discuss this approach in some detail in Section 3.

3   Setup

Let X be a subset of the Euclidean space $\mathbb{R}^d$, $d \geq 1$ equipped with a certain metric $\rho(\cdot, \cdot)$ belonging to the class of $\tilde{\rho}_p(x, y) := (\sum_i |x_i - y_i|^p)^{\frac{1}{p}}$, $p \geq 1$, and let $\{X_i\}$ be a partition of X into disjoint regions. By a *region* we mean a convex set with a nonempty interior. The union of boundary points of all regions we denote by Γ, which by definition is of zero d-dimensional Lebesgue measure.

Definition 3.1 A *piecewise isometry (PWI)* is a map $T : X \to X$, satisfying the property that its restriction to each region X_i is an isometry. We will refer to $\{X_i\}$ as a *special* partition, and to the maps $T_i|_{X_i}$ as *local* maps.

Definition 3.2 A restriction $T_{|X_i}$ is said to be an *extendable isometry* if it can be extended to an isometry T_i on the entire X such that $T_i|_{X_i} \equiv T_{|X_i}$. Correspondingly a piecewise isometry T is said to be *extendable* if its restriction to each region X_i is an extendable isometry.

As we will see the innocent looking assumption of extendability is in fact somewhat restricting. Collecting well known geometric facts (see e.g. [18, 1]) we get the following

Proposition 3.1 *An isometry from a region $Y \subset \mathbb{T}^d$ to $\mathbb{T}^d$ is extendable if and only if it can be represented as a finite superposition of basic torus isometries: translations and coordinate flips or exchanges.*

The main result of our study is the following theorem about extendable torus piecewise isometries.

Theorem 3.1 *Let T be an extendable PWI of the unit torus X with a finite special partition $\{X_i\}$. Then there exists at least one probabilistic T-invariant measure μ_T. Additionally, if the boundary set Γ contains no periodic points, this measure is non-atomic.*

We expect that the extendability assumption can be avoided or significantly weaken, but the methods used in this article make extensive use of this assumption. In the one-dimensional orientation preserving case the claim of Theorem 3.1 was proven in [14], but in the general (even one-dimensional) setting this claim is new. Moreover, in [14] the

---

[2]Basically this is a version of the Krylov-Bogolyubov theorem, specialized for discontinuous maps under study.

non-atomic property of invariant measures has been proven without the assumption of the absence of periodic points at Γ. One of the key-points there was the demonstration that a periodic trajectory of a PWI cannot be isolated.[3] It was expected that even in the one-dimensional non orientation preserving case the situation will be the same. This turns out to be wrong, and a counterexample will be discussed in the last part of the proof of Proposition 7.3.

To emphasize that the claim of Theorem 3.1 is far from being obvious, we will construct examples of extendable piecewise isometries without invariant measures, albeit with a countable special partition (see Section 4).

A general scheme of the proof of Theorem 3.1 is as follows. First, we consider a family of torus isometries $\{g_\theta\}$ satisfying the property that any finite subcollection of such isometries generates a group $G(g_{\theta_1}, \ldots, g_{\theta_n})$ having a finite number of elements. We refer to this property as weak periodicity.[4] In Section 5 we will analyze properties of weakly periodic maps. In the case of commuting generators the situation is rather trivial and an important point here is that we are able to study some important cases when at least some of the the generators g_θ do not commute.

On the 2-nd step we consider torus piecewise isometries T such that the restrictions T_{X_i} can be extended to the torus isometries belonging to the above family. In Section 5 we study properties of such maps and prove for them the claim of Theorem 3.1.

Finally in Section 6 we approximate an arbitrary extendable torus piecewise isometry by means of weakly periodic ones considered in Section 5, which will allow to finish the proof of Theorem 3.1.

It is worth noting, that this result does not contradict to the possibility of having attractors admitting singular invariant measures. Various numerical results (see e.g. [11, 1]) give some evidence of the possibility of such behavior. The point is that the corresponding measures might not be represented as limits of invariant measures of maps discussed in Section 5 and hence cannot be controlled by our construction.

A similar approach was used in [14], where a specialized version of the above construction was applied for the case of orientation preserving circle piecewise rotations. In that case the author used the approximation scheme based on rational translations and special partitions with rational boundary points. The last fact is a crucial point not allowing to use this approximation in the multidimensional setting. Indeed, for $d > 1$ the boundaries of regions cannot consist of points with rational coordinates only.

4 Focusing phenomenon and absence of invariant measures

A natural approach to prove the existence of an invariant measure of a piecewise continuous dynamical system $(T, \mathbb{T})$ is to consider a sequence of images $\{T^n m\}_n \geq 0$ of the Lebesgue measure m on the torus $\mathbb{T}$. Let μ be a limit point of this sequence. If the μ-measure of a set Γ of points of discontinuity of the map T is 0, then under some simple technical assumptions the measure μ is T-invariant. The opposite situation, when the images of the Lebesgue measure concentrates on a small set we refer as the focusing

---

[3]A trajectory is isolated if all nearby trajectories have other itineraries.
[4]See Section 5 for discussion.

phenomenon. To be precise let us introduce the *focusing set*:

$$S(T) := \liminf_{n \to \infty} T^n X,$$

where $\liminf_{n \to \infty} A_n := \cup_n \cap_{k \geq n} A_k$ is the set of points belonging to all but finitely many sets A_n. Then the "size" of the set $S(T)$ describes the focusing quantitatively.

The focusing or concentration phenomenon is well known for systems with (locally) stable periodic orbits. The focusing occurs exactly to these periodic orbits, and the main reason for this is the (locally) contracting nature of the map T. In the situation with piecewise isometries we do not observe the contraction anywhere even locally.

In this section we discuss how the focusing may occur for a PWI. In distinction to other parts of this paper we allow the special partition to be countable. The focusing possibilities for the case of the finite special partition will be discussed in the end of the Section. The main idea under the following two examples is that under the action of the maps under study any measure converges weakly (focuses) to a measure μ supported by a very small wandering set. Recall that the latter means that all its forward images under dynamics do not intersect with the set. Therefore the measure μ cannot be invariant. Examples of this sort are known in the case of piecewise contractive maps, where the phenomenon of the measure concentration is natural (see e.g. [2]), but look somewhat surprising for the piecewise isometry.

