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From formal to actual Puiseux series solutions of algebraic differential equations of first order

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Abstract. The existence, uniqueness and convergence of formal Puiseux series solutions of non-autonomous algebraic differential equations of first order at a nonsingular point of the equation is studied, including the case where the celebrated Painlevé theorem cannot be applied explicitly for the study of convergence. Several examples illustrating relationships to the Painlevé theorem and lesser-known Petrović's results are provided.

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1. Introduction

The solution of the algebraic equation

$$F(x, y) = \sum_{i=1}^{n} a_i \, x^{p_i} \, y^{q_i} = 0, \qquad x, y \in \mathbb{C}, \tag{1.1}$$

is a unique object, an algebraic function $y = y(x)$ if the polynomial F is irreducible or a set of algebraic functions if F is reducible. When one talks about local properties of the solution, then in a neighborhood of almost every point $x = x_0 \in \mathbb{C}$ there are finitely many holomorphic germs of the function $y(x)$, whereas in a neighborhood of each point $x = x_0$ from the finite set of ramification points there are finitely many germs of this function which can be presented as convergent Puiseux series in fractional powers of $x - x_0$ with a finite principal part. The construction of the Newton-Puiseux polygon of an algebraic equation

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allows to find all the germs. One more remark is that every Puiseux series, which formally satisfies the equation (1.1), has a nonzero radius of convergence, see [14] for a contemporary account and [17, Chapter 4.3] for a more classical approach. However, a non-algebraic function near its singular point $x = x_0$ also may have a presentation by a convergent Puiseux series with finite principal part, and any x_0 possessing such a property is referred to as a singular point of an *algebraic type.*

When we consider an algebraic *differential* equation

$$F(x, y, y') = \sum_{i=1}^{n} a_i(x)\, y^{p_i}\, (y')^{q_i} = 0, \tag{1.2}$$

a_i being polynomials, then the situation is quite different. The solution is not a unique object any more. In such a case, we consider an *entire collection of solutions* gathered under one notion – the *general solution* of the equation. In general, solutions of (1.2) may have singularities of non-algebraic type, and also formal power series in the variable $x - x_0$, which satisfy (1.2), may diverge. As a simple example, we consider the equation

$$xy' - 1 = 0.$$

Its general solution possesses a singularity of non-algebraic type at the point $x = 0$. Another, notable example is the Euler equation (see [6, Chapter II]),

$$x^2 y' - y + x = 0,$$

which has a formal solution in a form of the power series

$$\sum_{k=0}^{\infty} k!x^{k+1}$$

in the variable x with the radius of convergence equal to zero.

We come to two natural and important questions:

(i) *For which points* $x = x_0 \in \mathbb{C}$ *one can guarantee the algebraic local behaviour of the general solution of the equation* (1.2)*?*

(ii) *For which Puiseux series in the variable* $x - x_0$ *formally satisfying the equation* (1.2)*, one can guarantee convergence?*

(Throughout the paper we always mean by x_0 a finite point of $\mathbb{C}$. We don't speak about the infinite point separately, since it is studied, as usual, *via* the change $t = 1/x$ of the independent variable and considering the point $t = 0$ of the transformed equation.) These two questions are related to each other in fact, by mean of the "fundamental existence theorem" [15, 16]. The latter states that for each formal power series solution $\varphi \in \mathbb{C}[[x - x_0]]$ of (1.2) there exists an actual solution having φ as an asymptotic expansion in some sector with the vertex at x_0. Hence the existence of a *divergent* Puiseux series in the variable $x - x_0$ formally satisfying the equation (1.2) implies that the equation possesses a solution for which $x = x_0$ is a singular point of a non-algebraic type.