Example 4.1 *Let $X := [0,1)^2$ and the special partition $\{X_n\}$ be defined as follows: $X_n := (1 - 2^{-n+1}, 1 - 2^{-n}] \times [0,1), \quad n \geq 1$, while $Tx := (x_1 + 2^{-n-1}, x_2)$ if $x \in X_n$.*

$$X_n := (1 - 2^{-n+1}, 1 - 2^{-n}] \times [0,1)$$
$$X := \cup_{n \geq 1} X_n = [0,1) \times [0,1)$$
$$Tx := (x_1 + 2^{-n-1}, x_2) \text{ if } x \in X_n$$
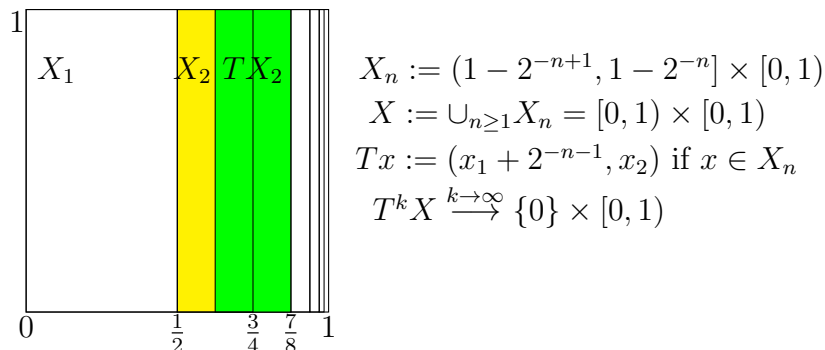$$T^k X \xrightarrow{k \to \infty} \{0\} \times [0,1)$$

Figure 3: Example of a piecewise torus isometry without invariant measures. The element X_2 of the special partition and its image are marked in yellow and green colors correspondingly.

Observe that after exactly two iterations each element of the partition $\{X_n\}$ will be mapped to the union of elements with larger indices. In this situation obviously any measure converges under the action of T to a measure supported by the unit segment $\{0\} \times [0,1)$. This segment is an wandering set, since its forward images under the action of T do not intersect with the segment, which proves the absence of invariant measures.

Nevertheless, in the Example 4.1 we do not have a complete concentration of measure, namely the situation when the limit (but not invariant) measure is a δ-measure at a single point. To achieve this goal instead of the basically one-dimensional example above, we consider a bit more involved d-dimensional map.

Example 4.2 *Let $X := [0,1)^d$ and the special partition $\{X_n\}$ be defined as follows:*

$$d(1 - \frac{1}{n}) \leq \sum_{i=1}^{d} x_i < d(1 - \frac{1}{n+1}),$$

while $Tx := x + \frac{1}{2n(n+1)}\bar{1} \,(\text{mod } 1)$ for $x \in X_n$, where $\bar{1}$ is the d-dimensional vector with unit coordinates.

$$X_n := \{x: \ 2(1 - \frac{1}{n}) \leq x_1 + x_2 < 2(1 - \frac{1}{n+1})\}$$
$$X := \cup_{n \geq 1} X_n = [0,1) \times [0,1)$$
$$Tx := x + \frac{1}{2n(n+1)}\bar{1} \,(\text{mod } 1) \text{ if } x \in X_n$$
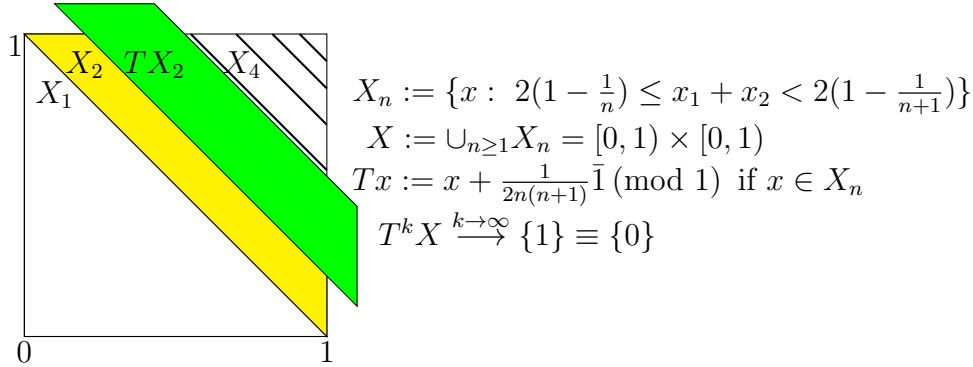$$T^k X \xrightarrow{k \to \infty} \{1\} \equiv \{0\}$$

Figure 4: Example of a piecewise torus isometry converging to $\bar{1}$. The element X_2 of the special partition and its image are marked in yellow and green colors correspondingly.

In distinction to the previous example the image of X_n may intersect with elements X_k with smaller indices $k < n$. Since the diameters of the elements of the partition decrease when their indices grow, after the implementation of the operation (mod 1) peripheral parts of the image may be located below the central part and intersect with the elements with lower indices. In fact, $TX_n \cap X_1 \neq \emptyset \ \forall n$. Nevertheless the Hausdorff distance from the set $T^k X_n$ to the point $\bar{1}$ goes to 0 when $k \to \infty$. Therefore under the action of T any measure weakly converges to $\delta_{\bar{1}}$, while the point $\bar{1} \equiv 0$ is again an wandering set.

If the special partition is finite we cannot provide examples with the focusing to the set of zero Lebesgue measure. Instead we try to estimate the "size" of the focusing set from bellow.

Example 4.3 *Let $X := [0,1)$ and the special partition $\{X_n\}$ consists of N intervals of equal length. Let $\alpha := \{\alpha_i\}_{i=1}^{n}$ be a collection of numbers and let $T_\alpha x := x + \alpha_i \,(\text{mod } 1)$ for $x \in X_i$. We are interested in the focusing for this family of maps, namely in*

$$\kappa := \inf_\alpha m(S(T_\alpha)).$$

For a given α, the set T_α is a typical one-dimensional orientation preserving piecewise isometry. Siince the images of the elements of the special partition may intersect, the value of κ needs not to be close to one. However, since the map is locally isometric, by the same telescoping construction as in the examples 4.1, 4.2 one can achieve $\kappa = \frac{1}{N}$. From the first sight it seems that this estimate cannot be significantly improved. To demonstrate that this is absolutely not the case choose an arbitrary small parameter $\varepsilon > 0$ and set $\alpha_1 := \varepsilon, \alpha_2 := -\varepsilon, \alpha_n := -\frac{1}{N}$ for $n > 2$. In other words, $T_\alpha X_{n+1} = X_n$ for $n > 3$ (telescoping), but the images of the first two elements intersect by the set of length 2ε. It is easy to calculate that $S(T_\alpha) := [\frac{1}{N} - \varepsilon, \frac{1}{N} + \varepsilon]$. Since $\varepsilon > 0$ is arbitrary, we get $\kappa = 0$.

5   Weakly periodic semigroups of maps

The main aim of this Section is to introduce the class of weakly periodic torus piecewise isometries and to prove Theorem 3.1 for them.

Definition 5.1 A collection of maps $\{g_i\}_{i=1}^n$ from a set X into itself is said to be *weakly periodic* (notation WPer) if the number of elements in the semigroup generated by them is finite, i.e. $\#G(g_1, \ldots, g_n) < \infty$.

The reason under this terminology is that a periodic homeomorphism T (i.e. $T^n = Id$) generates a semigroup with a finite number of elements, i.e. $T \in$ WPer.

The simplest (and very instructive) example of a weakly periodic semigroup of maps is a semigroup generated by a finite number of rational circle translations.

An important observation about this property gives the following result.

Proposition 5.1 *Let $g_i \circ g_j = g_j \circ g_i$ $\forall i, j$, then $\sum_i \#G(g_i) < \infty$ implies $\#G(g_1, \ldots, g_n) < \infty$. Otherwise there exists a pair of non-commuting maps g_1, g_2 such that $\sum_{i=1}^2 \#G(g_i) = 4$, but $\#G(g_1, g_2) = \infty$.*

Proof.   Each element $T \in G(g_1, \ldots, g_n)$ may be represented in the form of a finite superposition of the generators g_i, and due to their commutativity we get

$$Tx = g_1^{k_1} \circ \ldots \circ g_n^{k_n} x, \quad 0 \le k_i = k_i(x) < \infty \quad \forall x \in X.$$

By the assumption $\exists N < \infty$ such that $\#G(g_i) < N$ $\forall i$. Therefore,

$$\#(\cup_{T \in G} Tx) \le N^n < \infty \quad \forall x \in X,$$

which implies the first claim.

The 2nd claim is not so obvious as one might expect. In particular, in the one-dimensional case it is wrong for orientation preserving maps. The latter property turns out to be crucial here. Let $X := [0, 1)$, $g_1(x) := -x \,(\mathrm{mod}\ 1)$, $g_2(x) := -x + \alpha \,(\mathrm{mod}\ 1)$, $\alpha \notin \mathbb{Q}$. Observe that $\#G(g_i) = 2, i = 1, 2$, but $(g_1 g_2)^n x = x - n\alpha \,(\mathrm{mod}\ 1)$, which implies $\#G(g_1, g_2) = \infty$ since $\alpha \notin \mathbb{Q}$.   □

The definition of the weakly periodic maps implies the following simple technical result.

Proposition 5.2 *Let $\{g_i\}_{i=1}^n \in WPer$ and let a PWI map $T : X \to X$ with a finite special partition $\{X_i\}_{i=1}^n$ such that $T|_{X_i} \equiv g_i|_{X_i}$ $\forall i$. Then $\forall x \in X$ the trajectory $\{T^n x\}_{n \ge 0}$ starting at x is eventually periodic.*

Proof.   The claim follows immediately from the observation that the total number of different points obtained by succesive applications of the generators g_i is finite.   □

Despite that according to Proposition 5.1 in general the WPer property for individual maps does not imply this property for a collection of them, the situation becomes more optimistic when we restrict ourselves to torus isometries.

As we already noted in Proposition 3.1, any d-dimensional torus isometry may be represented by a superposition of basic isometries of the following 3 types:

(i) translation: $x \to x + v \,(\mathrm{mod}\ 1)\,, \quad v \in \mathbb{R}^d$

(ii) coordinate flip: $x_i \to -x_i$

(iii) coordinate exchange: $x_i \leftrightarrow x_j$

The maps belonging to different types do not commute, but at least the maps from the first two types commute with the maps of the same type, however the maps of type (iii) do not commute even between themselves. Weak periodicity of the maps from the first type is equivalent to their periodicity, while the maps of the other two types are always 2-periodic.

To apply the machinery developed in Proposition 5.4 one needs to show that a PWI generated by a finite collection of periodic basic isometries satisfy the weak periodicity property.

Theorem 5.1 *Let T be a torus PWI such that $T|_{X_i}$ is a finite composition of periodic basic isometries $\forall i$. Then the map $T \in WPer$.*

The proof of this intuitively obvious claim (to which we devote the remaining part of this Section) turns out to be surprisingly cumbersome. In particular, we are not able to perform the proof directly for the map T itself and are instead embed it to a certain semigroup generated by symmetrized torus extensions of the local isometries.

Definition 5.2 By a *symmetrized semigroup* $G^*(g_1, \ldots, g_n)$ generated by basic torus isometries $\{g_i\}$ we mean a semigroup generated by the maps $\{g_i\}$, where each translation map $x \to x + v \,(\mathrm{mod}\ 1)$ is considered together with all permutations and sign changes of the coordinates of the vector v, similarly each coordinate flip or exchange is considered together with all other coordinate flips or exchanges respectively. In the same manner by $(\{g_i\})^*$ we mean a symmetrized collection of basic torus isometries $\{g_i\}$.