At the same time, the answer to Question (i) partially follows from the celebrated Painlevé theorem [8–10]. According to this theorem every solution to the equation (1.2) can only have a singularity of an *algebraic type* at any point $x = x_0$, *with the exception of points of some fixed finite set* Σ *at most determined by the equation*. In other words, non-algebraic singular points of the solutions of a first order algebraic differential equation cannot fill domains in $\mathbb{C}$. Therefore, any Puiseux series in the variable $x - x_0$ formally satisfying the equation (1.2) converges near x_0, if $x_0 \in \mathbb{C} \setminus \Sigma$, which partially answers Question (ii). In particular, for an *autonomous* algebraic differential equation of first order the set Σ is empty and hence any formal Puiseux series solution of such an equation has a nonzero radius of convergence (although after the change $t = 1/x$ the transformed equation becomes non-autonomous, the last statement on convergence is still true for formal Puiseux series solutions in the variable $1/x$ considered in a neighbourhood of infinity, which was proved in [2]).

However, the question of detecting the set Σ of potentially non-algebraic singular points of the general solution of the equation (1.2) is quite delicate and elaborate. Writing the equation for a moment as

$$F(x, y, y') = A_0(x, y) + A_1(x, y)y' + \ldots + A_s(x, y)(y')^s = 0, \tag{1.3}$$

one definitely says that Σ contains the points $x = x_0$ for which

- Either $A_s(x_0, y) \equiv 0$;
- Or the equations $A_0(x_0, y) = 0$, $A_1(x_0, y) = 0, \ldots, A_s(x_0, y) = 0$ have a common solution;
- Or after the change of variable $y = 1/w$, for the new equation of the form (1.3) with the coefficients $\widetilde{A}_i(x, w)$, one has $\widetilde{A}_0(x_0, 0) = \widetilde{A}_1(x_0, 0) = \ldots = \widetilde{A}_s(x_0, 0) = 0$.

Yet Σ may contain additional points. In the case when the system

$$F(x, y, y') = 0, \quad \frac{\partial F}{\partial y'}(x, y, y') = 0, \quad \frac{\partial F}{\partial x}(x, y, y') + y' \frac{\partial F}{\partial y}(x, y, y') = 0 \tag{1.4}$$

has a finite set of isolated solutions (x_0, y_0, y_0'), such additional points are those $x = x_0$ coming from these solutions, see [8, page 42] (the explanation is also given by Picard [13, Chapter II, pages 40-41] based on the results of Briot and Bouquet [1]).

But in the case when the Painlevé-Picard system (1.4) is compatible for every $x \in \mathbb{C}$, detecting the set Σ is not clarified in general and thus Questions (i), (ii) have no an immediate answer. Moreover, even when one would succeed in obtaining the set Σ, for its points x_0 some Puiseux series in the variable $x - x_0$ formally satisfying the equation (1.2) could also converge (see Example 3.8 in the last section).

Therefore our aim is to present some general analysis of the question of convergence without appealing to the set Σ. We indeed propose such an analysis for any *nonsingular* point x_0 of the equation (1.2), in the sense of the following definition.

Definition 1.1. A point $x = x_0 \in \mathbb{C}$ will be referred to as a *singular point of the equation* (1.2) if it is a zero of any of the coefficients a_i.

We prove that every Puiseux series in powers of the variable $x - x_0$, starting with a generic term $c(x - x_0)^\lambda$, $\lambda \in \mathbb{Q}^*$, and formally satisfying (1.2), converges in a neighborhood of the nonsingular point $x = x_0$ of the equation (1.2). The last statement is content of Theorem 3.4(b).

Before we investigate the question of convergence of formal Puiseux series satisfying a *non-autonomous* algebraic differential equation of first order, we will study the existence and uniqueness of such formal Puiseux series solutions in Theorem 3.4(a). To that end we will employ the Newton-Puiseux polygons in the form in which it was proposed and applied by Mihailo Petrović to study the local behavior of solutions in a neighborhood of a *nonsingular* point of the equation. Thus, the Petrović method is exposed in the next section. As a side remark let us also mention that Petrović observed an interesting application of *non-autonomous* algebraic differential equations of first order in chemical dynamics in [12].