Fortunately to the finite dimension of the torus, the number of additional generators in the symmetrized semigroup is finite as well.[5]

Proposition 5.3 *Let $\{g_i\}_{i=1}^n$ be a collection of periodic basic torus isometries. Then $\#G^*(g_1, g_2, \ldots, g_n) \leq \prod_{i=1}^n \#G^*(g_i) < \infty$.*

Proof. We proceed by induction. The claim for $n = 1$ is trivial, except for its finiteness. If g_i belongs to (i) or (ii) types, then the finiteness follows from the periodicity assumption together with the commutation property. If g_i is of the 1st type, the last argument cannot be applied, but instead one gets a direct estimate $G^*(g_i) \leq d!$ since all possible coordinate permutations are taken into account.

Let us make the induction step from n to $n + 1$. Observe that the application of the semigroup $G^*(g_1, g_2, \ldots, g_n)$ to a given point $x \in X$, i.e. the set $G^*(g_1, g_2, \ldots, g_n)x$ is invariant with respect to each of the maps $(\{g_i\}_{i=1}^n)^*$. The latter means that

$$g(G^*(g_1, g_2, \ldots, g_n)x) = G^*(g_1, g_2, \ldots, g_n)x \qquad \forall g \in (\{g_i\}_{i=1}^n)^*.$$

We will show that for any periodic basic isometry g_{n+1} the set $G^*(g_{n+1})G^*(g_1, g_2, \ldots, g_n)x$ is invariant with respect to each of the maps $(\{g_i\}_{i=1}^{n+1})^*$, which implies the claim.

Denote $A := G^*(g_1, g_2, \ldots, g_n)x$, $B := G^*(g_{n+1})G(g_1, g_2, \ldots, g_n)x$ and consider all possibilities:

---

[5]It cannot be enlarged in more than $(d!)^3$ times.

- g_{n+1} is a translation. If A is invariant with respect to some translation g_i, then the the invariance of B with respect to g_i follows from the commutation property. The invariance of A with respect to a coordinate flip is equivalent to the fact that the set A is a uniform lattice on the torus. The latter property cannot be destroyed by a torus translation. If A is invariant with respect to a coordinate exchange, then this set is characterized by the symmetry with respect to the corresponding unit cube diagonal. Unfortunately the translation g_{n+1} *may destroy* this invariance for B. Indeed a symmetric pair $(a, b), (b, a)$ may by mapped to $(a+c, b), (b+c, a)$. This is exactly the point, where we need the symmetrization, because $G^*(g_{n+1})A$ clearly preserves the symmetry under question.

- g_{n+1} is a coordinate flip. If A was invariant with respect to some translation, $B := G^*(g_{n+1})A$ preserves this invariance due to the symmetrization. The invariance of A with respect to a coordinate flip g_i implies the same property for B, since coordinate flips commute. The invariance of A with respect to a coordinate exchange is inherited by B again helps to the symmetrization with respect to the coordinate flips.

- g_{n+1} is a coordinate exchange $x_i \leftrightarrow x_j$. This situation can be analyzed in exactly the same manner and we skip it.

The induction step is proven. $\qquad\square$

This result finishes the proof of Theorem 5.1. Indeed, denote the torus extensions of the local isometries $T|_{X_i}$ by g_i and assume that the maps g_i are finite superpositions of periodic basic torus isometries. Then the trajectory of length n under the action of T of any given point $x \in X$ is a subset of $G^*(g_1, \ldots, g_n)x$, which is uniformly finite on x and n by Proposition 5.3. This proves the weak periodicity of the map T. $\qquad\square$

Additionally from the weak periodicity of the map T in this setting we get some important information about its invariant measures.

Proposition 5.4 *Let $\{g_i\}_{i=1}^n$ be a collection of torus diffeomorphisms and let $\#G(g_1, \ldots, g_n) < \infty$. Let T be a piecewise isometry of the unit torus X with the finite partition $\{X_i\}$ and such that each $T|_{X_i}$ can be extended to a torus map $T_i \in G(g_1, \ldots, g_n)$. Then there exists at least one absolutely continuous invariant measure, being a restriction of the Lebesgue measure to a certain region.*

Proof. Denote by $\tilde{\Gamma} := \cup_{g \in G(g_1, \ldots, g_n)} g^{-1}\Gamma$ – the union of all pre-images under the action of all maps $g \in G(g_1, \ldots, g_n)$ of the boundary Γ of the elements of the partition $\{X_i\}$. By the weak periodicity assumption $\tilde{\Gamma}$ divides the torus X into a finite number of regions $\{Y_j\}$ of positive Lebesgue measure, such that points belonging to the same region $\{Y_j\}$ have the same itineraries under the action of the map T. Therefore each of the regions Y_i is either periodic under the action of T, or is wandering. The latter case is out of interest for us, while a collection of periodically exchanging regions $Y_{i_1}, \ldots, Y_{i_k}$ supports the Lebesgue measure on them being invariant with respect to the map T. Moreover, all absolutely continuous T-invariant measures may be described this way. $\qquad\square$

6 Approximation

In what follows to control the rate of recurrence we will need the following abstract result.

Lemma 6.1 *Let $(T, X, \mathcal{B}, \mu)$ be a measurable dynamical system, let $A \in \mathcal{B}$, $\mu(A) > 0$, and let $\tau_A(x)$ be the first return time for a point $x \in A$ to return to the set A under the action of T. Then $\mu(\{x \in A : \ \tau_A(x) \leq \frac{1}{\mu(A)}\}) > 0$.*

Before to prove this result, observe that according to the famous Kac Lemma (see e.g. [8]) in the ergodic case the mathematical expectation of the first return time is exactly equal to $1/\mu(A)$. On the other hand, in our setting we cannot a priori assume ergodicity without which the claim of Lemma 6.1 is as best as possible.

Proof. By the Poincare Recurrence Theorem μ-a.a. points return to A under the action of T. Let $A_n := \{x \in A : \ n \leq \tau_A(x) < \infty\}$ be the set of points returning to A at least after $n \geq 0$ iterations, and suppose $A \setminus A_n = \emptyset \implies \mu(A) = \mu(A_n)$. Since $T^{-k}A_n \cap A = \emptyset \ \forall k < n$, we get $T^{-i}A_n \cap T^{-j}A_n = \emptyset \ \forall i \neq j < n$ (otherwise the points from the intersection above will not return to A). Hence

$$1 \geq \mu(\cup_{k=0}^{n-1} T^{-k} A_n) = \sum_{k=0}^{n-1} \mu(T^{-k} A_n) = n\mu(A)$$

and thus $n \leq 1/\mu(A)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Let $\{T^{(n)}\}$ be a sequence of piecewise torus isometries with the common finite special partition $\{X_i\}$, $\Gamma := \cup_i \partial X_i$, satisfying the assumptions of Proposition 5.4, and let $\{\mu_n\}$ be a sequence of their absolutely continuous invariant measures.

Lemma 6.2 *Let $z \notin \mathrm{Per}(T)$. Then $\forall \varepsilon > 0 \ \exists \delta > 0 \ \limsup\limits_{n \to \infty} \mu_n(B_\delta(z)) < \varepsilon$.*

Proof. Assume from the contrary that there is $\varepsilon > 0$ and a pair of sequences $\{n_k\} \overset{k \to \infty}{\longrightarrow} \infty$ and $\{\delta_k\} \overset{k \to \infty}{\longrightarrow} 0$ such that $\mu_{n_k}(B_{\delta_k}(z)) \geq \varepsilon$. Then by Lemma 6.1

$$\exists N < 1/\varepsilon : \ \ (T^{(n_k)})^N B_{\delta_k}(z) \cap B_{\delta_k}(z) \neq \emptyset,$$

which implies that $z \in \mathrm{Per}(T)$. We came to the contradiction. $\qquad\qquad\qquad\qquad$ $\square$

Since we deal with a compact phase space, the sequence of the above mentioned absolutely continuous invariant measures of the approximating weakly periodic maps has a limit point. Denote this limit point by $\tilde{\mu}$ and choose a sequence of the measures $\{\mu_n\}$ converging to it in the weak topology.

Lemma 6.3 *Let $\Gamma \cap \mathrm{Per}(T) = \emptyset$. Then $\tilde{\mu}(\Gamma) = 0$ and $\forall k, \varphi \in C^0(X, \mathbb{R})$ we have*

$$\lim_{n \to \infty} \mu_n(1_{\mathrm{Clos}(X_k)} \cdot \varphi) = \tilde{\mu}(1_{\mathrm{Clos}(X_k)} \cdot \varphi).$$

Proof. The first statement follows from Lemma 6.2. To prove the second statement without loss of generality we assume that $|\varphi(x)| \leq 1$. Using again Lemma 6.2 we get: $\forall \varepsilon > 0 \, \exists \delta > 0$ such that $\mu_n(B_\delta(\partial X_k)) < \varepsilon$. Choose now a nonnegative function $h \in C^0(X, [0,1])$ such that $h_{|X_k \setminus B_\delta(\partial X_k)} \equiv 1$ and $h_{|X \setminus B_\delta(\partial X_k)} \equiv 0$. Then $\lim_{n\to\infty} \mu_n(h\varphi) = \tilde{\mu}(h\varphi)$. by the definition of the weak convergence of measures.

Therefore
$$\lim_{n\to\infty} |\mu_n(\varphi) - \tilde{\mu}(\varphi)| \leq m(B_\delta(\Gamma)),$$

which implies the result since the value of $\varepsilon > 0$ can be taken arbitrarily small. $\qquad\square$

Lemma 6.4 *Let* $\Gamma \cap \mathrm{Per}(T) = \emptyset$ *and let* $\{T^{(n)}\}$ *be a sequence of approximating weakly periodic maps, whose invariant measures* $\mu_n \overset{n\to\infty}{\Longrightarrow} \tilde{\mu}$. *Then*

$$\mu_n(\varphi \circ T^{(n)} - \varphi \circ T) \overset{n\to\infty}{\longrightarrow} 0 \quad \forall \varphi \in C^0(X, \mathbb{R}).$$

Proof. Given $\varepsilon > 0$ by Lemma 6.2 we can choose $\delta > 0$ such that $\mu_n(B_\delta(\Gamma)) < \varepsilon$.[6] From the approximation assumption $\exists n_0$ such that $|\varphi \circ T^{(n)}(x) - \varphi \circ T(x)| < 0 \, \forall n > n_0$ and $x \notin B_\delta(\Gamma)$. This implies the claim. $\qquad\square$

Now we are ready to finish the proof of Theorem 3.1.

First, observe that each map, satisfying the assumption of Theorem 3.1, can be approximated arbitrary well by weakly periodic ones. Indeed, by since $T|_{X_i}$ is extendable it can be represented by a finite superposition basic isometries. If there are isometries of type (i) among them, they can be approximated arbitrary well by rational translations (which are periodic). Basic isometries of other types are periodic from the very beginning.

The second step is to check that "good" invariant measures of the weakly periodic approximations, existing by Theorem 5.1 and Proposition 5.4, converge to a T-invariant measure.

Lemma 6.5 *If* $\exists \gamma \in \Gamma \cap \mathrm{Per}(T)$, *then the measure uniformly distributed on the trajectory of* γ *is* T-*invariant. Otherwise, if* $\Gamma \cap \mathrm{Per}(T) = \emptyset$ *the measure* $\tilde{\mu}$ *constructed in Lemmas above is* T-*invariant.*

Proof.
$$|\tilde{\mu}(\varphi - \varphi \circ T)| \leq |\tilde{\mu}(\varphi - \varphi \circ T) - \mu_n(\varphi - \varphi \circ T)|$$
$$+ |\mu_n(\varphi - \varphi \circ T^{(n)})| + |\mu_n(\varphi \circ T^{(n)} - \varphi \circ T)|.$$

The first term in the rhs goes to 0 as $n \to \infty$ by Lemma 6.3, the second term vanishes for each n since the measures μ_n are $T^{(n)}$-invariant, while the third term goes to 0 by Lemma 6.4. Hence $\tilde{\mu}(\varphi) = \tilde{\mu}(\varphi \circ T)$, which implies the claim. $\qquad\square$

---

[6]Note that the convexity of the elements of the special partition is important at this point. Otherwise one needs to assume that the boundaries of those elements are smooth enough.