2. Petrović's polygonal method and theorems

The Serbian mathematician Mihailo Petrović Alas, a student of Emile Picard and Charles Hermite, defended his thesis [11] in 1894. One of the chapters of the thesis is devoted to the study of the analytic properties of solutions of first-order algebraic differential equations. He developed a method, the *Petrović polygonal method*, applicable to algebraic differential equations of any order. His method uses the same principles as the Newton-Puiseux polygonal method for algebraic equations. Moreover, the Petrović polygon differs from the polygons of C. Briot and J. Bouquet, and of H. Fine [5], who also generalized the Newton-Puiseux polygonal method. In [4], a comparative analysis of constructions of polygons of Petrović and Fine was performed and Theorem 7 therein establishes the relationship between the two methods under certain conditions, see also [3, Theorem 2], where the notion of "the Fine-Petrović" polygons was coined.

Petrović investigated the local behavior of the general solution of the equation (1.2) in the neighborhood of its *nonsingular* point $x = x_0$. According to the Painlevé theorem, the solutions of the equation (1.2) generically have the local form $y = (x - x_0)^\lambda f(x)$, where $\lambda \in \mathbb{Q}$, and the limit of the function $f(x)$ is finite and nonzero at the point $x = x_0$. Petrović considered the case when $\lambda \neq 0$, *i.e.*, he investigated the question of *movable* zeros and poles (generally speaking, critical) of the general solution. He proved the following remarkable statements (see [11, Théorème II and Théorème V on pages 22-23]).

The first statement: *the presence of a slanted edge of the polygon of the equation* (1.2) *with the normal vector* $(\lambda, -1)$, $\lambda < 0$, *is a necessary and sufficient* *condition for the general solution of this equation to have a movable pole of order* $-\lambda$. The second statement can be considered as a consequence of the first: *the presence of a slanted edge of the polygon of the equation* (1.2) *with the normal vector* $(\lambda, -1)$, $\lambda > 0$, *is a necessary and sufficient condition that the general solution to this equation has a movable zero of order* λ.

The polygon of the algebraic differential equation (1.2) corresponding to any of its *nonsingular* points is defined by Petrović [11] as follows. To each summand $a_i(x)y^{p_i}(y')^{q_i}$ in (1.2) one relates the point $(M_i, N_i) = (p_i + q_i, q_i)$ of the plane OMN. Then the Petrović polygon of the algebraic differential equation (1.2) is *that part of the boundary of the convex hull* of the set of points (M_i, N_i), $(M_i, 0)$, $i = 1, \dots, n$, which is left after removing the horizontal segment of the OM axis and vertical segments from this boundary.

Let us give an example of the polygon of the equation

$$-2y^3y'^3 - y^2y'^2 + y - (x^3 + 1) = 0.$$

We take four points $(6, 3)$, $(4, 2)$, $(1, 0)$, $(0, 0)$ corresponding to the summands of the left-hand side of the equation and the additional points $(6, 0)$, $(4, 0)$. The convex hull of these six points is a right triangle with the slanted edge $[(0, 0), (6, 3)]$, the vertical edge $[(6, 3), (6, 0)]$, and the horizontal edge $[(0, 0), (6, 0)]$. After excluding the vertical and horizontal edges we obtain the Petrović polygon that consists of one slanted edge $[(0, 0), (6, 3)]$ with the normal vector $(1/2, -1)$, which is represented below in Figure 2.1.

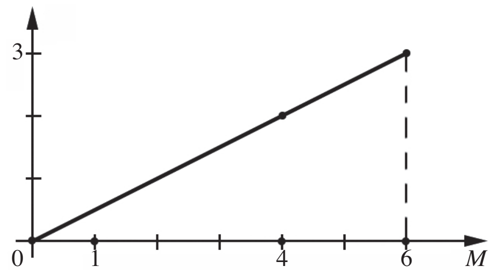

Figure 2.1. The Petrović polygon.