7   Torus restriction of a plane isometry

For the family of torus metrics $\rho_p(\cdot,\cdot)$ for all $p \neq 2$ the list of extendable isometries (translations, coordinate flips and coordinate exchanges) coincides with all possible isometries. However in the important Euclidean case $\rho_2(\cdot,\cdot)$ there is an additional isometry – rotation, which is not extendable. The techniques developed in this paper does not allow to consider local rotations, because we do not have their weakly periodic approximations.

In this section we consider actions of plane isometries restricted to the unit two-dimensional torus $X := \mathbb{T}^2$ as the simplest models local rotations. Some results in this direction (mainly about topological properties) may be found in [15] (see further references therein). Recall that a general plane isometry can be written as $x \to \mathcal{R}x + b$, where $x, b \in \mathbb{R}^2$ and $\mathcal{R}$ is an orthogonal matrix. Restricting this map to the torus we get the torus map $Tx := \mathcal{R}x + b \,(\mathrm{mod}\ 1) : \mathbb{T}^2 \to \mathbb{T}^2$.

Example 7.1 *Let $\mathcal{R} = \mathcal{R}(\varphi)$ be the rotation around the origin by the angle $\varphi \in [0, 2\pi)$ and let $b = 0$.*

Having in mind that that each trajectory of the plane rotation belongs to a single circle around the origin, while a typical torus translation $x \to x + b \,(\mathrm{mod}\ 1)$ fills the torus densely, one naively expects something of the same sort for the map T. It turns out that if the rotation angle $\varphi \neq 0$ the situation is completely different (see Fig. 1).

The map T defined above, considered as a map from the unit torus $X := \mathbb{T}^2$ into itself is continuous everywhere inside the open unit square $\tilde{X} := (0,1)^2$ and the set of discontinuities is represented by the boundary of this set $\partial \tilde{X}$. On the other hand, the same map may be considered as a map from the unit square $\hat{X} := [0,1]^2$ into itself. In the latter case the set of discontinuous is much more involved (see Fig. 7).
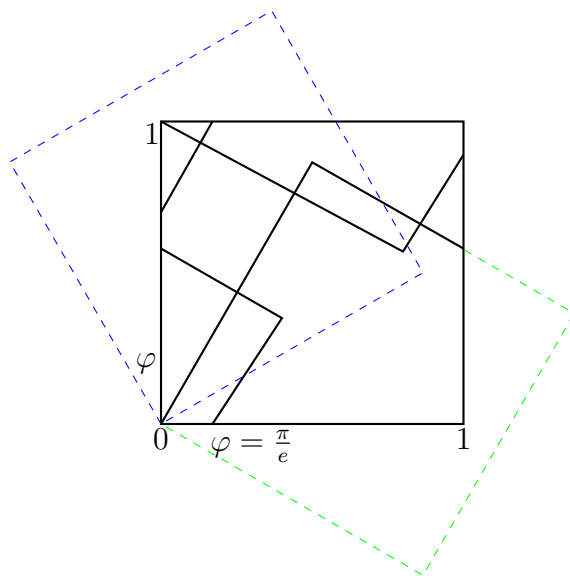

Figure 5: Thin blue/green lines – the boundaries of forward/backward images of the boundaries of the unit square under the rotation by the angle π/e around the origin. Thick black lines – boundaries of the special partition.

If $\det(I - \mathcal{R}) \neq 0$ there is at least one fixed point of the map $Tx := \mathcal{R}x + b \,(\mathrm{mod}\ 1)$. Here and in the sequel by I we denote the unit matrix. Indeed, making the change

of variables $x = y + a$, we rewrite the equation for the fixed point of the map T as $y + a = \mathcal{R}(y + a) + b \,(\mathrm{mod}\ 1)$, which gives a solution $y := 0$, $a := (I - \mathcal{R})^{-1}b \,(\mathrm{mod}\ 1)$. Thus $z := (I - \mathcal{R})^{-1}b \,(\mathrm{mod}\ 1)$ is the fixed point. If additionally, $b = 0$ (case of a pure rotation) the origin is always the fixed point even without the assumption about the absence of the degeneracy of the matrix $I - \mathcal{R}$

Therefore the main question is about the existence of nontrivial periodic trajectories. Our numerical observations allow to formulate the following

Hypothesis. *Let $Tx := j\mathcal{R}(\varphi)x + b \,(\mathrm{mod}\ 1)$, where $j = \pm 1$, $\varphi \in [0, 2\pi)$, $x, b \in \mathbb{R}^2$. Then $\forall j, \varphi, b\ \ \exists N = N(j, \varphi, b) \in \mathbb{Z}_+$ and $z = z(j, \varphi, b, N)$ such that $T^N z = z$. Moreover for a given triple j, φ, b one may find arbitrary large N satisfying the above property.*

An idea of the possible proof. We consider only the case of the pure rotation (i.e. $j = 1$, $b = 0$) by the irrational angle (i.e. $\varphi = \gamma\pi$, $\gamma \notin \mathbb{Q}$), leaving the general case (where a number of cumbersome combinatorial type calculations are necessary) for the future.