A distinctive feature of a differential equation of precisely the *first* order is that to each of the points (M_i, N_i) of its Petrović polygon, corresponds exactly one monomial of the sum (1.2), as for the Newton-Puiseux polygon of an algebraic equation.

3. The existence and convergence of formal Puiseux series solutions

Let x_0 be a nonsingular point of the equation (1.2). Suppose that the polygon of that equation has a slanted edge I_λ with a normal vector $(\lambda, -1)$, $\lambda \neq 0$. Consider the *approximate* equation

$$\hat{F}_{x_0,\lambda}(y, y') = \sum_{i:\, Q_i \in I_\lambda} a_i(x_0)\, y^{p_i} (y')^{q_i} = 0, \tag{3.1}$$

where only the monomials of the function F corresponding to the points $Q_i = (M_i, N_i) = (p_i + q_i, q_i)$ of this edge participate in the sum.

Definition 3.1. A polynomial

$$P_{x_0,\lambda}(c) = \sum_{i:\, Q_i \in I_\lambda} a_i(x_0)\, \lambda^{N_i} c^{M_i}$$

will be referred to as the *characteristic polynomial* corresponding to the edge I_λ.

First we note that each nonzero root c of the characteristic polynomial corresponding to the edge I_λ induces a solution $\varphi_0 = c(x - x_0)^\lambda \neq 0$ of the equation (3.1). Indeed, if we substitute φ_0 in the approximate equation (3.1) (in the polynomial $\hat{F}_{x_0,\lambda}(y, y')$) we will obtain:

$$\hat{F}_{x_0,\lambda}(\varphi_0, \varphi_0') = P_{x_0,\lambda}(c)\,(x - x_0)^\gamma,$$

where γ is the (constant) value of the linear function $L_\lambda(X, Y) = \lambda X - Y$ on the edge I_λ.

Definition 3.2. We will call the solution $\varphi_0 = c(x - x_0)^\lambda$ of (3.1) *non-exceptional* if c is a simple root of $P_{x_0,\lambda}$.

Note that the polynomial $P_{x_0,\lambda}$ necessarily has a nonzero root, since it contains at least two monomials and all M_i are different. Hence, the approximate equation (3.1) necessarily has a nonzero solution of the form $\varphi_0 = c(x - x_0)^\lambda$.

Definition 3.3. We will say that φ_0 is extended to the Puiseux series solution of (1.2) if it is the first term of a unique Puiseux series satisfying the equation (1.2) and having a nonzero radius of convergence.

Theorem 3.4. *Let x_0 be a nonsingular point of the equation* (1.2). *Suppose that the polygon of that equation has at least one slanted edge.*

(a) *Each* non-exceptional *solution $\varphi_0 = c(x - x_0)^\lambda \neq 0$ of the approximate equation* (3.1) *corresponding to a slanted edge I_λ with the normal vector* $(\lambda, -1)$, *$\lambda = r/s \neq 0$, r and s being coprime, is extended to the Puiseux series solution of* (1.2) *of the form*

$$\varphi = c(x - x_0)^{r/s} + \sum_{k \geqslant 1} c_k (x - x_0)^{(r+k)/s}; \tag{3.2}$$

(b) *Conversely, any formal Puiseux series solution of* (1.2) *whose first term* $\varphi_0 = c(x-x_0)^\lambda$ *is a non-exceptional solution of the approximate equation* (3.1) *corresponding to the slanted edge* I_λ*, has a nonzero radius of convergence.*

Remark 3.5. Generally speaking, there can exist several significantly different such solutions of the form $\varphi_0 = c(x-x_0)^\lambda \neq 0$ of the equation (3.1). Here the significant difference means that they are not transforming into each other after analytic continuation around the point $x = x_0$. Each such solution, if it is non-exceptional, is extended to its own Puiseux series solution of the original equation.

Remark 3.6. The case of $\lambda = 0$ ($\varphi_0 = c$) is not covered by Theorem 3.4 and needs an additional study.