Observe that $Tx = \mathcal{R}x \,(\mathrm{mod}\ 1) = \mathcal{R}x + \xi(x)$, where $\xi(x) \in \mathbb{Z}^2$. Therefore to find a periodic point of period 1 one needs to solve the equation:

$$z = \mathcal{R}z + \xi_1,$$

and for the period 2:

$$z = \mathcal{R}(\mathcal{R}z + \xi_1) + \xi_2 = \mathcal{R}^2 z + \mathcal{R}\xi_1 + \xi_2$$

Continuing this, for a point of period n we have

$$z = \mathcal{R}^n z + \sum_{j=1}^{n} \mathcal{R}^{n-j}\xi_j,$$

which is equivalent to

$$(I - \mathcal{R}^n)z = \sum_{j=1}^{n} \mathcal{R}^{n-j}\xi_j.$$

If the angle φ is irrational, then the inverse matrix $(I - \mathcal{R}^n)^{-1}$ is well defined. Thus we have an explicit solution

$$z := (I - \mathcal{R}^n)^{-1} \sum_{k=1}^{n} \mathcal{R}^{n-k}\xi_k.$$

Unfortunately this is not the end of the story, since in fact the integer vectors ξ_i depend on z and should be chosen in such a way that $\mathcal{R}^k z + \sum_{j=1}^{n} \mathcal{R}^{n-j}\xi_j$ belong to the unit square for each $1 \le k \le n$, otherwise further corrections need to be taken into account.

Let us formulate and prove a partial result leaving the general case for future studies.

Proposition 7.1 *Let $\varphi \notin [-\varphi_0, \varphi_0]$, where $\tan \frac{\varphi_0}{2} = \frac{1}{2}$. Then the equation $A(\varphi)x = x$ always has a nontrivial solution.*

Proof. Consider a point x lying on a vertical line with the 1st coordinate being equal to $\frac{1}{2}$, i.e. $x = (\frac{1}{2}, x_2)$. Then under the rotation around the origin by the angle $\varphi := -2\arctan(\frac{1}{2x_2})$ we get a point $\tilde{x} := (-\frac{1}{2}, x_2)$, which coincides with $x = (\frac{1}{2}, x_2)$ modulo 1. The assumption $\varphi \notin [-\varphi_0, \varphi_0]$ is explained by the choice of the point x. $\qquad \square$

Proposition 7.2 *Let z be a N-periodic point and let $R := \inf_k \text{dist}(T^k z, \partial X) > 0$. Then $S(r) := \cup_{i=0}^{N-1} B_r(z)$ is a forward invariant set $\forall r \in (0, R)$.*

Proof. Consider any point x close enough to z. Under the action of the map T the $\rho_2(\cdot, \cdot)$ distance between $T^n x$ and $T^n z$ is preserved. Therefore each circle centered at z with a small radius r is mapped into itself. The value of the radius r is bounded from above by the condition that the rotated circle does not intersect with the boundary of the unit square. If additionally φ/π is an irrational number, then the trajectory of the point x fills densely the corresponding circle, otherwise, it is periodic. $\qquad\square$

Observe that the number of ergodic invariant measures is always countable in distinction to the one-dimensional case, where it is known that in a "typical" situation the number of ergodic measures is bounded from above by the number of elements of the special partition.

A substantially more general result about the dynamics of a general PWI in a neighborhood of a given trajectory based on the Proposition 7.2 may be formulated as follows.

Let T be a PWI of $X \in \mathbb{R}^d$ with respect to a metric $\rho(\cdot, \cdot)$ and a special partition $\{X_i\}$.

Proposition 7.3 *Let a trajectory of a point $z \in X$ be separated fromcthe boundary of the special partition Γ, i.e. $\inf_{n \geq 0} \rho(T^n z, \Gamma) := R > 0$. Then $\forall x \in B_R(z)$ the itineraries of the trajectories started at the points x, z coincide, i.e. $T^n x \in X_i \Longleftrightarrow T^n z \in X_i \ \forall n \geq 0$. Moreover, $\forall r \in (0, R)$ the spheres $S_r(T^n z)$ of radius r centered at $T^n z$ are mapped one to another under dynamics, i.e. $T S_r(T^n z) = S_r(T^{n+1} z)$. If the separation property breaks down the trajectory stated at z may be completely isolated in the sense that $\liminf_{n \to \infty} \rho(T^n x, T^n z) > 0$ whenever $x \neq z$.*

Proof. The separation assumption means that the ball $B_R(z)$ at each iteration of the map T falls entirely to a single element of the special partition, which implies the claim about the itineraries. Now using the same argument as in the proof of Proposition 7.2 we see that the sequence of balls $B_R(T^n z)$ splits into the continuous on-parametric family of concentric spheres $S_r(T^n z)$ mapped one to another under dynamics.

Observe that in the case of a periodic trajectory the separation property becomes especially simple and trivially checkable: $\cup_{n \geq 0} T^n z \setminus \Gamma = \emptyset$.

To demonstrate the last claim we return to the pure rotation of the 2-dimensional torus around the origin by the angle $\pi/4$. Clearly, the point $z = 0$ is fixed under dynamics. Observe that a sufficiently small neighborhood O_z of this point on the torus consists of 4 triangular sectors at the corners of the unit square and that each of these sectors belong to at most 2 elements of the special partition. This allows to use a simple numerical simulations to check that $T^n(X \setminus \{z\}) \cap O_z = \emptyset \ \forall n \gg 1$, which finishes the proof.

Our numerical simulations indicate that the last claim remains valid for any rotation angle $\varphi \neq 0$ and any shift b, but this is not proven yet. Instead we consider an even more unexpected situation when a periodic trajectory of an one-dimensional non orientation preserving PWI is isolated.