Proof. (a) By technical reason connected with the further application of Malgrange's convergence theorem [7], we will consider the equation (1.2) rewritten in the form

$$\widetilde{F}(x,y,\delta y) = \sum_{i=1}^{n} a_i(x)(x-x_0)^{m-q_i}\, y^{p_i}(\delta y)^{q_i} = 0, \tag{3.3}$$

where $\delta = (x-x_0)(d/dx)$ and $m = \max_i q_i$.

Make the change of variable in the above equation (3.3):

$$y = \varphi_0 + (x-x_0)^\lambda u, \qquad \delta y = \delta\varphi_0 + (x-x_0)^\lambda(\delta+\lambda)u.$$

For simplicity we will use the notation $w := \delta y$. Then the Taylor formula yields the equation

$$\begin{aligned}\widetilde{F}(x,y,w) = {}& \widetilde{F}(x,\varphi_0,\delta\varphi_0) + (x-x_0)^\lambda \frac{\partial \widetilde{F}}{\partial y}(x,\varphi_0,\delta\varphi_0)u \\ & + (x-x_0)^\lambda \frac{\partial \widetilde{F}}{\partial w}(x,\varphi_0,\delta\varphi_0)(\delta+\lambda)u \\ & + \sum_{k+l\geqslant 2} \frac{(x-x_0)^{(k+l)\lambda}}{k!\,l!} \frac{\partial^{k+l}\widetilde{F}}{\partial y^k\,\partial w^l}(x,\varphi_0,\delta\varphi_0)u^k((\delta+\lambda)u)^l = 0.\end{aligned} \tag{3.4}$$

Recall that $M_i = p_i + q_i$ and $N_i = q_i$. Therefore,

$$\begin{aligned}\widetilde{F}(x,\varphi_0,\delta\varphi_0) &= \sum_{i=1}^{n} a_i(x)\,\lambda^{N_i} c^{M_i}(x-x_0)^{m+\lambda M_i - N_i} \\ &= \left(\sum_{i:\,Q_i\in I_\lambda} a_i(x_0)\,\lambda^{N_i} c^{M_i}\right)(x-x_0)^{m+\gamma} \\ &\quad + o\left((x-x_0)^{m+\gamma}\right) = P_{x_0,\lambda}(c)(x-x_0)^{m+\gamma} \\ &\quad + o\left((x-x_0)^{m+\gamma}\right) = o\left((x-x_0)^{m+\gamma}\right).\end{aligned} \tag{3.5}$$

This is due to the fact that the value $\lambda M_i - N_i = \gamma$ of the function $L_\lambda(X,Y) = \lambda X - Y$ is the same at each point $Q_i = (M_i, N_i)$ of the edge I_λ, whereas the values of this function at all $Q_i \notin I_\lambda$ are greater than γ. Similarly,

$$\begin{aligned}\frac{\partial \widetilde{F}}{\partial y}(x,\varphi_0,\delta\varphi_0) &= \sum_{i=1}^{n} a_i(x)\, p_i\, \lambda^{N_i} c^{M_i-1}(x-x_0)^{m+\lambda M_i - N_i - \lambda}\\ &= \Biggl(\sum_{i:\, Q_i\in I_\lambda} a_i(x_0)\, p_i\, \lambda^{N_i} c^{M_i-1}\Biggr)(x-x_0)^{m+\gamma-\lambda}\\ &\quad + o\Bigl((x-x_0)^{m+\gamma-\lambda}\Bigr).\end{aligned}$$

Since $p_i = (1-\lambda)M_i + \gamma$ for each $Q_i \in I_\lambda$, we have

$$\frac{\partial \widetilde{F}}{\partial y}(x,\varphi_0,\delta\varphi_0) = (1-\lambda)P'_{x_0,\lambda}(c)(x-x_0)^{m+\gamma-\lambda} + o\Bigl((x-x_0)^{m+\gamma-\lambda}\Bigr). \tag{3.6}$$

In an analogous way,

$$\begin{aligned}\frac{\partial \widetilde{F}}{\partial w}(x,\varphi_0,\delta\varphi_0) &= \Biggl(\sum_{i:\, Q_i\in I_\lambda} a_i(x_0)\, q_i\, \lambda^{N_i-1} c^{M_i-1}\Biggr)(x-x_0)^{m+\gamma-\lambda}\\ &\quad + o\Bigl((x-x_0)^{m+\gamma-\lambda}\Bigr)\\ &= P'_{x_0,\lambda}(c)(x-x_0)^{m+\gamma-\lambda} + o\Bigl((x-x_0)^{m+\gamma-\lambda}\Bigr),\end{aligned} \tag{3.7}$$

as $q_i = \lambda M_i - \gamma$ for each $Q_i \in I_\lambda$. Finally,

$$(x-x_0)^{(k+l)\lambda}\,\frac{\partial^{k+l}\widetilde{F}}{\partial y^k\,\partial w^l}(x,\varphi_0,\delta\varphi_0) = O\bigl((x-x_0)^{m+\gamma}\bigr). \tag{3.8}$$

Thus, in view of (3.5), (3.6), (3.7), (3.8), after dividing by $(x-x_0)^{m+\gamma}$ the equation (3.4) takes the form

$$\begin{aligned}&(1-\lambda)P'_{x_0,\lambda}(c)u + P'_{x_0,\lambda}(c)(\delta+\lambda)u\\ &= (x-x_0)^{1/s}\bigl(f(x) + g(x)u + h(x)(\delta+\lambda)u\bigr) + L_2(x,u,(\delta+\lambda)u),\end{aligned}$$

where the coefficients f, g, h of the linear part on the right-hand side are polynomials in $(x-x_0)^{1/s}$, whereas the function L_2 is polynomial in $(x-x_0)^{1/s}$, u, $(\delta+\lambda)u$ and contains only terms at least quadratic in u, $(\delta+\lambda)u$. Simplifying the left-hand side of the above equation we obtain

$$\begin{aligned}P'_{x_0,\lambda}(c)(\delta+1)u = (x-x_0)^{1/s}\bigl(&f(x) + g(x)u + h(x)(\delta+\lambda)u\bigr)\\ &+ L_2(x,u,(\delta+\lambda)u).\end{aligned}$$

Making the change of the independent variable, $x - x_0 = t^s$, we come to the following algebraic differential equation near $0 \in \mathbb{C}^3$:

$$\begin{aligned} P'_{x_0,\lambda}(c)((1/s)\delta_t + 1)u = t\big(\tilde{f}(t) + \tilde{g}(t)u + \tilde{h}(t)((1/s)\delta_t + \lambda)u\big) \\ + \widetilde{L}_2(t, u, ((1/s)\delta_t + \lambda)u), \end{aligned} \tag{3.9}$$

where $\delta_t = t(d/dt)$.

The equation (3.9) has a unique power series solution $\hat{u} = \sum_{k\geqslant 1} c_k t^k \in \mathbb{C}[[t]]$, where

$$P'_{x_0,\lambda}(c)(1 + 1/s)c_1 = \tilde{f}(0),$$

and other c_k's with $k>1$ are uniquely determined by the previous ones $c_1, \ldots, c_{k-1}$. By the Malgrange theorem [7], this series has a nonzero radius of convergence: if we write down (3.9) in the form $G(t, u, \delta_t u) = 0$ then we have $\mathrm{ord}_0\, G'_u(t, \hat{u}, \delta_t \hat{u}) = \mathrm{ord}_0\, G'_{\delta_t u}(t, \hat{u}, \delta_t \hat{u}) = 0$. Hence the initial equation (1.2) possesses the unique Puiseux series solution $\varphi = c(x - x_0)^{r/s} + \sum_{k\geqslant 1} c_k\, (x - x_0)^{(r+k)/s}$ starting with the term $\varphi_0 = c(x - x_0)^{r/s}$ and having a nonzero radius of convergence.