Example 7.2 *Consider a PWI $T : X \to X := [0, 1)$ with the special partition into 4 intervals $X_1 := [0, \frac{1}{4}), X_2 := [\frac{1}{4}, \frac{1}{2}), X_3 := [\frac{1}{2}, \frac{3}{4}], X_2 := (\frac{3}{4}, 1)$, where the map is defined as follows $T X_1 := X_4, T X_2 := -X_3, T X_3 := X_1, T X_4 = X_4$. The minus in $T X_2 := -X_3$ means the flip (orientation change) of the of the interval X_3, i.e. $T\frac{1}{4} = \frac{3}{4}, T\frac{1}{2} = \frac{1}{4}$.*

$X_1 := [0, \frac{1}{4}), X_2 := [\frac{1}{4}, \frac{1}{2}), X_3 := [\frac{1}{2}, \frac{3}{4}], X_4 := (\frac{3}{4}, 1)$

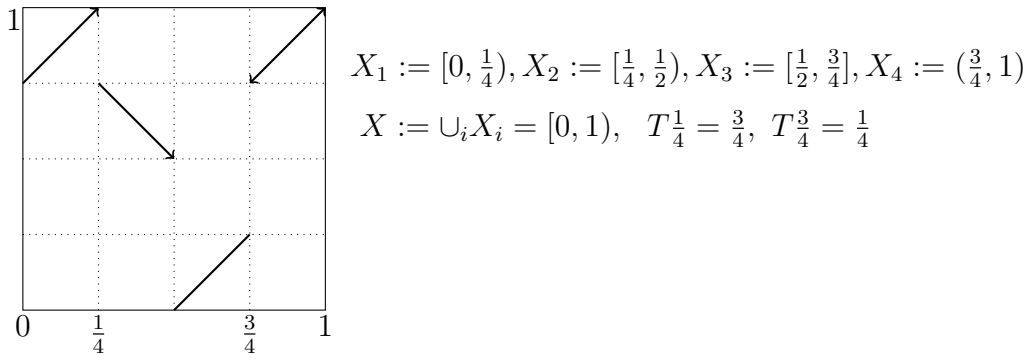$X := \cup_i X_i = [0, 1), \quad T\frac{1}{4} = \frac{3}{4}, \ T\frac{3}{4} = \frac{1}{4}$

Figure 6: A non orientation preserving circle PWI with an isolated periodic trajectory $1/2, 3/4$.

Observe that in this example the trajectory started at the point $\frac{1}{4}$ is 2-periodic, while all points from the surrounding intervals $(0, \frac{1}{4})$ and $(\frac{1}{4}, \frac{3}{4})$ in 2 iterations are mapped to the 4-th interval $(\frac{3}{4}, 1)$ and never return back. □

References

[1] R. Adler, B. Kitchens, C. Tresser, Dynamics of non-ergodic piecewise affine maps of the torus, Ergodic Theory and Dynamical Systems, 21:4 (2001), 959-999. DOI: 10.1017/S0143385701001468

[2] M. Blank, Ergodic averaging with and without invariant measures, Nonlinearity 30:12(2017), 4649-4664.

[3] M. Boshernitzan, I. Kornfeld, Interval translation mappings, Ergodic Theory Dynam. Systems, 15 (1995), 821-832.

[4] H. Bruin, Renormalization in a class of interval translation maps of d branches, Dynamical Systems, 22:1 (2007), 11-24.

[5] A. I. Bufetov, Decay of correlations for the Rauzy-Veech-Zorich induction map on the space of interval exchange transformations and the central limit theorem for the Teichmuller flow on the moduli space of abelian differentials, J. Amer. Math. Soc., 19:3 (2006), 579-623.

[6] J. Buzzi, P. Hubert, Piecewise monotone maps without periodic points: rigidity, measures and complexity, Ergodic Theory Dynam. Systems 24:2 (2004), 383-405.

[7] Z. Chen, R. Yu, X. Fu, Invariant measures for planar piecewise isometries. Journal of Shanghai University (English Edition), 14(3)(2010), 174-176. doi:10.1007/s11741-010-0625-3

[8] I.P. Cornfeld, S.V. Fomin, Y.G. Sinai, Ergodic Theory. New York: Springer-Verlag. 1982.

[9] J.H.B. Deane, Piecewise Isometries: Applications in Engineering. Meccanica 41 (2006), 241–252.

[10] X.-C. Fu, J. Duan, Global attractors and invariant measures for non-invertible planar piecewise isometric maps. Physics Letters A, 371:4 (2007), 285-290. doi:10.1016/j.physleta.2007.06.033

[11] A. Goetz, Dynamics of piecewise isometries, Illinois J. Math., 44 (2000), 465-478.

[12] A. Katok, Interval exchange transformations and some special flows are not mixing, Israel J. Math. 35 (1980) 301–310.

[13] M. Keane, Interval exchange transformations, Mathematische Zeitschrift, 141(1975), 25–31, doi:10.1007/BF01236981.

[14] S. Kryzhevich, Invariant measures for interval translations and some other piecewise continuous maps, Math. Model. Nat. Phenom. 15 (2020) DOI: 10.1051/mmnp/2019041

[15] J.H. Lowenstein, F. Vivaldi, Renormalization of one-parameter families of piecewise isometries, Dynamical Systems, 31:4 (2016), 393-465.

[16] H. Masur, Interval exchange transformations and measured foliations, Annals of Mathematics, Second Series, 115 (1)(1982), 169–200, doi:10.2307/1971341.

[17] S. Marmi, P. Moussa, J-C. Yoccoz, Linearization of generalized interval exchange maps, Ann. of Math. (2) 176 (2012), 1583-1646.

[18] S.P. Novikov, B.A. Dubrovin, A.T. Fomenko. Modern geometry. Methods and applications, Nauka, Moscow, 1979.

[19] V.I. Oseledets, The spectrum of ergodic automorphisms. Dokl. Akad. Nauk SSSR 168 (1966), 1009-1011.

[20] B. Pires, Invariant measures for piecewise continuous maps Comptes Rendus Mathematique, 354:7 (2016), 717-722.

[21] A. Skripchenko, S. Troubetzkoy, Polygonal billiards with one sided scattering, Ann. Inst. Fourier (Grenoble) 65:5 (2015), 1881–1896.

[22] S. Truong, Dynamics of piecewise translation maps, arXiv:1610.04700, (2016).

[23] W.A. Veech, Gauss measures for transformations on the space of interval exchange maps, Annals of Mathematics, Second Series, 115 (1)(1982), 201–242, doi:10.2307/1971391, MR 0644019.

[24] M. Viana, Ergodic theory of interval exchange maps, Revista Matematica Complutense 19:1 (2006), 7-100.

[25] D. Volk, Almost every interval translation map of three intervals is finite type, Discrete and Continuous Dynamical Systems - A, 34:5 (2014), 2307 - 2314.