(b) Since we assume that the first term $\varphi_0 = c(x - x_0)^{\lambda}$, $\lambda \in \mathbb{Q}^*$, of the formal Puiseux series φ satisfying the equation (1.2) is a *non-exceptional* solution of the approximate equation (3.1) corresponding to the slanted edge I_λ, by virtue of uniqueness and convergence of the Puiseux series starting with φ_0 and satisfying the equation (1.2), which was obtained in the item (a), the series φ has a nonzero radius of convergence. □

Let us conclude with several examples illustrating that, although the Painlevé theorem and Theorem 3.4 have an intersection in applications, they naturally complement each other on the important questions of convergence.

Example 3.7. Let us consider the first example, an algebraic differential equation for which the Painlevé-Picard system (1.4) is compatible for any $x \in \mathbb{C}$:

$$F(x, y, y') = y'^3 - \big(y - x^4\big)^2 = 0. \tag{3.10}$$

It follows that the fixed finite set Σ where solutions could have singular points $x = x_0$ of non-algebraic type and thus formal Puiseux series solutions in powers of $x - x_0$ could be divergent, is somehow hidden inside the equation. However, one can easily see that for any point $x_0 \neq 0$ the polygon of the equation consists of one edge connecting the points $(0, 0)$ and $(3, 3)$ thus having the normal vector $(1, -1)$. The characteristic polynomial corresponding to this edge $I_{\lambda=1}$ is

$$P_{x_0,1}(c) = c^3 - x_0^8,$$

and all its roots are simple ($x_0 \neq 0$). Hence Theorem 3.4 gives that, for each of the three values of $x_0^{8/3}$, there is a unique formal Puiseux series solution of (3.10) beginning with $x_0^{8/3}(x - x_0)$ and having a nonzero radius of convergence. This is a Taylor series in fact, which also follows from an analytic version of the implicit function theorem for ordinary differential equations, since $F'_{y'}(x_0, 0, x_0^{8/3}) \neq 0$.

One could expect the singular point $x = 0$ of the equation to be a candidate for a non-algebraic singular point of the general solution of (3.10). Actually at this point there is a formal Puiseux series solution $y = x^4 \sum_{k \geqslant 0} a_k x^{k/2} = x^4 + 8x^{9/2} + 108x^5 + 1863x^{11/2} + 37665x^6 + \dots$. The numerical evidences indicate that this series is rapidly diverging.

Example 3.8. The next example shows that in the case where the set Σ is simply detected by an equation, the convergence of some Puiseux series solutions in the variable $x - x_0$, $x_0 \in \Sigma$, can still be established by Theorem 3.4.

Consider the equation

$$F(x, y, y') = -2y^3 y'^3 - y^2 y'^2 + y - \big(x^3 + 1\big) = 0, \tag{3.11}$$

which already appeared in Section 2. In this case the Painlevé-Picard system (1.4) is compatible for $x = 0$ and this point thus being an element of Σ could be the fixed singular point of non-algebraic type for the general solution. The equation possesses a formal power series solution

$$1 + \sum_{k \geqslant 3} c_k (k-2)!\, x^k \tag{3.12}$$

in powers of x, starting with the constant term $c_0 = 1$ and having a zero radius of convergence, which says that there indeed exist solutions with the singularity of non-algebraic type at $x = 0$. Indeed, after the change $y = 1 + w$ of the dependent variable, we come to the equation

$$(1 + w)^2 (1 + 2(1 + w)w')w'^2 = w - x^3.$$

By substituting into the above equation $w = \sum_{k \geqslant 3} c_k (k-2)!\, x^k$, we get $c_3 = 1$ and, further,

$$\big(1 + 6x^2 + \dots\big)\, w'^2 = \sum_{k \geqslant 4} c_k (k-2)!\, x^k,$$

hence

$$\big(1 + 6x^2 + \dots\big) \left(\sum_{l \geqslant 3} c_l (l-2)!\, l\, x^{l-1} \right) \left(\sum_{m \geqslant 3} c_m (m-2)!\, m\, x^{m-1} \right)$$
$$= \sum_{k \geqslant 4} c_k (k-2)!\, x^k.$$

Thus, $c_4 = 9/2$. By comparing the coefficient $c_k(k-2)!$ of x^k on the right-hand side with the coefficient of x^k from the left-hand side, we get

$$c_k(k-2)! = 6c_{k-1}(k-3)!(k-1) + P_k\big(c_3, \dots, c_{k-2}\big), \qquad k = 5, 6, \dots,$$

where P_k is a polynomial with *positive* coefficients. From the relation

$$c_k = \frac{6(k-1)}{k-2}\, c_{k-1} + \frac{1}{(k-2)!}\, P_k\big(c_3, \dots, c_{k-2}\big), \qquad k = 5, 6, \dots,$$

the coefficients c_k are uniquely determined. Taking into account that $c_3 = 1$ and $c_4 = 9/2$, by applying induction we prove that $c_k \geqslant 1$ for all $k \geqslant 3$. Consequently, the series (3.7) has the radius of convergence equal to zero.

On the other hand, the polygon of the equation (3.11) corresponding to its nonsingular point $x = 0$ consists of one edge connecting the points $(0, 0)$ and $(6, 3)$, see Figure 1. The normal vector of this edge is $(1/2, -1)$ and the corresponding characteristic polynomial is

$$P_{0,1/2}(c) = -\frac{1}{4}\, c^6 - \frac{1}{4}\, c^4 - 1,$$

which has six simple roots $\pm c_i$, $i = 1, 2, 3$. Therefore we have three essentially different convergent Puiseux series solutions of (3.11) in the variable x, beginning with $c_i x^{1/2}$.

Concerning formal Puiseux series solutions in the variable $x - x_0$, where x_0 is a nonsingular point of an equation, that begin with an *exceptional* term $c(x - x_0)^\lambda$, $\lambda \neq 0$, they can also be convergent. This can be illustrated by quite simple examples, like the following one.

Example 3.9. Consider the equation

$$(y' - 1)^2 - 9x = 0.$$

For its nonsingular point $x_0 = 0$ and the edge $I_{\lambda=1}$ of the polygon, one has an exceptional solution $\varphi_0 = x$ of the corresponding approximate equation $(y'-1)^2 = 0$ (since $c = 1$ is a double root of the corresponding characteristic polynomial $P_{0,1}(c) = (c-1)^2$). However, this solution is extended to an actual solution $x + 2x^{3/2}$ of the initial equation. Though, note that the generator $1/2$ of the power exponents of the last truncated Puiseux series is not determined by that of the starting monomial x.

In general the assumption of non-exceptionality of a solution is essential and cannot be removed, as we see in the following example.

Example 3.10. Let us consider the differential equation

$$F(x, y, y') = y'^2 + 2y' - y + \big(1 - x + x^3\big) = 0.$$

Note that the point $x_0 = 0$ is a *nonsingular* point of this equation. The polygon has a slanted edge $I_{\lambda=1}$ containing the points $(0, 0)$, $(1, 1)$ and $(2, 2)$. The corresponding characteristic polynomial is $P_{0,1}(c) = (c + 1)^2$. The equation has a formal power series solution

$$-x + x^3 + 9x^4 + 216x^5 + 7776x^6 + \dots,$$

beginning with the *exceptional* term $\varphi_0 = -x$ (since $c = -1$ is a double root of $P_{0,1}(c)$). This formal power series has a zero radius of convergence indeed, which can be proved by making the change $y = -x + x^3 + w$ of the dependent variable and applying reasoning similar to that from Example 3.8, to the transformed equation

$$w'^2 + 6x^2 w' - w + 9x^4 = 0.$$

Note also that for the last equation the point $x_0 = 0$ becomes singular, which shows that the property of a point to be nonsingular for an equation is not invariant with respect to the change of the dependent variable.

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